An Infinite Loop Space Structure for K-theory of Bimonoidal Categories

by

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Abstract

In recent work of Baas-Dundas-Richter-Rognes, the authors introduce the notion of the K-theory of a bimonoidal category $\mathcal{R}$, and show that it is equivalent to the algebraic K-theory space of the ring spectrum $K\mathcal{R}$. In this thesis we show that $\mathcal{K}(\mathcal{R})$ is the group completion of the classifying space of the 2-category $Mod_{\mathcal{R}}$ of modules over $\mathcal{R}$, and show that $Mod_{\mathcal{R}}$ is a symmetric monoidal 2-category. We explain how to use this symmetric monoidal structure to produce a Γ-(2-category), which gives an infinite loop space structure on $\mathcal{K}(\mathcal{R})$. We show that the equivalence mentioned above is an equivalence of infinite loop spaces.

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Contents

Chapter 1

Introduction

Different cohomology theories detect different information about spaces. One way of distinguishing among some of them is by their chromatic level. This level depends on the ability of the cohomology to detect certain periodic phenomena. Rational cohomology has chromatic level 0, K-theory has chromatic level 1 and complex cobordism has chromatic level ∞.

For some time, topologists have been searching for a cohomology that is both geometrically flavored and has chromatic level 2. N. Baas, B. Dundas and J. Rognes introduce in [BDR] the notion of a complex 2-vector bundle over a topological space and construct a classifying space for these bundles, $\mathcal{K}(Vect_{\mathbb{C}})$.

The space $\mathcal{K}(Vect_{\mathbb{C}})$ is a particular case of $\mathcal{K}(\mathcal{R})$, the K-theory of a strict bimonoidal category $(\mathcal{R}, \oplus, \otimes)$. A strict bimonoidal category is one that roughly behaves as a semi-ring. The authors conjecture in [BDR], and then (with B. Richter) prove in [BDRR2], that there is a weak equivalence of spaces

$$\mathcal{K}(\mathcal{R}) \xrightarrow{\sim} K(K\mathcal{R}), \tag{1.1}$$

where $K\mathcal{R}$ denotes the K-theory spectrum of $\mathcal{R}$ (see [EM] for details). This is a ring spectrum, and $K(K\mathcal{R})$ denotes its algebraic K-theory.

In the case of $\mathcal{R} = Vect_{\mathbb{C}}$, the statement implies that the classifying space of virtual 2-vector bundles is equivalent to $K(ku)$, where ku is the connective complex K-theory spectrum. C. Ausoni and J. Rognes prove in [AR] that $K(ku_p^\wedge)$ has (at least) chromatic level 2.

As the right hand side of equation (1.1) is the algebraic K-theory of a ring spectrum, it is an infinite loop space. The authors use this structure to show that $\mathcal{K}(\mathcal{R})$ defines a cohomology theory.

The two-stage construction of $K(K\mathcal{R})$ makes the infinite loop space structure hard to handle or understand. On the other hand, $\mathcal{K}(\mathcal{R})$ is constructed in one step using both monoidal structures at once. Hence the following questions arise naturally:

1. Is there an infinite loop space structure on $\mathcal{K}(\mathcal{R})$ induced directly by the structure of $\mathcal{R}$?
2. If there is such a structure, is the map of equation (1.1) an infinite loop space map?

The purpose of this thesis is to answer these questions.

In order to answer the first question we construct $Mod_{\mathcal{R}}$, the 2-category of finitely generated free modules over $\mathcal{R}$. It turns out that the K-theory space of $\mathcal{R}$ is closely related to the classifying space of this 2-category. Moreover, we show that $Mod_{\mathcal{R}}$ is a symmetric monoidal 2-category.

We then adapt the techniques of [Seg] and [May1] to construct a special Γ-(2-category) $\widehat{Mod_{\mathcal{R}}}$, whose classifying space $|S\widehat{Mod_{\mathcal{R}}}|$ is an infinite loop space. This results in an infinite delooping of the classifying space $|SMod_{\mathcal{R}}|$ and hence of $\mathcal{K}(\mathcal{R})$.

