

GLOSSARY: A DICTIONARY FOR LINEAR ALGEBRA

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j ; otherwise $a_{ij} = 0$. $A = A^T$ for an undirected graph.

Affine transformation $T(\mathbf{v}) = A\mathbf{v} + \mathbf{v}_0 =$ linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC .

Augmented matrix $[A \ \mathbf{b}]$. $A\mathbf{x} = \mathbf{b}$ is solvable when \mathbf{b} is in the column space of A ; then $[A \ \mathbf{b}]$ has the same rank as A . Elimination on $[A \ \mathbf{b}]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1 .

Basis for V . Independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ whose linear combinations give every \mathbf{v} in V . A vector space has many bases!

Big formula for n by n determinants. $\det(A)$ is a sum of $n!$ terms, one term for each permutation P of the columns. That term is the product $a_{1\alpha} \cdots a_{n\omega}$ down the diagonal of the reordered matrix, times $\det(P) = \pm 1$.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit (the columns of A and rows of B must be in matching blocks).

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ zero matrix.

Change of basis matrix M . The old basis vectors \mathbf{v}_j are combinations $\sum m_{ij}\mathbf{w}_i$ of the new basis vectors. The coordinates of $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = d_1\mathbf{w}_1 + \cdots + d_n\mathbf{w}_n$ are related by $\mathbf{d} = M\mathbf{c}$. (For $n = 2$ set $\mathbf{v}_1 = m_{11}\mathbf{w}_1 + m_{21}\mathbf{w}_2$, $\mathbf{v}_2 = m_{12}\mathbf{w}_1 + m_{22}\mathbf{w}_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A .

Cholesky factorization $A = CC^T = (L\sqrt{D})(L\sqrt{D})^T$ for positive definite A .

Circulant matrix C . Constant diagonals wrap around as in cyclic shift S . Every circulant is $c_0I + c_1S + \cdots + c_{n-1}S^{n-1}$. $C\mathbf{x} =$ **convolution** $\mathbf{c} * \mathbf{x}$. Eigenvectors in F .

Cofactor C_{ij} . Remove row i and column j ; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\mathbf{x} = \mathbf{b}$. The vector \mathbf{b} becomes a combination of the columns of A . The system is solvable only when \mathbf{b} is in the column space $\mathbf{C}(A)$.

Column space $\mathbf{C}(A)$ = space of all combinations of the columns of A .

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put c_1, \dots, c_n in row n and put $n - 1$ 1's along diagonal 1. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots)$.

Complete solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ to $A\mathbf{x} = \mathbf{b}$. (Particular \mathbf{x}_p) + (\mathbf{x}_n in nullspace).

Complex conjugate $\bar{z} = a - ib$ for any complex number $z = a + ib$. Then $z\bar{z} = |z|^2$.

Condition number $cond(A) = \kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $A\mathbf{x} = \mathbf{b}$, the relative change $\|\delta\mathbf{x}\|/\|\mathbf{x}\|$ is less than $cond(A)$ times the relative change $\|\delta\mathbf{b}\|/\|\mathbf{b}\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $A\mathbf{x} = \mathbf{b}$ by minimizing $\frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{b}$ over growing Krylov subspaces.

Covariance matrix Σ . When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means \bar{x}_i , the matrix $\Sigma = \text{mean of } (\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T$ is positive (semi)definite; it is diagonal if the x_i are independent.

Cramer's Rule for $A\mathbf{x} = \mathbf{b}$. B_j has \mathbf{b} replacing column j of A , and $x_j = |B_j|/|A|$.

Cross product $\mathbf{u} \times \mathbf{v}$ in \mathbf{R}^3 . Vector perpendicular to \mathbf{u} and \mathbf{v} , length $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta =$ parallelogram area, computed as the “determinant” of $[\mathbf{i} \ \mathbf{j} \ \mathbf{k}; u_1 \ u_2 \ u_3; v_1 \ v_2 \ v_3]$.

