

## Course 18.06, Fall 2002: Final Exam, Solutions

1 (a)

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

(b) Four pivots  $\Rightarrow$  rank of  $A = 4$ . The row reduced echelon form is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The two special solutions  $(0, 0, 0, -1, 1, 0)$ ,  $(0, 0, 0, -1, 0, 1)$  form a basis for the nullspace.

2 (a) The dimension of the nullspace is 1, so the rank of  $A$  is  $4 - 1 = 3$ . The complete solution to  $A\mathbf{x} = 0$  is  $\mathbf{x} = c \cdot (2, 3, 1, 0)$  for any constant  $c$ .

(b) The row reduced echelon form has 3 pivots and the special solution  $\mathbf{x} = (2, 3, 1, 0)$ :

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3 (a) The projection matrix  $P$  projects onto the column space of  $P$  which is the line  $c \cdot (1, 2, -4)$ .

(b) The vector from  $\mathbf{b}$  to the subspace is

$$\mathbf{e} = \mathbf{b} - P\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{21} \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 22 \\ 23 \\ 17 \end{bmatrix}$$

and the distance is

$$\|\mathbf{e}\| = \frac{1}{21} \sqrt{22^2 + 23^2 + 17^2} = \frac{\sqrt{1302}}{21}.$$

(c) Since  $P$  projects onto a line, its three eigenvalues are 0, 0, 1. Since  $P$  is symmetric, it has a full set of (orthogonal) eigenvectors, and is then diagonalizable.

4 (a) When  $AB = 0$ , every column of  $B$  is in the nullspace of  $A$ . So the **null** space of  $A$  contains the **column** space of  $B$ . Also the **left null** space of  $B$  contains the **row** space of  $A$ .

(b) The dimension of the nullspace of  $A$  is  $n - r = 7 - r$ . The dimension of the column space of  $B$  is  $s$ . Since the first contains the second,  $7 - r \geq s$ , or  $r + s \leq 7$ .

5 (a) The least squares solution  $\hat{\mathbf{x}}$  to  $Q\mathbf{x} = \mathbf{b}$  is

$$\hat{\mathbf{x}} = (Q^T Q)^{-1} Q^T \mathbf{b} = (I)^{-1} Q^T \mathbf{b} = Q^T \mathbf{b}.$$

- (b) One approach:  $Q^T$  is 2 by 4 so there are free variables and many solutions to  $Q^T \mathbf{x} = 0$ . Then  $QQ^T \mathbf{x} = 0$  and  $QQ^T$  is singular; not positive definite.  
 Second approach: The rank of  $Q$  is 2, so the rank of  $QQ^T$  must be  $\leq 2$ . But  $QQ^T$  is a 4 by 4 matrix, so the dimension of its nullspace is 2. This means that it has 2 zero eigenvalues, and  $QQ^T$  is not positive definite.
- (c) The singular values of  $Q$  are the square roots of the eigenvalues of  $Q^T Q = I$ , that is, all 1.

- 6 (a) Let  $\mathbf{a} = (1, 2, 2)$  and  $\mathbf{b} = (5, 4, -2)$ . The orthonormal vectors are

$$\mathbf{q}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

$$\mathbf{q}_2 = \frac{\mathbf{b} - \frac{\mathbf{q}_1^T \mathbf{b}}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1}{\|\cdot\|} = \frac{1}{\sqrt{4^2 + 2^2 + (-4)^2}} \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

where  $\|\cdot\|$  means the norm of the numerator.

- (b) Let  $Q = [\mathbf{q}_1 \ \mathbf{q}_2]$ . The projection  $P$  is then

$$P = QQ^T = \frac{1}{9} \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$$

- (c) The properties of a projection matrix are  $P^2 = P$  and  $P^T = P$ . This gives

$$(P\mathbf{b})^T(\mathbf{b} - P\mathbf{b}) = \mathbf{b}^T P^T(\mathbf{b} - P\mathbf{b}) = \mathbf{b}^T(P\mathbf{b} - P^2\mathbf{b}) = 0.$$

- 7 (a) If  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is a basis for  $\mathbf{R}^3$  then the matrix with those three columns is **invertible** (non-singular, full rank).
- (b) If  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbf{R}^3$  then the column space is  $\mathbf{R}^3$ . The only possible rank for the matrix with those four columns is 3.
- (c) The transformations of the three basis vectors are

$$\begin{aligned} T(\mathbf{q}_1) &= \mathbf{q}_1 \\ T(\mathbf{q}_2) &= \mathbf{q}_2 \\ T(\mathbf{q}_3) &= \mathbf{0} \end{aligned}$$

so the transformation matrix  $T$  in the basis  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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$$\begin{vmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot 27 - 2 \cdot 8 = 65.$$

For general  $n > 4$ , the determinant is  $|A_n| = 3^n + (-1)^{n-1}2^n$ .

- 9 (a) The determinant of a permutation matrix  $P$  is 1 or  $-1$ . The only possible pivot is 1.  
 (b) For 3 by 3 permutations, the trace of  $P$  is 0, 1, or 3. The eigenvalues are 1 and  $-1$  when two rows are exchanged. Otherwise

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \quad (\text{or } e^{2\pi i/3}, e^{4\pi i/3}),$$

so the four possible eigenvalues are 1,  $-1$ ,  $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ , and  $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ .

- 10 (a)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns 1, 2, and 4 have pivots. Because of the nonzero bottom-right element in  $A - 3I$ , the fourth component of  $\mathbf{x}_3$  is definitely zero.

- (b) In the same way as above, the special solutions for the matrices  $A - 1I$ ,  $A - 2I$ ,  $A - 3I$ , and  $A - 4I$  must have 3, 2, 1, and 0 zeros as the last components. The eigenvector matrix  $S$  is then upper triangular.