Math 18.06 Exam 1 Solutions

1 (30 pts.)

(a) Because row 3 of R is all zeros, row 3 of A must be a linear combination of rows 1 and 2 of A. The three rows of A are linearly dependent.

(b) After one step of elimination we have

$$\begin{bmatrix} 1 & 2 & 1 & b \\ 0 & a-4 & -1 & 8-2b \\ & (\text{row} & 3) & \end{bmatrix}$$

Looking at R we see that the second column of A is not a pivot column, so $a = 4$. Continuing with elimination, we get to

$$\begin{bmatrix} 1 & 2 & 0 & 8-b \\ 0 & 0 & 1 & 2b-8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Comparing this to R we see that $b = 5$

(c) Setting the free variables x_2 and x_4 to 1 and 0, and vice versa, and solving $Rx = 0$, we get the nullspace solution

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The row space and the nullspace are always the same for A and R.

2 (30 pts.) (a) After elimination, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & c-8 \end{bmatrix}$$

So this matrix will not be invertible when $c = 8$

(b) When c is not equal to 8, the matrix is invertible, its rank is 3. So its nullspace is just the zero vector, and its columnspace is all of $\mathbb{R}^3$. The same logic and answers apply to A^{-1}.

(c) Using our multipliers from elimination,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 12 \end{bmatrix}$$

3 (40 pts.)

(a) There must be a pivot in every row, so $r = m$ and the column space of A is all of $\mathbb{R}^m$

(b) We always have $r \leq n$. From (a) we know $r = m$. From these we deduce also that $m \leq n$

(c) Just use a multiple of [2,5] for the other rows also. For example

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 10 \\ 0 & 0 \end{bmatrix}$$

The column space will be the line in $\mathbb{R}^3$ consisting of all multiples of your first column. The nullspace will be the line in $\mathbb{R}^2$ consisting of all multiples of the nullspace solution $\begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$

(d) Adding the particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the nullspace solution from (c) we get the complete solution

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$$