Cyclic shift S . Permutation with $s_{21} = 1, s_{32} = 1, \dots$, finally $s_{1n} = 1$. Its eigenvalues are n th roots $e^{2\pi i k/n}$ of 1; eigenvectors are columns of the Fourier matrix F .

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^T| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n - 1$, volume of box = $|\det(A)|$.

Diagonal matrix D . $d_{ij} = 0$ if $i \neq j$. **Block-diagonal:** zero outside square blocks D_{ii} .

Diagonalizable matrix A . Must have n independent eigenvectors (in the columns of S ; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda =$ eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. $\Lambda =$ eigenvalue matrix and $S =$ eigenvector matrix. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(\mathbf{V}) =$ number of vectors in any basis for \mathbf{V} .

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product $\mathbf{x}^T \mathbf{y} = x_1 y_1 + \dots + x_n y_n$. Complex dot product is $\bar{\mathbf{x}}^T \mathbf{y}$. Perpendicular vectors have zero dot product. $(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$.

Echelon matrix U . The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ and eigenvector \mathbf{x} . $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ so $\det(A - \lambda I) = 0$.

Eigshow. Graphical 2 by 2 eigenvalues and singular values (MATLAB or Java).

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \text{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L , or $PA = LU$ with row exchanges in P , or $EA = R$ with an invertible E .

Elimination matrix = Elementary matrix E_{ij} . The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i .

Ellipse (or ellipsoid) $\mathbf{x}^T A\mathbf{x} = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A , with lengths $1/\sqrt{\lambda}$. (For $\|\mathbf{x}\| = 1$ the vectors $\mathbf{y} = A\mathbf{x}$ lie on the ellipse $\|A^{-1}\mathbf{y}\|^2 = \mathbf{y}^T (AA^T)^{-1} \mathbf{y} = 1$ displayed by eigshow; axis lengths σ_i .)

Exponential $e^{At} = I + At + (At)^2/2! + \dots$ has derivative Ae^{At} ; $e^{At}\mathbf{u}(0)$ solves $\mathbf{u}' = A\mathbf{u}$.

Factorization $A = LU$. If elimination takes A to U without row exchanges, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A .

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n \mathbf{x}$ and $F_n^{-1} \mathbf{c}$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers $0, 1, 1, 2, 3, 5, \dots$ satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1 + \sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Four fundamental subspaces of $A = \mathbf{C}(A), \mathbf{N}(A), \mathbf{C}(A^T), \mathbf{N}(A^T)$.

Fourier matrix F . Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^T F = nI$. Then $\mathbf{y} = F \mathbf{c}$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A . Columns without pivots; combinations of earlier columns.

Free variable x_i . Column i has no pivot in elimination. We can give the $n - r$ free variables any values, then $A \mathbf{x} = \mathbf{b}$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $\mathbf{N}(A) = \{\mathbf{0}\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $A \mathbf{x} = \mathbf{b}$, column space is all of \mathbf{R}^m . *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $\mathbf{N}(A)$ and row space $\mathbf{C}(A^T)$ are orthogonal complements (perpendicular subspaces of \mathbf{R}^n with dimensions r and $n - r$) from $A \mathbf{x} = \mathbf{0}$. Applied to A^T , the column space $\mathbf{C}(A)$ is the orthogonal complement of $\mathbf{N}(A^T)$.

Gauss-Jordan method. Invert A by row operations on $[A \ I]$ to reach $[I \ A^{-1}]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A , orthonormal columns in Q . Each column \mathbf{q}_j of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\text{diag}(R) > \mathbf{0}$.

Graph G . Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n - 1)/2$ edges between nodes. A **tree** has only $n - 1$ edges and no closed loops. A **directed graph** has a direction arrow specified on each edge.

Hankel matrix H . Constant along each antidiagonal; h_{ij} depends on $i + j$.

Hermitian matrix $A^H = \overline{A}^T = A$. Complex analog of a symmetric matrix: $\overline{a_{ji}} = a_{ij}$.

