18.06 Professor Strang Exam 2 Solutions April 8, 1998

1. (a) $\|a_1\| = 2$ so $q_1 = a_1/2$. Then subtract from a_2 its projection onto a_1:

$$B = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

This also has length $\|B\| = 2$ so $q_2 = B/2$. The vector $a_3 = a_1 + a_2$ does not affect the dimension of S or its basis.

(b) $p = QQ^T b = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}.$

2. (a) The orthogonal complement of S is the nullspace of A:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix} \to U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \end{bmatrix}.$$

The special solutions give a basis for $S^\perp$ (you may find another basis!):

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Since b is split into perpendicular pieces $p + q$, we know immediately that

$$q = b - p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}.$$

3. (a)

$$A_4 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \to U = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}$$

The product of the pivots is 9.

(b) There are four nonzero terms: 16, -4, -4 and 1.

(c) $\det A_n = 2\det A_{n-1} - \det M_{n-2}$. The matrix M_{n-2} starts with $\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$ in its upper left corner and after that it continues like A_{n-2}. With $n = 4$ we only see that 2 by 2 corner from the cofactor rule used twice (which removes rows 1, 3 and columns 1, 3).

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Note for the future: by continuing on M_{n-2} I finally arrived at

$$\det A_n = 2\det A_{n-1} - 2\det A_{n-3} + \det A_{n-4}\,.$$

4. (a) $P = A(A^TA)^{-1}A^T$: the matrix A^TA is invertible if and only if A has independent columns.

(b) The properties give $PP^T = P$. Compare the $(1,1)$ entry on both sides of this equation to find $v^Tv = v_1$.