18.06 Midterm Exam 3, Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.
There are 3 problems. Good luck.

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

   (a) Given that one eigenvalue of A is $\lambda = 6$, find the remaining eigenvalues.

   (b) Find three linearly independent eigenvectors of A.

   (c) Find an *orthogonal* matrix Q and a diagonal matrix Λ, so that $A = Q\Lambda Q^T$.

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$ and $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$ of this system and hence find the solution $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. (40pts.) Let A_n be the $n \times n$ tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for $n \geq 3$ that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \tag{1}$$

(b) Show that eq.(1) can equivalently be written as $\mathbf{x}_n = B\,\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for any n. (*Hint:* One method starts by writing B in the form $B = S\Lambda S^{-1}$, where Λ is a diagonal matrix.)