Hessenberg matrix H . Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix $\text{hilb}(n)$. Entries $H_{ij} = 1/(i + j - 1) = \int_0^1 x^{i-1} x^{j-1} dx$. Positive definite but extremely small λ_{\min} and large condition number.

Hypercube matrix P_L^2 . Row $n + 1$ counts corners, edges, faces, . . . of a cube in \mathbf{R}^n .

Identity matrix I (or I_n). Diagonal entries = 1, off-diagonal entries = 0.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j .

Indefinite matrix. A symmetric matrix with eigenvalues of both signs (+ and -).

Independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. No combination $c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}$ unless all $c_i = 0$. If the \mathbf{v} 's are the columns of A , the only solution to $A \mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Inverse matrix A^{-1} . Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\text{rank}(A) < n$ and $A\mathbf{x} = \mathbf{0}$ for a nonzero vector \mathbf{x} . The inverses of AB and A^T are $B^{-1}A^{-1}$ and $(A^{-1})^T$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its “generalized” eigenvector matrix M gives $J = \text{diag}(J_1, \dots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1’s on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector $(1, 0, \dots, 0)$.

Kirchhoff’s Laws. *Current law:* net current (in minus out) is zero at each node. *Voltage law:* Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, \mathbf{b})$. The subspace spanned by $\mathbf{b}, A\mathbf{b}, \dots, A^{j-1}\mathbf{b}$. Numerical methods approximate $A^{-1}\mathbf{b}$ by \mathbf{x}_j with residual $\mathbf{b} - A\mathbf{x}_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\hat{\mathbf{x}}$. The vector $\hat{\mathbf{x}}$ that minimizes the error $\|\mathbf{e}\|^2$ solves $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$. Then $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to all columns of A .

Left inverse A^+ . If A has full column rank n , then $A^+ = (A^T A)^{-1} A^T$ has $A^+ A = I_n$.

Left nullspace $N(A^T)$. Nullspace of $A^T =$ “left nullspace” of A because $\mathbf{y}^T A = \mathbf{0}^T$.

Length $\|\mathbf{x}\|$. Square root of $\mathbf{x}^T \mathbf{x}$ (Pythagoras in n dimensions).

Linear combination $c\mathbf{v} + d\mathbf{w}$ or $\sum c_j \mathbf{v}_j$. Vector addition and scalar multiplication.

Linear transformation T . Each vector \mathbf{v} in the input space transforms to $T(\mathbf{v})$ in the output space, and linearity requires $T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w})$. Examples: Matrix multiplication $A\mathbf{v}$, differentiation in function space.

Linearly dependent $\mathbf{v}_1, \dots, \mathbf{v}_n$. A combination other than all $c_i = 0$ gives $\sum c_i \mathbf{v}_i = \mathbf{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \dots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with eigenvalues $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ of the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Compare $L_0 = 2$ with Fibonacci.

Markov matrix M . All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $M\mathbf{s} = \mathbf{s} > \mathbf{0}$.

Matrix multiplication AB . The i, j entry of AB is (row i of A) · (column j of B) = $\sum a_{ik} b_{kj}$. By columns: Column j of $AB = A$ times column j of B . By rows: row i of A multiplies B . Columns times rows: $AB =$ sum of (column k)(row k). All these equivalent definitions come from the rule that AB times \mathbf{x} equals A times $B\mathbf{x}$.

Minimal polynomial of A . The lowest degree polynomial with $m(A) =$ zero matrix. The roots of m are eigenvalues, and $m(\lambda)$ divides $\det(A - \lambda I)$.

Multiplication $A\mathbf{x} = x_1(\text{column } 1) + \dots + x_n(\text{column } n) =$ combination of columns.

Multiplicities AM and GM . The algebraic multiplicity AM of an eigenvalue λ is the number of times λ appears as a root of $\det(A - \lambda I) = 0$. The geometric multiplicity GM is the number of independent eigenvectors (= dimension of the eigenspace for λ).

Multiplier ℓ_{ij} . The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: $\ell_{ij} = (\text{entry to eliminate})/(\textit{jth pivot})$.

