18.06 Professor Strang Final Exam May 20, 2004

**Your name is:** ______________________________

**Please circle your recitation:**

**Problems 1–8 are 12 points each; Problem 9 is 4 points.**

**Thank you for taking 18.06!**

**1** Suppose $A$ is an $m$ by $n$ matrix of rank $r$. You multiply it by any $m$ by $n$ invertible matrix $E$ to get $B = EA$.

**(a)** Circle if true and cross out if false (three parts):

$$A \text{ and } B \text{ have the } \begin{cases} \textit{same nullspace} \\ \textit{same column space} \\ \textit{same bases for row space.} \end{cases}$$

**(b)** Suppose the right $E$ gives the row-reduced echelon matrix

$$EA = R = \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(**1**) Find a basis for the nullspace of $A$.

(**2**) True statement: *The nullspace of a matrix is a vector space.*

What does it mean for a set of vectors to be a vector space?

**(c)** What is the nullspace of a 5 by 4 matrix with linearly independent columns?

What is the nullspace of a 4 by 5 matrix with linearly independent columns?

**2** This matrix $A$ has column 1 + column 2 = column 3:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

**(a)** Describe the column space $\boldsymbol{C}(A)$ in two ways:

(**1**) Give a basis for $\boldsymbol{C}(A)$.

(**2**) Find all vectors that are *perpendicular* to $\boldsymbol{C}(A)$.

**(b)** The projection matrix $P$ onto the column space does not come from the usual formula $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. *Why not*—what goes wrong with this formula?

**(c)** Find that matrix $P$ for projection onto the column space of $A$.

**3** Suppose $P$ is the 3 by 3 projection matrix (so $P = P^{\mathrm{T}} = P^2$) onto the plane $2x + 2y - z = 0$. You do not have to compute this matrix $P$ but you can if you want.

**(a)** What is the rank of $P$? What are its three eigenvalues? What is its column space?

**(b)** Is $P$ diagonalizable—why or why not? Find a nonzero vector in its nullspace.

**(c)** If $b$ is any unit vector in $\mathbf{R}^3$, find the number $q$. *Explain your thinking in 1 sentence and 1 equation*:

$$q = \|Pb\|^2 + \|b - Pb\|^2 .$$

**4** **(a)** If $a \neq c$, find the eigenvalue matrix $\Lambda$ and eigenvector matrix $S$ in

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = S \Lambda S^{-1}.$$

**(b)** Find the *four entries* in the matrix $A^{1000}$.

**5** **(a)** Suppose $A^{\mathrm{T}}Ax = 0$. This tells us that $Ax$ is in the _____ space of $A^{\mathrm{T}}$. Always $Ax$ is in the _____ space of $A$. Why can you conclude that $Ax = 0$?

**(b)** Supposing again that $A^{\mathrm{T}}Ax = 0$ we immediately get $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$.

From this, *show directly that* $Ax = 0$.

**Every matrix $A^{\mathrm{T}}A$ is symmetric and** ______________.

**(c)** The rectangular $m$ by $n$ matrix $A$ always has the same nullspace as the square matrix $A^{\mathrm{T}}A$ (this is proved above). Now deduce that $A$ and $A^{\mathrm{T}}A$ have the *same rank*.

**6** Suppose $A = \textsf{ones}(3,5)$ and $A^{\mathrm{T}} = \textsf{ones}(5,3)$ are the 3 by 5 and 5 by 3 matrices of all 1's.

**(a)** Find the trace of $AA^{\mathrm{T}}$ and the trace of $A^{\mathrm{T}}A$.

**(b)** Find the eigenvalues of $AA^{\mathrm{T}}$ and the eigenvalues of $A^{\mathrm{T}}A$.

**(c)** What is the matrix $\Sigma$ in the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$?

**7** **(a)** By elimination or otherwise, *find the determinant of* $A$:

$$A = \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ v_1 & v_2 & v_3 & 0 \end{bmatrix}$$

**(b)** If that *zero* in the lower right corner of $A$ changes to 100, what is the change (if any) in the determinant of $A$? (You can consider its cofactors)

**(c)** If $(u_1, u_2, u_3)$ is the same as $(v_1, v_2, v_3)$ so $A$ is symmetric, decide if $A$ is or is not positive definite—and why?

**(d)** Show that this block matrix $M$ is singular for any $u$ and $v$ in $\mathbf{R}^n$, by finding a vector in its nullspace:

$$M = \begin{bmatrix} I & u \\ v^{\mathrm{T}} & v^{\mathrm{T}}u \end{bmatrix}.$$

**8** Suppose $q_1, q_2, q_3$ are orthonormal vectors in $\mathbf{R}^4$ (not $\mathbf{R}^3$ !).

**(a)** What is the length of the vector $v = 2q_1 - 3q_2 + 2q_3$?

**(b)** What four vectors does Gram-Schmidt produce when it orthonormalizes the vectors $q_1, q_2, q_3, u$?

**(c)** If $u$ in part (b) is the vector $v$ in part (a), why does Gram-Schmidt break down? Find a nonzero vector in the nullspace of the 4 by 4 matrix

$$A = \begin{bmatrix} q_1 & q_2 & q_3 & v \end{bmatrix} \quad \text{with columns } q_1, q_2, q_3, v\,.$$

**9** **(4 points)** PROVE (give a clear reason): If $A$ is a symmetric invertible matrix then $A^{-1}$ is also symmetric.