18.06 Professor Strang Quiz 1 Solutions March 6, 1998

1. (a) The nullspace has dimension 2. Therefore $3 - r = 2$ and $r = 1$.

   (b) The first column of A comes from knowing the particular solution. The other columns come from knowing the two special solutions in the nullspace:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

   (c) The vector b must be a multiple of $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$.

2. (a) One basis for the row space is

$$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \\ 6 \end{bmatrix}.$$

   (b) One basis for the column space (since columns 1 and 3 have pivots) is

$$B \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix}.$$

   (c) All answers are different (because the rank is different) when $b_{33} = 0$:

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

3. (a) The row-reduced form is

$$R = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   This form was reached by a product of elementary matrices, including a permutation:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

   The matrix E^{-1} that recovers A from R is

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(b) The third row of R is zero! So the two column-row multiplications are from columns 1 and 2 of E^{-1} and rows 1 and 2 of R:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 0 & 6 & 6 \\ 1 & 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = A\,.$$

4. (a) The matrix Z is m by n. All its entries are zero except for r ones at the start of the main diagonal. If A is 3 by 4 of rank $r = 2$, then

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The matrix ZZ is the same as Z, because A and A^T always have the same rank.