Course 18.06, Fall 2002: Quiz 3, Solutions

1 (a) One eigenvalue of $A =$ones(5) is $\lambda_1 = 5$, corresponding to the eigenvector $\boldsymbol{x}_1 = (1, 1, 1, 1, 1)$. Since the rank of A is 1, all the other eigenvalues $\lambda_2, \ldots, \lambda_5$ are zero. Check: The trace of A is 5.

(b) The initial condition $\boldsymbol{u}(0)$ can be written as a sum of the two eigenvectors $\boldsymbol{x}_1 = (1, 1, 1, 1, 1)$ and $\boldsymbol{x}_2 = (-1, 0, 0, 0, 1)$, corresponding to the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 0$:

$$\boldsymbol{u}(0) = (0, 1, 1, 1, 2) = (1, 1, 1, 1, 1) + (-1, 0, 0, 0, 1) = \boldsymbol{x}_1 + \boldsymbol{x}_2.$$

The solution to $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ is then

$$\boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2 = (1, 1, 1, 1, 1)e^{5t} + (-1, 0, 0, 0, 1).$$

(c) The eigenvectors of $B = A - I$ are the same as for A, and the eigenvalues are smaller by 1:

$$B\boldsymbol{x} = (A - I)\boldsymbol{x} = A\boldsymbol{x} - \boldsymbol{x} = \lambda\boldsymbol{x} - \boldsymbol{x} = (\lambda - 1)\boldsymbol{x},$$

where $\boldsymbol{x}, \lambda$ are an eigenvector and an eigenvalue of A. The eigenvalues of B are then $4, -1, -1, -1, -1$, the trace is $\sum_i \lambda_i = 0$, and the determinant is $\prod_i \lambda_i = 4$.

2 (a) B is similar to A when $B = M^{-1}AM$, with M invertible. The exponential of A is

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots.$$

Every power B^k of B is similar to the same power A^k of A:

$$B^k = M^{-1}AMM^{-1}AM \cdots M^{-1}AM = M^{-1}A^kM.$$

Then

$$e^B = I + B + \frac{1}{2}B^2 + \cdots = M^{-1}\left(I + A + \frac{1}{2}A^2 + \cdots\right)M = M^{-1}e^AM.$$

It is also OK to show this using $e^A = Se^\Lambda S^{-1}$, although that assumes that the matrices are diagonalizable.

(b) The exponential of A is

$$e^A = Se^\Lambda S^{-1} = S\begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^4 \end{bmatrix}S^{-1}.$$

But this is an eigenvalue decomposition of e^A, so the eigenvalues are $1, e^2, e^4$.
More generally, the eigenvalues of e^A are the exponentials of the eigenvalues of A, and

$$\det(e^A) = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n} = e^{\mathrm{tr}(A)}.$$

3 (a) For A to be symmetric, U has to be equal to V (notice V^T in the matrices):

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Together with the restrictions on θ, α this requires that $\theta = \alpha$. A is then a positive definite symmetric matrix, since it is similar to $\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) The eigenvalues of A^TA are the square of the singular values, that is, 81 and 16. The eigenvectors of A^TA are the columns of V, that is, $(\cos\alpha, \sin\alpha)$ and $(-\sin\alpha, \cos\alpha)$. This can also be shown by multiplying $A^TA = V\Sigma^2V^T$ and identifying this as the eigenvalue decomposition of A^TA.

4 (a) A is singular, so one eigenvalue is 0. It is also a Markov matrix, so another eigenvalue is 1 (Motivation: Each column of A sums to 1, so each column of $A - I$ sums to 0. $A - I$ then has an eigenvalue 0, and A has an eigenvalue 1). The last eigenvalue is 0.5 since $\text{trace}(A) = \sum_i \lambda_i = 1.5$.
The eigenvectors are found by solving the following systems:

$$\lambda_1 = 1: \qquad (A - \lambda_1 I)\boldsymbol{x}_1 = \begin{bmatrix} -.5 & .5 & .5 \\ .25 & -.5 & 0 \\ .25 & 0 & -.5 \end{bmatrix} \boldsymbol{x}_1 = 0 \Longrightarrow \boldsymbol{x}_1 = (2, 1, 1),$$

$$\lambda_2 = 0.5: \qquad (A - \lambda_2 I)\boldsymbol{x}_2 = \begin{bmatrix} 0 & .5 & .5 \\ .25 & 0 & 0 \\ .25 & 0 & 0 \end{bmatrix} \boldsymbol{x}_2 = 0 \Longrightarrow \boldsymbol{x}_2 = (0, 1, -1),$$

$$\lambda_3 = 0: \qquad (A - \lambda_3 I)\boldsymbol{x}_3 = \begin{bmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{bmatrix} \boldsymbol{x}_3 = 0 \Longrightarrow \boldsymbol{x}_3 = (2, -1, -1).$$

(b) Write the initial value as a linear combination of the eigenvectors:

$$\boldsymbol{u}_0 = (6, 0, 6) = 3\boldsymbol{x}_1 - 3\boldsymbol{x}_2.$$

The distribution after k steps is then

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = 3\lambda_1^k\boldsymbol{x}_1 - 3\lambda_2^k\boldsymbol{x}_2 = 3\boldsymbol{x}_1 - 3\cdot 0.5^k\boldsymbol{x}_2 \to 3\boldsymbol{x}_1 = (6, 3, 3) \text{ as } k \to \infty.$$