Network. A directed graph that has constants c_1, \dots, c_m associated with the edges.

Nilpotent matrix N . Some power of N is the zero matrix, $N^k = 0$. The only eigenvalue is $\lambda = 0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$ **of a matrix**. The “ ℓ^2 norm” is the maximum ratio $\|A\mathbf{x}\|/\|\mathbf{x}\| = \sigma_{\max}$. Then $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$ and $\|AB\| \leq \|A\|\|B\|$ and $\|A+B\| \leq \|A\| + \|B\|$. **Frobenius norm** $\|A\|_F^2 = \sum \sum a_{ij}^2$; ℓ^1 and ℓ^∞ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$. Gives the least squares solution to $A\mathbf{x} = \mathbf{b}$ if A has full rank n . The equation says that $(\text{columns of } A) \cdot (\mathbf{b} - A\hat{\mathbf{x}}) = 0$.

Normal matrix N . $NN^T = N^T N$, leads to orthonormal (complex) eigenvectors.

Nullspace $N(A) = \text{Solutions to } A\mathbf{x} = \mathbf{0}$. Dimension $n - r = (\# \text{ columns}) - \text{rank}$.

Nullspace matrix N . The columns of N are the $n - r$ special solutions to $A\mathbf{s} = \mathbf{0}$. **Orthogonal matrix** Q . Square matrix with orthonormal columns, so $Q^T Q = I$ implies $Q^T = Q^{-1}$. Preserves length and angles, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ and $(Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^T \mathbf{y}$. All $|\lambda| = 1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces. Every \mathbf{v} in \mathbf{V} is orthogonal to every \mathbf{w} in \mathbf{W} .

Orthonormal vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$. Dot products are $\mathbf{q}_i^T \mathbf{q}_j = 0$ if $i \neq j$ and $\mathbf{q}_i^T \mathbf{q}_i = 1$. The matrix Q with these orthonormal columns has $Q^T Q = I$. If $m = n$ then $Q^T = Q^{-1}$ and $\mathbf{q}_1, \dots, \mathbf{q}_n$ is an **orthonormal basis** for \mathbf{R}^n : every $\mathbf{v} = \sum (\mathbf{v}^T \mathbf{q}_j) \mathbf{q}_j$.

Outer product $\mathbf{u}\mathbf{v}^T = \text{column times row} = \text{rank one matrix}$.

Partial pivoting. In elimination, the j th pivot is chosen as the largest available entry (in absolute value) in column j . Then all multipliers have $|\ell_{ij}| \leq 1$. Roundoff error is controlled (depending on the *condition number* of A).

Particular solution \mathbf{x}_p . Any solution to $A\mathbf{x} = \mathbf{b}$; often \mathbf{x}_p has free variables = 0.

Pascal matrix $P_S = \text{pascal}(n)$. The symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S = P_L P_U$ all contain Pascal’s triangle with $\det = 1$ (see index for more properties).

Permutation matrix P . There are $n!$ orders of $1, \dots, n$; the $n!$ P ’s have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij} ; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A . Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d . The diagonal entry (*first nonzero*) when a row is used in elimination.

Plane (or hyperplane) in \mathbf{R}^n . Solutions to $\mathbf{a}^T \mathbf{x} = 0$ give the plane (dimension $n - 1$) perpendicular to $\mathbf{a} \neq \mathbf{0}$.

Polar decomposition $A = QH$. Orthogonal Q , positive (semi)definite H .

Positive definite matrix A . Symmetric matrix with positive eigenvalues and positive pivots. Definition: $\mathbf{x}^T A \mathbf{x} > 0$ unless $\mathbf{x} = \mathbf{0}$.

Projection $\mathbf{p} = \mathbf{a}(\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a})$ onto the line through \mathbf{a} . $P = \mathbf{a} \mathbf{a}^T / \mathbf{a}^T \mathbf{a}$ has rank 1.