To answer question (2) we would like to construct a map of Γ-spaces $|S\widehat{Mod_{\mathcal{R}}}| \to |N\widehat{Mod_{K\mathcal{R}}}|$, where $Mod_{K\mathcal{R}}$ is the category of modules over the ring spectrum $K\mathcal{R}$. Although we cannot build this map directly, we build an alternative construction for the Γ-category $\widetilde{\mathcal{D}}$ of a simplicial symmetric monoidal category $\mathcal{D}$. We then show that we get a zigzag of maps

$$|S\widehat{Mod_{\mathcal{R}}}| \longrightarrow |N\widetilde{Mod_{K\mathcal{R}}}| \xleftarrow{\sim} |N\widehat{Mod_{K\mathcal{R}}}|,$$

where the right-hand map is a levelwise equivalence of Γ-spaces. At level 1, the right-hand map is the identity, and the left-hand map corresponds to the equivalence in equation (1.1).

1.1 Organization and general conventions

In Chapter 2, we recall the definitions of 2-category as well other related notions, and set the notation that will be used in subsequent chapters. We also give the construction of the classifying space of a 2-category. In Chapter 3 we recall the construction of the K-theory of a bimonoidal category from [BDR] and show how it is related to the 2-category

$Mod_{\mathcal{R}}$. In Chapter 4 we show that the 2-category $Mod_{\mathcal{R}}$ is symmetric monoidal. Chapter 5 contains the construction of the special Γ-(2-category) $\widehat{Mod_{\mathcal{R}}}$ and the result about the infinite delooping of the space $\mathcal{K}(\mathcal{R})$. Finally, in Chapter 6 we show that the map in equation (1.1) is an infinite loop map by showing an alternative construction of the infinite loop space structure on $K(K\mathcal{R})$.

Throughout the document we will assume categories and 2-categories are enriched over simplicial sets without explicitly saying it. Whenever we worked with a non-enriched category we will indicate it explicitly. We sometimes use the term space to refer to a simplicial set.

Chapter 2

2-categories and their classifying spaces

In this thesis we will use the term 2-category to refer to the non-strict version, that is, what other authors -in particular category theorists- call bicategories. Since some of the notation is non-standard, in this chapter we will review the definitions and theorems that will be used in subsequent chapters. We will omit the proofs of the theorems and refer the reader to the earlier papers on 2-categories [Bén, Str] and some more recent accounts [Lei, SP].

In Section 2.1 we review the definitions of 2-categories, functors, natural transformations and modifications. In Section 2.2 we give the definition of an equivalence of 2-categories, and give an equivalent characterization. Section 2.3 is devoted to pasting diagrams, which are a way of representing 2-morphisms in 2-categories. We end the chapter with Section 2.4, where we give the construction of the classifying space of a 2-category and establish some of its properties.

2.1 Basic definitions and conventions

Definition 2.1. A *2-category* $\mathcal{C}$ consists of the following data:

1. A class of objects $Ob(\mathcal{C})$;
2. for any $A, B \in Ob(\mathcal{C})$, a category $\mathcal{C}(A, B)$. The objects of this category are called *1-morphisms* and are denoted by $1Hom_{\mathcal{C}}(A, B)$. The morphisms of the category are called *2-morphisms*, and given 1-morphisms f, g, the space of 2-morphisms between

them is denoted by $2Hom_{\mathcal{C}}(f,g)$.

The composition in the category $\mathcal{C}(A,B)$ will be denoted by $\circ$.

3. for any objects A, B, C, a composition functor

$$\mathcal{C}(B,C) \times \mathcal{C}(A,B) \xrightarrow{*} \mathcal{C}(A,C);$$

4. for all objects A, B, C, D, a natural associativity isomorphism

$$\begin{tikzcd}
\mathcal{C}(C,D) \times \mathcal{C}(B,C) \times \mathcal{C}(A,B) \arrow[r, "1\times *"] \arrow[d, "*\times 1"'] \arrow[dr, phantom, "\Leftarrow\alpha"] & \mathcal{C}(C,D) \times \mathcal{C}(A,C) \arrow[d, "*"] \\
\mathcal{C}(B,D) \times \mathcal{C}(A,B) \arrow[r, "*"'] & \mathcal{C}(A,D),
\end{tikzcd}$$

