18.06 Exam 1 Solutions March 1, 2000

Problem 1 (a) After forming the augmented matrix and doing row reduction, the third row becomes $\begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}$, which corresponds to the equation $0 = -1$, so there is no solution.
(b) The same argument shows that in order for $Ax = b$ to have a solution, b must satisfy $b_3 = b_1 + b_2$.
(c) If A were invertible, there would always be a solution $Ax = b$.

Problem 2 (a)

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) From U in part (a) we see that every column is a pivot column. The pivot columns from A are a basis for the column space: $(2,2,0)$, $(2,5,3)$, $(1,0,2)$. Since the rank is three, the column space is all of $\mathbf{R}^3$, so another basis would be the standard basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. In fact, any three independent vectors in $\mathbf{R}^3$ will do.
(c) The rank is three because there are three pivots.

Problem 3 (a) This is an LU factorization. The U is the echelon form of A, so you can see that there are three pivots, so the rank of A is three.
(b) A basis for $N(A)$ consists of the special solutions. These are $(-1,-2,1,0,0)$ and $(-1,1,0,-1,1)$.
(c) A particular solution is $(-30,-15,0,10,0)$ so the complete solution is

$$\begin{bmatrix} -30 \\ -15 \\ 0 \\ 10 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Problem 4 (a) One basis would be $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
(b) The subspace consisting of all multiples of A is a subspace which contains A but not B.
(c) True: If a subspace V contains A and B, then it contains $A - B = I$.
(d) Same answer as (b) will work.

Problem 5 There are many different proofs. One is to say that if $A^2 = 0$ then obviously A^2 is not invertible. Therefore A isn't invertible, because the product of invertible matrices is invertible. I.e., if A were invertible, then A^2 would be invertible.