18.06 Professor Strang Quiz 3 May 5, 2004

Your name is: SOLUTIONS

Please circle your recitation:

Grading
1
2
3

1 (40 pts.) This question deals with the following symmetric matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

One eigenvalue is $\lambda = 1$ with the line of eigenvectors $x = (c, c, 0)$.

(a) That line is the nullspace of what matrix constructed from A?

(b) Find (in any way) the other two eigenvalues of A and two corresponding eigenvectors.

(c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^{\mathrm{T}}$. Write all entries in the three matrices in the nice symmetric diagonalization of A.

(d) Give a reason why e^A *is or is not* a symmetric positive definite matrix.

Solution:

(a) The eigenvectors for $\lambda = 1$ make up the nullspace of $A - I$.

(b) First method: A has trace 2 and determinant -2. So the two eigenvalues after $\lambda_1 = 1$ will add to 1 and multiply to -2. *Those are $\lambda_2 = 2$ and $\lambda_3 = -1$.*

Second method: Compute $\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + \lambda - 2$ and find the roots $1, 2, -1$: (divide by $\lambda - 1$ to get $\lambda^2 - \lambda - 2 = 0$ for the roots λ_2 and λ_3).

Eigenvectors: $\lambda_2 = 2$ has $x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\lambda_3 = -1$ has $x_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

(c) Every symmetric matrix has the nice form $A = Q\Lambda Q^{\mathrm{T}}$ with orthogonal matrix Q. The columns of Q are orthonormal eigenvectors. (They could be multiplied by -1.)

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 1 & & \\ & 2 & \\ & & -1 \end{bmatrix}.$$

(d) e^A *is* symmetric and all its eigenvalues e^λ *are* positive—so e^A is positive definite.

2 (30 pts.) (a) Find the *eigenvalues* and *eigenvectors* (depending on c) of

$$A = \begin{bmatrix} .3 & c \\ .7 & 1-c \end{bmatrix}.$$

For which value of c is the matrix A *not diagonalizable* (so $A = S\Lambda S^{-1}$ is impossible)?

(b) What is the *largest range of values of* c (real number) so that A^n approaches a limiting matrix A^∞ as $n \to \infty$?

(c) What is that limit of A^n (still depending on c)? You could work from $A = S\Lambda S^{-1}$ to find A^n.

Solution:

(a) Both columns add to 1. As we know for Markov matrices, $\lambda = 1$ is an eigenvalue. From trace$(A) = .3 + (1-c)$ the other eigenvalue is $\lambda = .3 - c$. Check: $\det A = \lambda_1\lambda_2 = (1)(.3-c)$ is correct.

The eigenvector for $\lambda = 1$ is in the nullspaces of

$$A - I = \begin{bmatrix} -.7 & c \\ .7 & -c \end{bmatrix} \qquad \text{so } x_1 = \begin{bmatrix} c \\ .7 \end{bmatrix}$$

$$A - (.3-c)I = \begin{bmatrix} c & c \\ .7 & .7 \end{bmatrix} \qquad \text{so } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A is *not diagonalizable* when its eigenvalues are equal: $1 = .3 - c$ or $c = -.7$. (The two eigenvectors above become dependent at $c = -.7$)

(b) $A^n = S\Lambda^n S^{-1} = S \begin{bmatrix} 1 & 0 \\ 0 & (.3-c)^n \end{bmatrix} S^{-1}$

This approaches a limit if $|.3 - c| < 1$. You could write that out as $-.7 < c < 1.3$ (Small note: at $c = -.7$ the eigenvalues are 1 and 1, at $c = 1.3$ the eigenvalues are 1 and -1.)

(c) The eigenvectors are in S. As $n \to \infty$ the smaller eigenvalue λ_2^n goes to zero, leaving

$$\begin{aligned} A^\infty = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1} &= \begin{bmatrix} c & 1 \\ .7 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .7 & -c \end{bmatrix} /(c+.7) \\ &= \begin{bmatrix} c \\ .7 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} /(c+.7) = \begin{bmatrix} c & c \\ .7 & .7 \end{bmatrix} /(c+.7) \end{aligned}$$

3 (30 pts.) Suppose A (3 by 4) has the Singular Value Decomposition (with real orthogonal matrices U and V)

$$A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} \\ u_1 & u_2 & u_3 \\ \\ \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \\ v_1 & v_2 & v_3 & v_4 \\ \\ \end{bmatrix}^{\mathrm{T}}.$$

(a) Find the *rank* of A and a *basis* for its column space $C(A)$.

(b) What are the eigenvalues and eigenvectors of $A^{\mathrm{T}}A$? (You could first multiply A^{T} times A.)

(c) What is Av_1? You could start with $V^{\mathrm{T}}v_1$ and then multiply by Σ and U to get $U\Sigma V^{\mathrm{T}}v_1$.

Solution:

(a) Rank $= 2 = \operatorname{rank}(A^{\mathrm{T}}A) = \#$ of nonzero singular values. The vectors u_1 and u_2 (very sorry about the typo) are a basis for the column space of A.

(b) $A^{\mathrm{T}}A = (V\Sigma^{\mathrm{T}}U^{\mathrm{T}})(U\Sigma V^{\mathrm{T}}) = V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}$. The eigenvalues of $A^{\mathrm{T}}A$ are $4, 1, 0, 0$ in the diagonal matrix $\Sigma^{\mathrm{T}}\Sigma$. The eigenvectors are v_1, v_2, v_3, v_4 in the matrix V.

(c) $V^{\mathrm{T}}v_1 = \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \\ v_3^{\mathrm{T}} \\ v_4^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \\ v_1 \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ by orthogonality of the v's.

Multiply by Σ to get $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Then multiply by U to get the final answer $2u_1$.

Thus $Av_1 = 2u_1$, which was a main point of the SVD.