that gives 2-isomorphisms

$$\alpha_{h,g,f} : h * (g * f) \Rightarrow (h * g) * f;$$

5. for any object A of $\mathcal{C}$, a 1-automorphism I_A, called the identity, and natural isomorphisms

$$\begin{tikzcd}
\mathcal{C}(A,B) \times \mathbf{1} \arrow[d, "1\times I_A"'] \arrow[dr, "\simeq"] & \\
\mathcal{C}(A,B) \times \mathcal{C}(A,A) \arrow[r, "*"'] \arrow[ur, phantom, "\rho\Uparrow"] & \mathcal{C}(A,B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C}(A,B) \times \mathbf{1} \arrow[d, "I_B\times 1"'] \arrow[dr, "\simeq"] & \\
\mathcal{C}(B,B) \times \mathcal{C}(A,B) \arrow[r, "*"'] \arrow[ur, phantom, "\lambda\Uparrow"] & \mathcal{C}(A,B)
\end{tikzcd}$$

These transformations are given by 2-isomorphisms

$$\rho_f : f * I_A \Rightarrow f \quad \text{and} \quad \lambda_f : I_B * f \Rightarrow f.$$

For all composable 1-morphisms f, g, h, k, the following diagrams must commute:

$$\begin{tikzcd} k*(h*(g*f)) \arrow[rr, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "1*\alpha"'] & & (k*h)*(g*f) \arrow[d, Rightarrow, "\alpha"] \\ k*((h*g)*f) \arrow[r, Rightarrow, "\alpha"'] & (k*(h*g))*f \arrow[r, Rightarrow, "\alpha*1"'] & ((k*h)*g)*f \end{tikzcd}$$

$$\begin{tikzcd} g*(I_B*f) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "1*\lambda"'] & & (g*I_B)*f \arrow[dl, Rightarrow, "\rho*1"] \\ & g*f & \end{tikzcd}$$

If the natural transformations α, ρ, λ are the identity, then we say the 2-category is *strict.*

Example 2.2. Let <u>*Cat*</u> be the 2-category of small categories, functors and natural transformations. This 2-category is strict.

Example 2.3. A monoidal category $\mathcal{M}$ can be viewed as a 2-category $\Sigma\mathcal{M}$ with one object $*$, and the category of morphisms

$$\Sigma\mathcal{M}(*,*) = \mathcal{M},$$

with the composition given by the monoidal product.

Theorem 2.4 (Coherence Theorem for 2-categories, [MLP])**.** *Given a string of n composable 1-morphisms f_i and two bracketings b, b' (which might also insert identity 1-morphisms), there is a unique 2-isomorphism $\sigma : b(f_i) \Rightarrow b'(f_i)$ that is a composition of instances of α, ρ, λ.*

This theorem shows that the axioms required on the definition of 2-category imply the commutativity of any other diagram whose vertices are given by different bracketings of the same string of composable 1-morphisms and whose edges are given by instances of α, ρ, λ.

Let $f : A \to B$ be a 1-morphism in $\mathcal{C}$, we can define functors

$$f^* : \mathcal{C}(B, C) \to \mathcal{C}(A, C),$$

$$f_* : \mathcal{C}(C, A) \to \mathcal{C}(C, B)$$

given by pre- and post-composition with f.

Definition 2.5. Let $\mathcal{C}$ be a 2-category. A *1-equivalence* (or *internal equivalence*) is a 1-morphism $f : A \to B$ such that there exists a 1-morphism $g : B \to A$ and 2-isomorphisms $\eta : I_A \Rightarrow g * f$ and $\epsilon : f * g \Rightarrow I_B$. We say that the objects A and B are *equivalent.*

Definition 2.6. Let $\mathcal{C}, \mathcal{D}$ be 2-categories. A *functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ consists of the following data:

1. An assignment of objects $\mathcal{F} : Ob(\mathcal{C}) \to Ob(\mathcal{D})$;

2. for all $A, B \in Ob(\mathcal{C})$, a functor

$$\mathcal{F}_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(\mathcal{F}A, \mathcal{F}B);$$

