18.06 Quiz 1 March 10, 1995 Professor Strang

Your name is ______________________________

Your recitation is with (circle one) Professor Kac Professor Axelrod

Your recitation time is (circle one) M2 M3 T10 (Kac) T10 (A) T12 T1

Grading

1.

2.

3.

4.

1. *Given a* 5 *by* 3 *matrix* A. (a) How would you decide if the column vector $c = [1\ 1\ 1\ 1\ 1]'$ is a linear combination of the columns of A? One sentence please.

   (b) How would you decide *by row operations* (not fair to transpose A) if the row vector $r = [1\ 1\ 1]$ is a combination of the rows of A?

   (c) If the decisions in (a) and (b) are both *yes*, what information do you have about the rank of A? *Full* information in box please.

   (d) If the decisions in (a) and (b) are both *no*, what information do you have about the rank of A? Give *reason* also in the box.

2. The 3×3 matrix A reduces to I by the following row operations:

(1) Subtract -2(row 1) from row 2
(2) Subtract 3(row 1) from row 3
(3) Subtract row 3 from row 2
(4) Subtract 3(row 2) from row 1.

(a) What is A^{-1}?

(b) What is A?

3. Suppose A is the matrix L times U:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Give a basis for the row space of A and a basis for the column space of A.

(b) Describe explicitly all solutions to $Ax = 0$.

(c) Find all solutions (if any, depending on c) to $Ax = \begin{bmatrix} 3 \\ 1 \\ c \end{bmatrix}$.

4. This question is about an m by n matrix for which

$Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has **no** solution and $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ has exactly **one** solution.

(a) Give all possible information about m and n and the rank r of A.

(b) If $Ax = 0$ state one specific fact about x (not just that x is in the nullspace).

(c) Write down an example of a matrix A that fits the description in this question.

(d) [Not related to parts a–c] How do you know that the rank of a matrix stays the same if its first and last rows are exchanged?

18.06 Quiz 2 April 10, 1995 Professor Strang

Your name is ____________________________

Your recitation is with (circle one) Professor Kac Professor Axelrod

Your recitation time is (circle one) M2 M3 T10 (Kac) T10 (A) T12 T1

Grading

1.

2.

3.

4.

1. (18pts) (a) If an m by n matrix Q has orthonormal columns is the matrix Q necessarily invertible? Give a reason or a counterexample.

   (b) What is the nullspace of Q (and WHY)?

   (c) What is the projection matrix onto the column space of Q? Avoid inverses where possible.

2. (30pts) We look for the line $y = C+Dt$ closest to 3 points $(t, y) = (0, -1)$ and $(1, 2)$ and $(2, -1)$.

   (a) If the line went through those points (it doesn't), what three equations would be solved?

   (b) Find the best C and D by the least squares method.

   (c) Explain the result you get for C and D: How is the vector $b = (-1, 2, -1)$ related to the plane you are projecting onto?

   (d) What is the length of the error vector e (= distance to plane = $\|b - A\bar{x}\|$).

3. (22pts) The problem is to find the determinants of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) Find $\det A$ and give a reason.

(b) Find $\det B$ using elimination.

(c) Find $\det C$ for any value of x. For this you could use Property 1 of the determinant.

4. (30pts) (a) Decide if A is singular or invertible. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & -4 \end{bmatrix}.$

(b) Find an orthonormal basis for its column space (if such a basis exists).

(c) Why does $P = A(A^TA)^{-1}A^T$ *not* give the projection matrix onto the column space of A?

Find that projection matrix somehow.

18.06 Quiz 3 May 10, 1995 Professor Strang

Your name is ______________________________

Your recitation is with (circle one) Professor Kac Professor Axelrod

Your recitation time is (circle one) M2 M3 T10 (Kac) T10 (A) T12 T1

Grading

1.

2.

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4.

1. (30 pts) (a) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ -\frac{1}{6} & \frac{1}{6} \end{bmatrix}.$$

(b) Suppose the solution to $u_{k+1} = Au_k$ after 100 steps is $u_{100} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. What was the starting vector u_0? (Same matrix A.)

(c) If B is any other 2 by 2 matrix, explain clearly why AB and BA have the same eigenvalues.

