18.06 Professor Strang Final Exam May 20, 2004

Your name is: SOLUTIONS

Please circle your recitation:

Problems 1–8 are 12 points each; Problem 9 is 4 points.

Thank you for taking 18.06!

1 Suppose A is an m by n matrix of rank r. You multiply it by any m by n invertible matrix E to get $B = EA$.

(a) Circle if true and cross out if false (three parts):

$$A \text{ and } B \text{ have the } \begin{cases} \textit{same nullspace} \\ \textit{same column space} \\ \textit{same bases for row space.} \end{cases}$$

(b) Suppose the right E gives the row-reduced echelon matrix

$$EA = R = \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) Find a basis for the nullspace of A.

(2) True statement: *The nullspace of a matrix is a vector space.*

What does it mean for a set of vectors to be a vector space?

(c) What is the nullspace of a 5 by 4 matrix with linearly independent columns?

What is the nullspace of a 4 by 5 matrix with linearly independent columns?

Solutions

(a) True, false, and true.

(b) A possible basis for $\boldsymbol{N}(A)$ is $(6, 0, 5, -1)$ and $(4, -1, 0, 0)$.

(c) Nullspace = zero vector $(0, 0, 0, 0)$; no 4 by 5 matrix has independent columns

2 This matrix A has column 1 + column 2 = column 3:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Describe the column space $\boldsymbol{C}(A)$ in two ways:

(**1**) Give a basis for $\boldsymbol{C}(A)$.

(**2**) Find all vectors that are *perpendicular* to $\boldsymbol{C}(A)$.

(b) The projection matrix P onto the column space does not come from the usual formula $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. *Why not*—what goes wrong with this formula?

(c) Find that matrix P for projection onto the column space of A.

Solutions

(a) (**1**) $(1, 1, 0)$ and $(1, 1, 1)$

(**2**) multiples of $(1, -1, 0)$ are perpendicular to $\boldsymbol{C}(A)$

(b) Columns of A are not linearly independent, so $A^T A$ is not invertible.

(c) The matrix $B' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same column space as A.

Even better: take $B = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$, that has orthonormal columns.

The second one has a much simplified formula: $P = BB^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3 Suppose P is the 3 by 3 projection matrix (so $P = P^{\mathrm{T}} = P^2$) onto the plane $2x + 2y - z = 0$. You do not have to compute this matrix P but you can if you want.

(a) What is the rank of P? What are its three eigenvalues? What is its column space?

(b) Is P diagonalizable—why or why not? Find a nonzero vector in its nullspace.

(c) If b is any unit vector in $\mathbf{R}^3$, find the number q. *Explain your thinking in 1 sentence and 1 equation*:

$$q = \|Pb\|^2 + \|b - Pb\|^2 .$$

Solutions

(a) Rank$(P) = 2$, since the column space is a plane ($2x + 2y - z = 0$). The eigenvalues can only be 0 or 1—since it has rank 2, $\lambda = 0, 1$ and 1.

(b) Being symmetric, P is diagonalizable. A non-zero vector in $N(P) = N(P^{\mathrm{T}})$ is $(2, 2, -1)$: it is orthogonal to the column space.

(c) $b = Pb + (b - Pb)$, and since Pb and $(b - Pb)$ are orthogonal, we use Pythagorean Theorem! $1 = \|b\|^2 = \|Pb\|^2 + \|(b - Pb)\|^2$. Or expand $b^{\mathrm{T}} P^{\mathrm{T}} P b + \cdots$

4 **(a)** If $a \neq c$, find the eigenvalue matrix Λ and eigenvector matrix S in

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = S\Lambda S^{-1}.$$

(b) Find the *four entries* in the matrix A^{1000}.

Solutions

(a) $$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & c-a \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} c-a & -b \\ 0 & 1 \end{bmatrix} \frac{1}{c-a}$$

$$= \begin{bmatrix} 1 & b/(c-a) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & -b/(c-a) \\ 0 & 1 \end{bmatrix}$$

(b) $$A^{1000} = S\Lambda^{1000}S^{-1} = \begin{bmatrix} a^{1000} & c^{1000}b \\ 0 & c^{1000}(c-a) \end{bmatrix} \begin{bmatrix} c-a & -b \\ 0 & 1 \end{bmatrix} \frac{1}{c-a}$$

$$= \begin{bmatrix} a^{1000} & (c^{1000}-a^{1000})b/(c-a) \\ 0 & c^{1000} \end{bmatrix}$$

5 **(a)** Suppose $A^{\mathrm{T}}Ax = 0$. This tells us that Ax is in the _____ space of A^{T}. Always Ax is in the _____ space of A. Why can you conclude that $Ax = 0$?

(b) Supposing again that $A^{\mathrm{T}}Ax = 0$ we immediately get $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$.

From this, *show directly that* $Ax = 0$.

Every matrix $A^{\mathrm{T}}A$ is symmetric and ______________.

(c) The rectangular m by n matrix A always has the same nullspace as the square matrix $A^{\mathrm{T}}A$ (this is proved above). Now deduce that A and $A^{\mathrm{T}}A$ have the *same rank*.