Projection matrix P onto subspace \mathcal{S} . Projection $\mathbf{p} = P\mathbf{b}$ is the closest point to \mathbf{b} in \mathcal{S} , error $\mathbf{e} = \mathbf{b} - P\mathbf{b}$ is perpendicular to \mathcal{S} . $P^2 = P = P^T$, eigenvalues are 1 or 0, eigenvectors are in \mathcal{S} or \mathcal{S}^\perp . If columns of $A =$ basis for \mathcal{S} then $P = A(A^T A)^{-1} A^T$.

Pseudoinverse A^+ (**Moore-Penrose inverse**). The n by m matrix that “inverts” A from column space back to row space, with $\mathbf{N}(A^+) = \mathbf{N}(A^T)$. $A^+ A$ and AA^+ are the projection matrices onto the row space and column space. $\text{Rank}(A^+) = \text{rank}(A)$.

Random matrix $\text{rand}(n)$ or $\text{randn}(n)$. MATLAB creates a matrix with random entries, uniformly distributed on $[0 \ 1]$ for rand and standard normal distribution for randn .

Rank one matrix $A = \mathbf{u} \mathbf{v}^T \neq 0$. Column and row spaces = lines $c\mathbf{u}$ and $c\mathbf{v}$.

Rank $r(A) =$ number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x}$ for symmetric A : $\lambda_{\min} \leq q(\mathbf{x}) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors \mathbf{x} for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \text{rref}(A)$. Pivots = 1; zeros above and below pivots; r nonzero rows of R give a basis for the row space of A .

Reflection matrix $Q = I - 2\mathbf{u} \mathbf{u}^T$. The unit vector \mathbf{u} is reflected to $Q\mathbf{u} = -\mathbf{u}$. All vectors \mathbf{x} in the plane mirror $\mathbf{u}^T \mathbf{x} = 0$ are unchanged because $Q\mathbf{x} = \mathbf{x}$. The “Householder matrix” has $Q^T = Q^{-1} = Q$.

Right inverse A^+ . If A has full row rank m , then $A^+ = A^T(AA^T)^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ rotates the plane by θ and $R^{-1} = R^T$ rotates back by $-\theta$. Orthogonal matrix, eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors $(1, \pm i)$.

Row picture of $A\mathbf{x} = \mathbf{b}$. Each equation gives a plane in \mathbf{R}^n ; planes intersect at \mathbf{x} .

Row space $\mathcal{C}(A^T) =$ all combinations of rows of A . Column vectors by convention.

Saddle point of $f(x_1, \dots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix $(\partial^2 f / \partial x_i \partial x_j =$ **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$.

Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$. Then $|\mathbf{v}^T A \mathbf{w}|^2 \leq (\mathbf{v}^T A \mathbf{v})(\mathbf{w}^T A \mathbf{w})$ if $A = C^T C$.

Semidefinite matrix A . (Positive) semidefinite means symmetric with $\mathbf{x}^T A \mathbf{x} \geq 0$ for all vectors \mathbf{x} . Then all eigenvalues $\lambda \geq 0$; no negative pivots.

Similar matrices A and B . Every $B = M^{-1} A M$ has the same eigenvalues as A .

Simplex method for linear programming. The minimum cost vector \mathbf{x}^* is found by moving from corner to lower cost corner along the edges of the feasible set (where the constraints $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ are satisfied). Minimum cost at a corner!

Singular matrix A . A square matrix that has no inverse: $\det(A) = 0$.

Singular Value Decomposition (SVD) $A = U \Sigma V^T =$ (**orthogonal** U) **times** (**diagonal** Σ) **times** (**orthogonal** V^T). First r columns of U and V are orthonormal bases of $\mathcal{C}(A)$ and $\mathcal{C}(A^T)$ with $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ and singular value $\sigma_i > 0$. Last columns of U and V are orthonormal bases of the nullspaces of A^T and A .

Skew-symmetric matrix K . The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $A\mathbf{x} = \mathbf{b}$. The right side \mathbf{b} is in the column space of A .