2. (30 pts) Suppose that A is a positive definite matrix:

$$A = \begin{bmatrix} 1 & b & 0 \\ b & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

(a) What are the possible values of b?

(b) How do you know that the matrix $A^2 + I$ is positive definite for *every* *b*?

(c) Complete this sentence correctly for a general matrix M, possibly rectangular:

The matrix $M^T M$ is symmetric positive definite unless

________________________________.

3. (32 pts) (a) P is the projection matrix onto the line through $a = (1, 2, 2)$:

$$P = \frac{aa^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

What are its eigenvalues? *Describe all of the corresponding eigenvectors.*

(b) Circle *True or False*: There is a matrix S so that $S^{-1}PS$ is a diagonal matrix (and thus P is diagonalized).

(c) Solve the differential equation $\frac{du}{dt} = Pu$ to find $u(t)$ starting from

$$u(0) = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}.$$

(d) For the difference equation $u_{k+1} = Pu_k$ starting from $u_0 = (1, 0, 0)$, what is the vector u_{100}?

4. (8 pts) Give an example of a linear transformation from four-dimensional space R^4 to two-dimensional space R^2.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1995: Final Examination in 18.06: Linear Algebra

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Your name (printed) Recitation Secret code (optional)	4
(Kac) (Axelrod)	5
	6
	7

Closed book exam (and no Calculators). Answer all 7 questions in the space provided (or direct us to the answer). Solutions will be posted outside the offices of Professor Kac (2–178) and Professor Axelrod (2–247) who are completely in charge of grades. Best wishes to the whole class.

The first questions are about the symmetric matrices with entries $1, 2, 3, \ldots, n-1$

just above and just below the main diagonal. All other entries are zero:

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \qquad A_5 = \ldots$$

1. (18 points) (a) Find a permutation matrix P_3, a lower triangular L_3 with unit diagonal,
   and an echelon matrix U_3 so that $P_3A_3 = L_3U_3$.

1. (b) What is the general (complete) solution to $A_3x = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$?

1. (c) Give a basis for the left nullspace of A_3. Describe that whole nullspace.

1.(d) Find the projection matrix (call it P) onto the column space of A_3.

2. (12 points) (a) Find the eigenvalues and eigenvectors of A_3.

2. (b) For which initial vectors $u(0)$, if any, will the solution of $\frac{du}{dt} = A_3u$ decay to zero?

2. (c) Two eigenvalues of A_4 are approximately 3.65 and $.822$. Find the other two eigenvalues using

$$M^{-1}A_4M = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} = -A_4.$$

3. (12 points) (a) Prove that A_5 is not invertible. $A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$

3. (b) Is A_5 diagonalizable (similar to a diagonal matrix)? Why or why not?

3. (c) Find the determinant of A_6 (cofactors recommended).

4. (15 points) The least squares solution to

$$Ax = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = b \quad \text{is} \quad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 11/3 \\ -1 \end{pmatrix}.$$

4. (a) Find the projection p of b onto the column space of A.

4. (b) Draw the straight-line fit corresponding to this least squares problem. Show on your graph where to see the three components of p.

4. (c) By Gram-Schmidt, find an orthonormal basis q_1, q_2 for the column space of A. *Factor A into QR.* (More space next page.)

5. (12 points) Suppose that the general solution to $Ax = \begin{pmatrix} 3 \\ 1/2 \\ -1 \end{pmatrix}$ is $x = \begin{pmatrix} 5 \\ 11 \\ -9 \\ 7 \end{pmatrix} + t \begin{pmatrix} e \\ \pi \\ 1 \\ 0 \end{pmatrix}$.

(a) What are the dimensions of $R(A)$ and $N(A)$ and $N\left(A^T\right)$?

(b) True or False or Undecidable for this A: $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is solvable.

Give a reason!

(c) How do you know that $A^T A$ is not positive definite?

6. (12 points) The column vector $u_k = (R_k, D_k, I_k)$ gives the number of Republicans, Democrats, and Independents in election k. For the next election all Republicans become Independent, while $\frac{1}{3}$ of the Democrats and $\frac{1}{3}$ of the Independents go into each component of $u_{k+1} = (R_{k+1}, D_{k+1}, I_{k+1})$.

(a) What matrix A gives $u_{k+1} = Au_k$? Check your answer for $u_k = (1, 0, 0)$ and $u_k = (0, 0, 1)$.

