18.06 Solutions to Midterm Exam 3, Spring, 2001

1. (40pts.) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

(a) Given that one eigenvalue of A is $\lambda = 6$, find the remaining eigenvalues.

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$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & -1 & 1 \\ -1 & 4-\lambda & -1 \\ 1 & -1 & 4-\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow -\lambda^3 + 12\lambda^2 - 45\lambda + 54 &= 0 \\ \Leftrightarrow \lambda_1 = 6, \ \lambda_2 = \lambda_3 = 3 & \end{aligned}$$

(b) Find three linearly independent eigenvectors of A.

• For $\lambda_1 = 6$, we have $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}$, and so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = \lambda_3 = 3$, we have $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, and two linearly independent solutions are $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(c) Find an *orthogonal* matrix Q and a diagonal matrix Λ, so that $A = Q\Lambda Q^T$.

• To obtain the first column of Q, let $\mathbf{q}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$. The second column of Q is given by $\mathbf{q}_2 = \mathbf{v}_1/\|\mathbf{v}_2\| = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$. The vector $\mathbf{v}_3$ is not orthogonal to $\mathbf{q}_2$, so we need to use Gram-Schmidt to make it so:

$$\tilde{\mathbf{q}}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix}.$$

Normalising gives $\mathbf{q}_3 = \tilde{\mathbf{q}}_3/\|\tilde{\mathbf{q}}_3\| = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$. Hence,

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

2. (20pts.) Consider the system of first order linear ODEs

$$\frac{dx}{dt} = -7x + 2y \quad \frac{dy}{dt} = -6x.$$

Find two independent real-valued solutions $\begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix}$ and $\begin{pmatrix} x^{(2)} \\ y^{(2)} \end{pmatrix}$ of this system and hence find the solution $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- We can write this system as $\frac{d}{dt}\mathbf{x} = \begin{pmatrix} -7 & 2 \\ -6 & 0 \end{pmatrix}\mathbf{x}$. The eigenvalues of this matrix are given by

$$\begin{aligned} \det\begin{pmatrix} -7-\lambda & 2 \\ -6 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 + 7\lambda + 12 &= 0 \\ \Leftrightarrow \lambda_1 = -4,\ \lambda_2 = -3. \end{aligned}$$

The corresponding eigenvectors are: for $\lambda_1 = -4$, $\begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; and for $\lambda_2 = -3$, $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence the general solution to this system of equations is

$$\mathbf{x}(t) = C_1 e^{-4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

To satisfy the initial condition, we need to solve

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives $C_1 = 1$ and $C_2 = -1$.

3. (40pts.) Let A_n be the $n \times n$ tridiagonal matrix

$$A_n = \begin{pmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}.$$

(a) Show for $n \geq 3$ that

$$\det(A_n) = \det(A_{n-1}) - a^2 \cdot \det(A_{n-2}). \tag{1}$$

- Let us expand the determinant of A_n along the first column,

$$\begin{aligned} \det(A_n) &= 1 \cdot \det \begin{pmatrix} 1 & -a & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} + a \det \begin{pmatrix} -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -a & 1 & -a \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det \begin{pmatrix} 1 & -a & 0 & \cdots \\ -a & 1 & -a & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & -a & 1 \end{pmatrix} \\ &= \det(A_{n-1}) - a^2 \det(A_{n-2}). \end{aligned}$$

Here, we have expanded the determinant in the second term of line 1 along the first row.

(b) Show that eq.(1) can equivalently be written as $\mathbf{x}_n = B\,\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix}.$$

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$$\begin{aligned} \mathbf{x}_n = \begin{pmatrix} \det(A_n) \\ \det(A_{n-1}) \end{pmatrix} &= \begin{pmatrix} \det(A_{n-1}) - a^2 \det(A_{n-2}) \\ \det(A_{n-1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \det(A_{n-1}) \\ \det(A_{n-2}) \end{pmatrix} = \begin{pmatrix} 1 & -a^2 \\ 1 & 0 \end{pmatrix} \mathbf{x}_{n-1}. \end{aligned}$$

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for any n. (*Hint:* One method starts by writing B in the form $B = S\Lambda S^{-1}$, where Λ is a diagonal matrix.)

• The answer is given by $\mathbf{x}_n = B^{n-2}\mathbf{x}_2$. To determine $\mathbf{x}_2$, we need $\det(A_1) = 1$, and $\det(A_2) = \det\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} = 1 - a^2 = 1 - \frac{3}{16} = \frac{13}{16}$. To find B^{n-2}, we first need to diagonalise B. Eigenvalues are given by

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & -\frac{3}{16} \\ 1 & -\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow \lambda^2 - \lambda + \frac{3}{16} &= 0 \\ \Leftrightarrow \lambda_1 = \frac{3}{4},\ \lambda_2 = \frac{1}{4}. \end{aligned}$$

The eigenvector corresponding to $\lambda_1 = \frac{3}{4}$ is given by $\begin{pmatrix} 1/4 & -3/16 \\ 1 & -3/4 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$, so that $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The eigenvector corresponding to $\lambda_2 = \frac{1}{4}$ is given by $\begin{pmatrix} 3/4 & -3/16 \\ 1 & -1/4 \end{pmatrix}\mathbf{v}_2 = \mathbf{0}$, and hence $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. So $B = S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad S^{-1} = \frac{1}{8}\begin{pmatrix} 4 & -1 \\ -4 & 3 \end{pmatrix}.$$

Hence,

$$\mathbf{x}_n = S\Lambda^{n-2}S^{-1}\mathbf{x}_2 = \frac{1}{8}\begin{pmatrix} \frac{27}{4}\left(\frac{3}{4}\right)^{n-2} - \frac{1}{4}\left(\frac{1}{4}\right)^{n-2} \\ 9\left(\frac{3}{4}\right)^{n-2} - \left(\frac{1}{4}\right)^{n-2} \end{pmatrix}$$

and

$$\det(A_n) = \frac{2}{4^{n+1}}\left(3^{n+1} - 1\right).$$