Spanning set $\mathbf{v}_1, \dots, \mathbf{v}_m$ for \mathbf{V} . Every vector in \mathbf{V} is a combination of $\mathbf{v}_1, \dots, \mathbf{v}_m$.

Special solutions to $A\mathbf{s} = \mathbf{0}$. One free variable is $s_i = 1$, other free variables = 0.

Spectral theorem $A = Q\Lambda Q^T$. Real symmetric A has real λ_i and orthonormal \mathbf{q}_i with $A\mathbf{q}_i = \lambda_i\mathbf{q}_i$. In mechanics the \mathbf{q}_i give the *principal axes*.

Spectrum of A = the set of eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. **Spectral radius** = $|\lambda_{\max}|$.

Standard basis for \mathbf{R}^n . Columns of n by n identity matrix (written $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in \mathbf{R}^3).

Stiffness matrix K . If \mathbf{x} gives the movements of the nodes in a discrete structure, $K\mathbf{x}$ gives the internal forces. Often $K = A^T C A$ where C contains spring constants from Hooke's Law and $A\mathbf{x}$ = stretching (strains) from the movements \mathbf{x} .

Subspace \mathbf{S} of \mathbf{V} . Any vector space inside \mathbf{V} , including \mathbf{V} and $\mathbf{Z} = \{\text{zero vector}\}$.

Sum $\mathbf{V} + \mathbf{W}$ of subspaces. Space of all $(\mathbf{v}$ in $\mathbf{V}) + (\mathbf{w}$ in $\mathbf{W})$. **Direct sum**: $\dim(\mathbf{V} + \mathbf{W}) = \dim \mathbf{V} + \dim \mathbf{W}$ when \mathbf{V} and \mathbf{W} share only the zero vector.

Symmetric factorizations $A = LDL^T$ and $A = Q\Lambda Q^T$. The number of positive pivots in D and positive eigenvalues in Λ is the same.

Symmetric matrix A . The transpose is $A^T = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric. All matrices of the form $R^T R$ and LDL^T and $Q\Lambda Q^T$ are symmetric. Symmetric matrices have real eigenvalues in Λ and orthonormal eigenvectors in Q .

Toeplitz matrix T . Constant-diagonal matrix, so t_{ij} depends only on $j - i$. Toeplitz matrices represent linear time-invariant filters in signal processing.

Trace of A = sum of diagonal entries = sum of eigenvalues of A . $\text{Tr } AB = \text{Tr } BA$.

Transpose matrix A^T . Entries $A_{ij}^T = A_{ji}$. A^T is n by m , $A^T A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^T A^T$ and $(A^T)^{-1}$.

Triangle inequality $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. For matrix norms $\|A + B\| \leq \|A\| + \|B\|$.

Tridiagonal matrix T : $t_{ij} = 0$ if $|i - j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^H = \bar{U}^T = U^{-1}$. Orthonormal columns (complex analog of Q).

Vandermonde matrix V . $V\mathbf{c} = \mathbf{b}$ gives the polynomial $p(x) = c_0 + \dots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$ at n points. $V_{ij} = (x_i)^{j-1}$ and $\det V = \text{product of } (x_k - x_i) \text{ for } k > i$.

Vector \mathbf{v} in \mathbf{R}^n . Sequence of n real numbers $\mathbf{v} = (v_1, \dots, v_n) = \text{point in } \mathbf{R}^n$.

Vector addition. $\mathbf{v} + \mathbf{w} = (v_1 + w_1, \dots, v_n + w_n) = \text{diagonal of parallelogram}$.

Vector space \mathbf{V} . Set of vectors such that all combinations $c\mathbf{v} + d\mathbf{w}$ remain in \mathbf{V} . Eight required rules are given in Section 3.1 for $c\mathbf{v} + d\mathbf{w}$.

Volume of box. The rows (or columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$ or **vectors** \mathbf{w}_{jk} . Stretch and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$. Vectors from $\mathbf{w}_{00} = (1, 1, -1, -1)$ would be $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$.