(b) What fractions of the voters are in $R_\infty, D_\infty, I_\infty$ at steady state?

(c) Find all eigenvalues and eigenvectors of A and find u_k (after k years) if nobody is for Perot at the start:

$$u_0 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

7. (19 points) (a) If you can solve $Ax = b$, then b must be perpendicular to every vector y in the ______________. Give a 3 by 2 example of A and b and y.

7. (b) Find the projection of $b = (1, 2, 2, 7)$ onto the plane $x_1+x_2+x_3+x_4 = 0$. You could project b first onto the line through $a = (1, 1, 1, 1)$.

7. (c) Circle True or False: if A has repeated eigenvalues, it is always possible to find an orthonormal basis for its column space.

7. (d) Circle True or False: if $v_1, \ldots, v_n$ is a basis for R^n and $b = c_1 v_1 + \cdots + c_n v_n$ then $c_1 = \frac{b^T v_1}{b^T b}$.

7. (e) Construct a matrix with eigenvalues $\lambda = 0, 0, 1$ and rank 2. Why can't it be a projection matrix?

Tuesday May 23, 1995 **Time: 9 AM–12 NOON**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Conflict Examination in 18.06: Linear Algebra

1 2 3 4 5 6 7

1 In parts (a) and (b), find a matrix A with the given property or explain why it is impossible.

(a) The only solution to $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) The only solution to $Ax = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(c) Suppose A is a 3×5 matrix such that

$$A^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$

Find a vector b such that $Ax = b$ has no solution.

2 Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -1 & -5 \end{pmatrix}$. You may work on the back of the preceding page.

(a) Find the LU decomposition of A.

(b) Find a basis for each of the four fundamental subspaces $\mathcal{R}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A)$, $\mathcal{N}(A^T)$.

(c) Find the general solutions of both

$$Ax = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad Ax = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

3 Suppose J = reverse identity matrix = n by n matrix with 1's on the *anti*diagonal:

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{for} \quad n = 3.$$

(a) For any n, what are J^2 and $(I+J)^2$ in terms of I and J?

(b) What numbers might be eigenvalues of $I+J$? For $n=5$ find a full set of eigenvectors.

(c) What is the rank r of $I+J$? The answer depends on n.

(d) What numbers are eigenvalues of J if n is large?

4 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & a & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & b \\ 1 & -1 \end{pmatrix}$. The following questions are about the standard matrix decompositions: L is lower triangular with ones on the diagonal, U and R are upper triangular with *non-zero* pivots, S is invertible, Q is orthogonal, and Λ is diagonal.

(a) $A = LU$ is impossible if $a =$?

(b) $A = QR$ is impossible if $a =$?

(c) $B = Q\Lambda Q^T$ is impossible if $b =$?

(d) $B = S\Lambda S^{-1}$ is impossible if $b =$?

5 Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 8 \\ -6 \\ 2 \\ 0 \end{pmatrix}$. Let V be the vector space spanned by v_1 and v_2.

(a) Find a matrix B whose nullspace is V.

(b) Let $A = \begin{pmatrix} 1 & 8 \\ 1 & -6 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$. Apply Gram-Schmidt to the columns and factor A into QR.

(c) Find the projection of $b = (7, -6, 4, -5)^T$ onto the column space of A.

6 Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Markov matrix.

(a) Give the requirements on a, b, c, d. Then find the two eigenvalues as simply as possible in terms of a, b, c, d.

(b) For $a = b = 3c$, find M and diagonalize it into $S\Lambda S^{-1}$.

(c) For the matrix in (b), what is the limit of $A^k u_0$ as $k \to \infty$ for $u_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

7 Let $A = \begin{pmatrix} -2 & -6 \\ -6 & 7 \end{pmatrix}$.

(a) Find the eigenvalues of A, and an eigenvector for each eigenvalue.

(b) Give a diagonalization $Q\Lambda Q^T$ of A, with Q orthogonal.

(c) Solve $\frac{du}{dt} = Au$ when $u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Give a formula for $u(t)$. As $t \to +\infty$, $u(t)$ goes to a multiple of what vector?

(d) Find the cofactor matrix for $A = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{pmatrix}$ and compute A^{-1}. What conditions on a, b, c, d, e make A invertible?