Solutions

(a) Nullspace of A^{T} and column space of A. Then $Ax = 0$ because $\boldsymbol{C}(A) \perp \boldsymbol{N}(A^T)$.

(b) $x^{\mathrm{T}}A^{\mathrm{T}}Ax = (Ax)^{\mathrm{T}}(Ax) = \|Ax\|^2 = 0$, hence $Ax = 0$. Then $A^{\mathrm{T}}A$ is positive semidefinite.

(c) $\mathrm{Rank}(A^{\mathrm{T}}A) + \dim \boldsymbol{N}(A^{\mathrm{T}}A) = n = \mathrm{Rank}(A) + \dim \boldsymbol{N}(A)$. With equal nullspaces we get equal ranks.

6 Suppose $A = \mathsf{ones}(3,5)$ and $A^{\mathrm{T}} = \mathsf{ones}(5,3)$ are the 3 by 5 and 5 by 3 matrices of all 1's.

(a) Find the trace of AA^{T} and the trace of $A^{\mathrm{T}}A$.

(b) Find the eigenvalues of AA^{T} and the eigenvalues of $A^{\mathrm{T}}A$.

(c) What is the matrix Σ in the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$?

Solutions

(a) $AA^{\mathrm{T}} = 5 * \mathsf{ones}(3,3)$ and $A^{\mathrm{T}}A = 3 * \mathsf{ones}(5,5)$, so both traces are 15.

(b) Since both ranks are 1, $\mathrm{eig}(AA^{\mathrm{T}}) = \{15, 0, 0\}$, and $\mathrm{eig}(AA^{\mathrm{T}}) = \{15, 0, 0, 0, 0\}$.

(c) $\Sigma = \begin{bmatrix} \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

7 **(a)** By elimination or otherwise, *find the determinant of* A:

$$A = \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ v_1 & v_2 & v_3 & 0 \end{bmatrix}$$

(b) If that *zero* in the lower right corner of A changes to 100, what is the change (if any) in the determinant of A? (You can consider its cofactors)

(c) If (u_1, u_2, u_3) is the same as (v_1, v_2, v_3) so A is symmetric, decide if A is or is not positive definite—and why?

(d) Show that this block matrix M is singular for any u and v in $\mathbf{R}^n$, by finding a vector in its nullspace:

$$M = \begin{bmatrix} I & u \\ v^{\mathrm{T}} & v^{\mathrm{T}}u \end{bmatrix}.$$

Solutions

(a) The determinant is $-v^{\mathrm{T}}u$:

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ 0 & v_2 & v_3 & -u_1v_1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & u_2 \\ 0 & 1 & u_3 \\ v_2 & v_3 & -u_1v_1 \end{bmatrix} = -v^{\mathrm{T}}u\,.$$

(b) The cofactor of the $(4, 4)$ entry is 100, so det changes by 100:

$$\det(A') = \det \begin{bmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ 0 & v_2 & v_3 & 100 - u_1v_1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & u_2 \\ 0 & 1 & u_3 \\ v_2 & v_3 & 100 - u_1v_1 \end{bmatrix} = 100 - v^{\mathrm{T}}u.$$

(c) Since in this case $\det(A) = -\|u\|^2 \leq 0$, at least one of the eigenvalues is not positive. Hence A cannot be positive definite.

(d) The vector is $\begin{bmatrix} -u \\ 1 \end{bmatrix}$.

8 Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbf{R}^4$ (not $\mathbf{R}^3$!).

(a) What is the length of the vector $v = 2q_1 - 3q_2 + 2q_3$?

(b) What four vectors does Gram-Schmidt produce when it orthonormalizes the vectors q_1, q_2, q_3, u?

(c) If u in part (b) is the vector v in part (a), why does Gram-Schmidt break down? Find a nonzero vector in the nullspace of the 4 by 4 matrix

$$A = \begin{bmatrix} q_1 & q_2 & q_3 & v \end{bmatrix} \quad \text{with columns } q_1, q_2, q_3, v\,.$$

Solutions

(a) By orthogonality (the Pythagorean Theorem) $\|v\|^2 = \|2q_1 - 3q_2 + 2q_3\|^2 = 4+9+4 = 17$.

(b) q_1, q_2, q_3 and

$$q_4 = \frac{u - (q_1^{\mathrm{T}}u)q_1 - (q_2^{\mathrm{T}}u)q_2 - (q_3^{\mathrm{T}}u)q_3}{\|u - (q_1^{\mathrm{T}}u)q_1 - (q_2^{\mathrm{T}}u)q_2 - (q_3^{\mathrm{T}}u)q_3\|}\,.$$

(c) Gram-Schmidt fails because v is a linear combination of the q_i's: Not independent. A vector in $\boldsymbol{N}(A)$ is $(2, -3, 2, -1)$.

9 **(4 points)** PROVE (give a clear reason): If A is a symmetric invertible matrix then A^{-1} is also symmetric.

Solutions

$AA^{-1} = I$ leads to $(AA^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I$. Hence always $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$. If $A = A^{\mathrm{T}}$, then $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1}$ and A^{-1} is symmetric.

Proof 2: The i, j cofactor of A equals the j, i cofactor. Then $A^{-1} = (\text{cofactor matrix})/\det A$ is symmetric.