18.06 Fall 2003 Quiz 1 October 1, 2003

Your name is:

Please circle your recitation:

Grading:

Question	Points	Maximum
Name + rec		5
1		25
2		15
3		5
4		35
5		15
Extra credit:		(10)
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation name correctly is worth 5 points.
The exam is worth a total of 100 points.

1. a) (15 points) Find an LU-decomposition of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution:

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} L &= (E_{32}E_{31}E_{21})^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{aligned}$$

Therefore we have,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) (10 points) Solve $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

From 1(a) we have $A = LU$. Let $c = Ux$ and solve for $Lc = b$ using back substitution to get

$$c = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Now, solve for $Ux = c$ using back substitution to get

$$x = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

2. (15 points) Let A be an unknown 3×3 matrix, and let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the augmented matrix $B = [A \,|\, P\,]$. After performing row operations on B we get the following matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right].$$

What is A^{-1}?

Solution:

By performing 2 more row operations on B we get the following augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & -3 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] = \left[\, I \,\middle|\, A^{-1}P \right].$$

Since $P^{-1} = P$, we have

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 2 & -3 & -3 \\ -1 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 & -3 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3. (5 points) Find a matrix A such that

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - y \\ x + y + 2w \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

4. All of the questions below refer to the following matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

a) (5 points) What is the rank of A?

Solution:
The rank of A is equal to the number of pivots which is 2.

b) (5 points) Do all pairs of columns span the column space, $C(A)$, of A? If yes, explain. If no, give a pair of columns that do not span the column space.

Solution:
No! The column space of A is all of $\mathbb{R}^2$. However, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent and hence only span a one-dimensional subspace of $\mathbb{R}^2$.

c) (10 points) Find a basis for the nullspace $N(A)$ of A.

Solution:
Let $x_2 = 1$ and $x_4 = 0$. We solve for the pivot variables: $x_1 = -2$ and $x_3 = 0$.

Let $x_2 = 0$ and $x_4 = 1$. We solve for the pivot variables: $x_1 = -1$ and $x_3 = -2$.

A basis for the nullspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

d) (5 points) Does there exist a vector $b \in R^2$ such that $Ax = b$ has no solution?

Solution:

No! One possible solution to $Ax = b$ is $x = \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{bmatrix}$.

e) (10 points) Find all solutions of

$$Ax = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Express your solution in the form

$$x = x_{particular} + c_1 x_1 + c_2 x_2$$

where x_1, x_2 are special solutions.

Solution:

$$x = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

5. a) (6 points) How many 3×3 permutation matrices are there (including I)?

Solution: 3!=6

b) (9 points) Is there a 3×3 permutation matrix P, besides $P = I$, such that $P^3 = I$? If yes, give one such P. If no, explain why.

Solution: Yes,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. **Extra Credit (10 points)** The matrix in question 1 is a Pascal matrix. Find an LU-decomposition of the 6×6 Pascal matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

Note: you don't need to write the entire matrix again, just explain how to get the LU-decomposition.

Solution:

Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $L = U^T$ then $A = LU$.