Final Examination in Linear Algebra: 18.06

Dec 21, 2000 9:00 – 12:00 Professor Strang

Your name is: ______________________________

Grading 1
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Answer all 8 questions on these pages (25 parts, 4 points each). This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor.* Best wishes for the holidays and thank you for taking 18.06. GS

1 (a) Explain why every eigenvector of A is either in the column space $C(A)$ or the nullspace $N(A)$ (or explain why this is false).

(b) From $A = S\Lambda S^{-1}$ find the eigenvalue matrix and the eigenvector matrix for A^{T}. How are the eigenvalues of A and A^{T} related?

(c) Suppose $Ax = 0$ and $A^{\mathrm{T}}y = 2y$. Deduce that x is orthogonal to y. You may prove this directly or use the subspace ideas in (a) or the eigenvector matrices in (b). Write a clear answer.

2 (a) Suppose A is a symmetric matrix. If you first subtract 3 times row 1 from row 3, and after that you subtract 3 times column 1 from column 3, is the resulting matrix B still symmetric? Yes or not necessarily, with a reason.

(b) Create a symmetric positive definite matrix (but not diagonal) with eigenvalues $1, 2, 4$.

(c) Create a nonsymmetric matrix (if possible) with those eigenvalues. Create a rank-one matrix (if possible) with those eigenvalues.

3 Gram-Schmidt is $A = QR$ (start from rectangular A with independent columns, produce Q with orthonormal columns and upper triangular R). The problem is to produce the same Q and R from ordinary (symmetric) elimination on $A^{\mathrm{T}}A$ which gives

$$A^{\mathrm{T}}A = LDL^{\mathrm{T}} = R^{\mathrm{T}}R \qquad \text{(with } R = \sqrt{D}L^{\mathrm{T}}\text{)}.$$

(a) How do you know that the pivots are positive, so $\sqrt{D}$ gives real numbers?

(b) From $A^{\mathrm{T}}A = R^{\mathrm{T}}R$ show that the matrix $Q = AR^{-1}$ has orthonormal columns (what is the test?). Then we have $A = QR$.

(c) Apply Gram-Schmidt to these vectors a_1 and a_2, producing q_1 and q_2. Write your result as QR:

$$a_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad a_2 = \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix}.$$

4 The Fibonacci numbers $F_0, F_1, F_2, F_3, F_4, \ldots$ are $0, 1, 1, 2, 3, \ldots$ and they obey the rule $F_{k+2} = F_{k+1} + F_k$. In matrix form this is

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \quad \text{or} \quad u_{k+1} = Au_k.$$

The eigenvalues of this particular matrix A will be called a and b.

(a) What quadratic equation connected with A has the solutions (the roots) a and b?

(b) Find a matrix that has the eigenvalues a^2 and b^2. What quadratic equation has the solutions a^2 and b^2?

(c) If you directly compute A^4 you get

$$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Make a guess at the entries of A^k, involving Fibonacci numbers. Then multiply by A to show why your guess is correct. What is the determinant of A^k (not a hard question!)?

5 Suppose A is 3 by 4 and its reduced row echelon form is R:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The four subspaces associated with the original A are $N(A)$, $C(A)$, $N(A^{\mathrm{T}})$, and $C(A^{\mathrm{T}})$. Give the <u>dimension</u> of each subspace and if possible give a <u>basis</u>.

(b) Find the complete solution (when is there a solution?) to the equations

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

(c) Find a matrix A with <u>no zero entries</u> (if possible) whose reduced row echelon form is this same R.

6 Suppose A is a 3 by 3 matrix and you know the three outputs $y_1 = Ax_1$ and $y_2 = Ax_2$ and $y_3 = Ax_3$ from three independent input vectors x_1, x_2, x_3.

(a) Find the matrix A using this hint: Put the vectors x_1, x_2, x_3 into the columns of a matrix X and multiply AX. Why did I require the x's to be independent?

(b) Under what condition on A will the outputs y_1, y_2, y_3 be a basis for R^3? Explain your answer.

(c) If x_1, x_2, x_3 is the input basis and y_1, y_2, y_3 is the output basis, what is the matrix M that represents this same linear transformation (defined by $T(x_1) = y_1, \quad T(x_2) = y_2, \quad T(x_3) = y_3$)?

7 (a) Find the eigenvalues of the antidiagonal matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Find as many eigenvectors as possible, with the best possible properties. Are there 4 <u>independent</u> eigenvectors? Are there 4 <u>orthonormal</u> eigenvectors?

(c) What is the rank of $A + 2I$? What is the determinant of $A + 2I$?

8 (a) If $U\Sigma V^{\mathrm{T}}$ is the singular value decomposition of A (m by n) give a formula for the best least squares solution $\bar{x}$ to $Ax = b$. (Simplify your formula as much as possible).

(b) Write down the equations for the straight line $b = C + Dt$ to go through all four of the points (t_1, b_1), (t_2, b_2), (t_3, b_3), (t_4, b_4). Those four points lie on a line provided the vector $b = (b_1, b_2, b_3, b_4)$ lies in ____ .

(c) Suppose S is the subspace spanned by the columns of some m by n matrix A. Give the formula for the projection matrix P that projects each vector in R^m onto the subspace S. Explain where this formula comes from and any condition on A for it to be correct.

(d) Suppose x and y are both in the row space of a matrix A, and $Ax = Ay$. Show that $x - y$ is in the nullspace of A. Then prove that $x = y$.