18.06 Professor Strang Quiz 2 April 2, 2004

Grading

1

2

3

4

Your name is: SOLUTIONS

Please circle your recitation:

1 (20 pts.) We are given two vectors a and b in $\mathbb{R}^4$:

$$a = \begin{bmatrix} 2 \\ 5 \\ 2 \\ 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

(a) Find the projection p of the vector b onto the line through a. **Check** (!) that the error $e = b - p$ is perpendicular to (<u>**what**?</u>)

(b) The subspace S of all vectors in $\mathbb{R}^4$ that are *perpendicular to this* a is 3-dimensional. Find the projection q of b onto this perpendicular subspace S. The numerical answer (it doesn't need a big computation!) is $q =$ ______.

Answer:

(a) $p = \frac{a^{\mathrm{T}}b}{a^{\mathrm{T}}a}a = \frac{2}{7}a$. $e = b - p = \frac{1}{7}(3 \quad 4 \quad 3 \quad -8)^{\mathrm{T}}$. Easy to check that $e^{\mathrm{T}}p = 0$.

(b) Since $b = e + p$, the projection of b onto S is just $q = e$, with error p.

2 (30 pts.) Suppose q_1, q_2, q_3 are 3 orthonormal vectors in $\mathbb{R}^n$. They go in the columns of an n by 3 matrix Q.

(a) What inequality do you know for n?

Is there any condition on n for $Q^{\mathrm{T}}Q = I$ (3 by 3)?

Is there any condition on n for $QQ^{\mathrm{T}} = I$ (n by n)?

(b) Give a nice matrix formula involving b and Q, for the projection p of a vector b onto the column space of Q.

Complete this sentence: p **is the closest vector** ______________ ...

(c) Suppose the projection of b onto that column space is $p = c_1q_1 + c_2q_2 + c_3q_3$. Find a formula for c_1 that only involves b and q_1. (You could take dot products with q_1.)

Answer:

(a) $n \geq 3$, no condition, and $n = 3$.

(b) $p = QQ^{\mathrm{T}}b$. p is the closest vector to b in the column space of Q.

(c) $c_1 =$ length of the projection of b onto q_1, i.e., $c_1 = q_1^{\mathrm{T}}b$.

3 (20 pts.) Suppose the 4 by 4 matrix M has four equal rows all containing a, b, c, d. We know that $\det(M) = 0$. The problem is to find by any method

$$\det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

Note If you can't find $\det(I+M)$ in general, partial credit for the determinant when $a = b = c = d = 1$.

Answer:

One Solution: Subtracting row 1 from the other rows leaves

$$\begin{vmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}$$

Adding columns $2, 3, 4$ to column 1 leaves a triangular matrix:

$$\begin{vmatrix} 1+a+b+c+d & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1+a+b+c+d\,.$$

There are many other possible solutions, here is another:

$$\det(I+M) = \begin{vmatrix} a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} + \begin{vmatrix} 1 & b & c & d \\ 0 & 1+b & c & d \\ 0 & b & 1+c & d \\ 0 & b & c & 1+d \end{vmatrix}.$$

The first determinant, by subtracting the first line from all others, is just a, while the second determinant is just the determinant of the 3×3 analog of this situation. Proceeding by induction, we get the same answer.

4 (30 pts.) We are looking for the parabola $y = C + Dt + Et^2$ that gives the least squares fit to these four measurements:

$$y_1 = 1 \text{ at } t_1 = -2,\ y_2 = 1 \text{ at } t_2 = -1,\ y_3 = 1 \text{ at } t_3 = 1,\ y_4 = 0 \text{ at } t_4 = 2\,.$$

(a) Write down the four equations (not solvable!) for the parabola $C + Dt + Et^2$ to go through those four points. This is the system $Ax = b$ to solve by least squares:

$$A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = b\,.$$

What equations would you solve to find the best C, D, E?

(b) Compute $A^{\mathrm{T}}A$. Compute its determinant. Compute its inverse. NOT ASKING FOR C, D, E.

(c) The first two columns of A are already orthogonal. From column 3, subtract its projection onto the plane of the first two columns. That produces what third orthogonal vector v? Normalize v to find the third orthonormal vector q_3 from Gram-Schmidt.

Answer:

(a)

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We want to solve $A^{\mathrm{T}}Ax = b$.

(b)

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}.$$

$\det(A^{\mathrm{T}}A) = 40.34 - 1000 = 360$; $(A^{\mathrm{T}}A)^{-1} = \frac{1}{\det(A^{\mathrm{T}}A)}C^{\mathrm{T}}$, where C (cofactor matrix, symmetric in this case) is given by:

$$C = \begin{bmatrix} 340 & 0 & -100 \\ 0 & 36 & 0 \\ -100 & 0 & 34 \end{bmatrix}.$$

(c) Since the third and second columns are already orthogonal, suffices to subtract from the third column its projection onto the first column:

$$C = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \\ -3/2 \\ 3/2 \end{bmatrix}.$$

To find q_3, just divide C by its length, 3. So $q_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.