18.06 Final Exam in Linear Algebra

13 December 1993: Professor Strang

Part I (short questions)

1. Find all possible values for the determinant of the given type of 3×3 real matrix.

[2] (a) A matrix with independent columns.

[2] (b) A matrix with $A^2 = A$.

[2] (c) A matrix with pivots 1, 2 and 3.

[2] (d) A Markov matrix.

[3] 2.Why is there no orthogonal matrix Q such that $Q^{-1}AQ = \Lambda$ if

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}?$$

[2] 3. The left null space $\mathcal{N}(A^T)$ of a matrix A is spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. The set of solutions of the equation $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is (circle one)

the empty set, a point, a line, a plane, a three-dimensional hyperplane in $\mathbb{R}^4$, all of $\mathbb{R}^4$.

[4] 4. Apply the Gram-Schmidt algorithm to the columns of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ to obtain three orthonormal vectors.

[3] 5. For what values of a and b is the quadratic form $ax^2 + 2xy + by^2 = [x\ y] \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ positive definite?

6. Suppose A is an m by n matrix, with independent columns.

[2] • What can you deduce about the relation of m and n?

[2] • What can you deduce about the set of solutions to $A\mathbf{x} = 0$?

[2] • For which m and n are there nonzero solutions to $A^T\mathbf{y} = 0$?

[2] • Give two properties of the matrix A^TA (other than the fact that it is square).

[2] 7. Give an example of a matrix with exactly two zero eigenvalues and no zero entries.

[3] 8. Consider the square matrix $A = \begin{bmatrix} 7 & -3 \\ 9 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$. Find the solution to the differential equation $\frac{du(t)}{dt} = Au(t)$ with initial condition $u(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

[3] 9. Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

10. Are the following statements true or false? You get 2 points for a correct answer and -2 points for an incorrect answer.

T F (a) If $M^{-1}AM = B$, then A and B must have the same eigenvectors.

T F (b) The matrices A and A^T always have the same rank.

T F (c) There is a matrix with column space is spanned by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and with row space spanned by vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

T F (d) If a square matrix has a repeated eigenvalue, it cannot be diagonalizable.

T F (e) The set of vectors in $\mathbb{R}^3$ with integer (whole number) components is a subspace of $\mathbb{R}^3$.

Part II

1. Let A be the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

[3] (a) Find a factorization $A = LU$, where L is a lower triangular matrix, and U is in echelon form.

[4] (b) Find the general solution of $Ax = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

[4] (c) The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the column space of A if a, b and c satisfy what linear conditions?

[4] 2. The vector space $\mathbb{R}^2$ has bases $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \mathbf{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$. Write the matrix for the identity linear transformation I from the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$.

3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$.

[3] (a) Find a matrix Q with orthonormal columns and an upper triangular matrix R such that $A = QR$.

[3] (b) Find the closest vector to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the column space of A.

[3] (c) If $\mathbf{v}$ and $\mathbf{w}$ are any two linearly independent vectors, find a nonzero linear combination that is perpendicular to $\mathbf{v}$.

[3] (d) Compute the matrix P which projects onto the column space of A.

4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the Euclidean space $\mathbb{R}^n$, and let A be the square matrix $\mathbf{u}\mathbf{v}^T$.

[3] (a) Describe the row space and nullspace of A in terms of $\mathbf{u}$ and $\mathbf{v}$.

[2] (b) Show that $\mathbf{u}$ is an eigenvector of A, and find the corresponding eigenvalue.

[2] (c) What condition must be satisfied by $\mathbf{u}$ and $\mathbf{v}$ for A to be skew-symmetric ($A = -A^T$)?

[2] (d) What condition must be satisfied by $\mathbf{u}$ and $\mathbf{v}$ so that $A^2 = A$?

5. Let A_n be the $n \times n$ matrix with entries $a_{ij} = \begin{cases} -2, & i = j-1, \\ 1, & i = j, \\ 1, & i = j+1, \\ 0, & \text{other entries.} \end{cases}$ For example,

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

[4] (a) Let $d_n = \det(A_n)$. Find numbers a and b such that for $n = 3, 4, 5, \ldots,$

$$d_n = a d_{n-1} + b d_{n-2}.$$

[1] (b) What is d_4?

[4] (c) Write the matrix A such that $\begin{bmatrix} d_{n+1} \\ d_n \end{bmatrix} = A \begin{bmatrix} d_n \\ d_{n-1} \end{bmatrix}$, and calculate its eigenvalues and eigenvectors.

[2] (d) Find the number λ such that d_n/λ^n tends to a non-zero, finite limit as n tends to infinity.

6. (a) If a, b, c are real numbers, find the eigenvalues and null space of the skew-symmetric

[5] matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$.

[2] (b) Explain why the matrix A in part (a) of this problem cannot be orthogonal.

11. Diagonalize the matrix $\begin{bmatrix} 13 & 4 \\ 4 & 7 \end{bmatrix}$.

2. The left null space $\mathcal{N}(A^T)$ of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

(a) What is the rank of A? What is the determinant of A?

(b) Find a linear equation or equations for a, b and c whose solutions are those values for which $Ax = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$ can be solved.

(c) The set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where a, b and c satisfy the equation(s) of part (b), is (circle one)

the empty set, a point, a line, a plane, all of $\mathbb{R}^3$.

(d) The set of solutions of the equation $Ax = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is (circle one)

the empty set, a point, a line, a plane, a three-dimensional hyperplane in $\mathbb{R}^4$, all of $\mathbb{R}^4$.

Explain why.

[2] 2. What is the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$?