18.06 Final Exam, Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This final exam is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 6 problems.

Additional paper for your calculations is provided at the back of this booklet.

Good luck.

Problem	Maximum Points	Your Points
1.	15	
2.	15	
3.	15	
4.	20	
5.	20	
6.	15	
Total	100	

1. (15pts.)

(a) Show that the system S:

$$\begin{array}{rcrcrcr} x & + & y & & & = & 3 \\ x & + & y & + & bz & = & 2 \\ ax & + & by & + & (b-a)z & = & 1+3a \end{array}$$

has no solutions if $b = 0$ or $a = b$.

(b) Calculate the solution for $a = 2$, $b = 1$.

(c) Find a general formula for the solution of the system S for $b \neq 0$ and $a \neq b$.

Additional paper for your calculations at the back of this booklet.

2. (15pts.)

(a) Decide whether or not the following vectors form a basis for $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.$$

(b) Find an orthonormal basis for $\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Additional paper for your calculations at the back of this booklet.

3. (15pts.) Let

$$A_n = \begin{pmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & a_2 & -1 & 0 & \cdots & 0 \\ 0 & 1 & a_3 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & -1 \\ 0 & 0 & \cdots & 0 & 1 & a_n \end{pmatrix}.$$

(a) Show for $n \geq 3$ that $\det A_n = a_n \det A_{n-1} + \det A_{n-2}$.

(b) Calculate $\det A_6$ for the cases that (i) $a_j = j$ for all $j = 1, \ldots, 6$, and (ii) $a_j = 6 - j$, for all $j = 1, \ldots, 6$.

Additional paper for your calculations at the back of this booklet.

4. (20pts.) Let U and V be vector spaces.

(a) Define the *kernel* and *image* of a linear transformation $T : U \to V$.

(b) Show that the kernel of T is a subspace of U.

(c) Let T be a linear transformation from U to V and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a basis of Ker T. The following steps help you to show that if $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$ form a basis of U, then $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ form a basis of Im T. So assume that $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1} \ldots, \mathbf{u}_n$ are a basis of U.

i. Argue that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are elements of Im T.

ii. Show that any element of Im T can be expressed as a linear combination of $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$.

iii. Show that $T\mathbf{u}_{k+1}, \ldots, T\mathbf{u}_n$ are linearly independent.

(d) Deduce a formula relating the dimensions of U, Ker T and Im T.

Additional paper for your calculations at the back of this booklet.

5. (20pts.) Let V be the vector space of polynomials of degree at most 3 with real coefficients. Let T be the map defined by

$$T(f(x)) = \frac{d^2 f}{dx^2} + 2\frac{df}{dx}$$

for all $f(x) \in V$.

(a) Show that T is a linear transformation.

(b) Find the matrices ${}_B(T)_B$ and ${}_C(T)_C$ representing T with respect to the the bases $B = \{1, x, x^2, x^3\}$ and $C = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$, respectively.

(c) Find the matrix ${}_C(I)_B$ representing the change of basis from B to C, and verify that ${}_C(T)_C = {}_C(I)_{B\,B}(T)_{B\,B}(I)_C$.

Additional paper for your calculations at the back of this booklet.

6. (15pts.) Let

$$A = \begin{pmatrix} -3 & 2 & 4 \\ 2 & -6 & 2 \\ 4 & 2 & -3 \end{pmatrix}.$$

(a) Given that one eigenvalue of A is $\lambda_1 = 2$, find the remaining two eigenvalues of A and an eigenvector for each eigenvalue.

(b) Find an orthogonal matrix P such that $P^T AP$ is diagonal.

(c) Find an expression for $\exp(A)$.

Additional paper for your calculations at the back of this booklet.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.

Your calculations for problem ____.