18.06 Midterm Exam 1, Spring, 2001

Name ______________________ Optional Code ______________

Recitation Instructor ______________ Email Address ______________

Recitation Time ________________

This midterm is closed book and closed notes. No calculators, laptops, cell phones or other electronic devices may be used during the exam.

There are 3 problems. Good luck.

1. (20pts.) Find a general formula for the solutions of the following linear system of equations,

$$\begin{array}{rrrrcr} -x_1 & +3x_2 & & +2x_4 & = & 1 \\ 4x_2 & -12x_2 & +2x_3 & -4x_4 & = & -4 \\ -7x_1 & +21x_2 & +2x_3 & +18x_4 & = & 7 \end{array}$$

- The augmented matrix is

$$\left(\begin{array}{rrrr|r} -1 & 3 & 0 & 2 & 1 \\ 4 & -12 & 2 & -4 & -4 \\ -7 & 21 & 2 & 18 & 7 \end{array}\right).$$

The corresponding row reduced matrix is

$$\left(\begin{array}{rrrr|r} -1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

If we let $x_4 = b$ and $x_2 = a$, then $x_3 = -2b$ and $x_1 = -1 + 3a + 2b$. The general solution thus takes the form

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

2. (40pts.) Let $A = \begin{pmatrix} 1 & 1 & b \\ a & b & b-a \\ 1 & 1 & 0 \end{pmatrix}$.

(a) For $a = 2$ and $b = 1$, find the inverse of A.

(b) For which values of a and b is the matrix A not invertible, i.e. it has less than three pivots?

(c) For what values of a and b is the rank of A equal to 3 ? For what values is it equal to 2, equal to 1 ?

(d) For $a = b = 2$, describe the nullspace of A.

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(a) The augmented matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

We perform row operations to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right).$$

The matrix on the right is the inverse of A.

(b) If we do row operations on the matrix

$$\begin{pmatrix} 1 & 1 & b \\ a & b & b-a \\ 1 & 1 & 0 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} 1 & 1 & b \\ 0 & b-a & b-a-ab \\ 0 & 0 & -b \end{pmatrix}.$$

There are less than three pivot columns if $a = b$ or $b = 0$.

(c) $\mathrm{rk}A = 3$ if $a \neq b \neq 0$. $\mathrm{rk}A = 2$ if $b = 0$ or $a = b \neq 0$. $\mathrm{rk}A = 1$ if $b = a = 0$.

(d) For $a = b = 2$ the row reduced matrix is

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $x_3 = 0$. If we let $x_2 = a$, then $x_1 = -a$. The general solution takes the form

$$\mathbf{x} = a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

3. (40pts.) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

(a) For what vectors $\mathbf{b} = (b_1, b_2, b_3)^T$ does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution?

(b) Prove that the column space of A is made up of those vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = 0$.

(c) Prove that the vectors $(x, y, z)^T \in \mathbb{R}^3$ that satisfy $x + y + z = c$ form a subspace of $\mathbb{R}^3$ if and only if $c = 0$.

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(a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -1 & 1 & 0 & b_2 \\ 0 & -1 & 1 & b_3 \end{array}\right).$$

Performing row operations we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -1 & b_1 + b_2 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{array}\right).$$

Hence, the vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution must satisfy $b_1 + b_2 + b_3 = 0$.

(b) See part (a).

(c) Suppose that the vectors $\mathbf{x} \in \mathbb{R}^3$, which satisfy $x + y + z = c$ form a subspace. Since $\mathbf{0}$ must be in that subspace, and $0 + 0 + 0 = 0$, it follows that $c = 0$.

Now suppose that $c = 0$. Since $0 + 0 + 0 = 0$, $\mathbf{0}$ is in the space. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be in the subspace, and let a and b be real numbers. Consider $a\mathbf{x}_1 + b\mathbf{x}_2$,

$$(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = a(x_1 + y_1 + z_1) + b(x_2 + y_2 + z_2) = a \cdot 0 + b \cdot 0 = 0.$$

So $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the space, and therefore this is a subspace of $\mathbb{R}^3$.