3. for all objects A, B, C, a natural isomorphism

$$\begin{tikzcd} \mathcal{C}(B,C) \times \mathcal{C}(A,B) \arrow[r, "\mathcal{F}_{B,C} \times \mathcal{F}_{A,B}"] \arrow[d, "*"'] \arrow[dr, phantom, "\Swarrow \mathcal{F}^2_{A,B,C}"] & \mathcal{D}(\mathcal{F}B, \mathcal{F}C) \times \mathcal{D}(\mathcal{F}A, \mathcal{F}B) \arrow[d, "*"] \\ \mathcal{C}(A,C) \arrow[r, "\mathcal{F}_{A,C}"'] & \mathcal{D}(\mathcal{F}A, \mathcal{F}C), \end{tikzcd}$$

that is represented by 2-isomorphisms

$$\mathcal{F}^2(g, f) : \mathcal{F}(g) * \mathcal{F}(f) \Rightarrow \mathcal{F}(g * f);$$

4. for every object A, a natural isomorphism

$$\begin{tikzcd} \mathbf{1} \arrow[d, "I_A"'] \arrow[dr, "I_{\mathcal{F}A}"] & \\ \mathcal{C}(A,A) \arrow[r, "\mathcal{F}_{A,A}"'] \arrow[ur, phantom, "\Swarrow \mathcal{F}^0_A"] & \mathcal{D}(\mathcal{F}A, \mathcal{F}A), \end{tikzcd}$$

that gives a 2-isomorphism

$$\mathcal{F}^0 : I_{\mathcal{F}A} \Rightarrow \mathcal{F}(I_A).$$

The data above must satisfy the commutativity of the following diagrams for composable 1-morphisms f, g, h:

$$
\begin{tikzcd}
\mathcal{F}h * (\mathcal{F}g * \mathcal{F}f) \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "1*\mathcal{F}^2"'] & (\mathcal{F}h * \mathcal{F}g) * \mathcal{F}f \arrow[d, Rightarrow, "\mathcal{F}^2*1"] \\
\mathcal{F}h * \mathcal{F}(g * f) \arrow[d, Rightarrow, "\mathcal{F}^2"'] & \mathcal{F}(h * g) * \mathcal{F}f \arrow[d, Rightarrow, "\mathcal{F}^2"] \\
\mathcal{F}(h * (g * h)) \arrow[r, Rightarrow, "\mathcal{F}(\alpha)"'] & \mathcal{F}((h * g) * f);
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{F}f * I_{\mathcal{F}A} \arrow[r, Rightarrow, "1*\mathcal{F}^0"] \arrow[d, Rightarrow, "\rho"'] & \mathcal{F}f * \mathcal{F}I_A \arrow[d, Rightarrow, "\mathcal{F}^2"] \\
\mathcal{F}f & \mathcal{F}(f * I_A) \arrow[l, Rightarrow, "\mathcal{F}\rho"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I_{\mathcal{F}B} * \mathcal{F}f \arrow[r, Rightarrow, "\mathcal{F}^0*1"] \arrow[d, Rightarrow, "\lambda"'] & \mathcal{F}I_B * \mathcal{F}f \arrow[d, Rightarrow, "\mathcal{F}^2"] \\
\mathcal{F}f & \mathcal{F}(I_B * f). \arrow[l, Rightarrow, "\mathcal{F}\lambda"]
\end{tikzcd}
$$

When the natural isomorphisms $\mathcal{F}^2$ and $\mathcal{F}^0$ are identities we say the functor is *strict*. If only $\mathcal{F}^0$ is the identity, then the functor is called *normal*.

Remark 2.7. There does not seem to be a standard convention for the terminology of these different morphisms between 2-categories. Other authors use the terminology "functor" and "pseudo-functor" to refer to what we have called here "strict functor" and "functor", respectively. There is also the notion of *lax functor*, where one does not require $\mathcal{F}^2$ and $\mathcal{F}^0$ to be invertible.

Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ be functors. We can define the composition $\mathcal{GF}$ as usual composition of functions at the level of objects and usual composition of functors at the level of categories of morphisms. The natural isomorphisms $(\mathcal{GF})^2$ and $(\mathcal{GF})^0$ are given on 1-morphisms by the following compositions, respectively:

$$\mathcal{GF}g * \mathcal{GF}f \xRightarrow{\mathcal{G}^2_{\mathcal{F}g,\mathcal{F}f}} \mathcal{G}(\mathcal{F}g * \mathcal{F}f) \xRightarrow{\mathcal{G}(\mathcal{F}^2_{g,f})} \mathcal{GF}(g * f);$$

$$I_{\mathcal{GF}A} \xRightarrow{\mathcal{G}^0_{\mathcal{F}A}} \mathcal{G}(I_{\mathcal{F}A}) \xRightarrow{\mathcal{G}(\mathcal{F}^0_A)} \mathcal{GF}(I_A).$$

We will denote by $2Cat$, $2Cat^u$ the categories whose objects are small 2-categories and whose morphisms are functors, normal functors, respectively.

Definition 2.8. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between 2-categories. A *natural transformation* $\eta : \mathcal{F} \to \mathcal{G}$ consists of the following data:

1. For every $A \in Ob(\mathcal{C})$, a 1-morphism

$$\eta_A : \mathcal{F}A \to \mathcal{G}A;$$

2. for every pair A, B, a natural isomorphism

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[r, "\mathcal{G}_{A,B}"] \arrow[d, "\mathcal{F}_{A,B}"'] \arrow[dr, phantom, "\swArrow\ \eta^2_{A,B}"] & \mathcal{D}(\mathcal{G}A, \mathcal{G}B) \arrow[d, "(\eta_A)^*"] \\
\mathcal{D}(\mathcal{F}A, \mathcal{F}B) \arrow[r, "(\eta_B)_*"'] & \mathcal{D}(\mathcal{F}A, \mathcal{G}B),
\end{tikzcd}$$

given by 2-isomorphisms:

$$\eta^2_f : \mathcal{G}f * \eta_A \Rightarrow \eta_B * \mathcal{F}f.$$

For all $A \xrightarrow{f} B \xrightarrow{g} C$, the following diagrams must commute:

$$\begin{tikzcd}
\mathcal{G}g * (\mathcal{G}f * \eta_A) \arrow[r, Rightarrow, "1*\eta^2_f"] \arrow[d, Rightarrow, "\alpha"'] & \mathcal{G}g * (\eta_B * \mathcal{F}f) \arrow[d, Rightarrow, "\alpha"] \\
(\mathcal{G}g * \mathcal{G}f) * \eta_A \arrow[d, Rightarrow, "\mathcal{G}^2 * 1"'] & (\mathcal{G}g * \eta_B) * \mathcal{F}f \arrow[d, Rightarrow, "\eta^2_g * 1"] \\
\mathcal{G}(g * f) * \eta_A \arrow[d, Rightarrow, "\eta^2_{g*f}"'] & (\eta_C * \mathcal{F}g) * \mathcal{F}f \arrow[d, Rightarrow, "\alpha^{-1}"] \\
\eta_C * \mathcal{F}(g * f) & \eta_C * (\mathcal{F}g * \mathcal{F}f) \arrow[l, Rightarrow, "1*\mathcal{F}^2"]
\end{tikzcd} \tag{2.9}$$

$$\begin{tikzcd}
I_{\mathcal{G}A} * \eta_A \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\mathcal{G}^0 * 1"'] & \eta_A \arrow[r, Rightarrow, "\rho^{-1}"] & \eta_A * I_{\mathcal{F}A} \arrow[d, Rightarrow, "1*\mathcal{F}^0"] \\
\mathcal{G}(I_A) * \eta_A \arrow[rr, Rightarrow, "\eta^2_{I_A}"'] & & \eta_A * \mathcal{F}(I_A)
\end{tikzcd} \tag{2.10}$$

Natural transformations can be composed. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{C} \to \mathcal{D}$ be functors between 2-categories, and $\eta : \mathcal{F} \to \mathcal{G}$, $\epsilon : \mathcal{G} \to \mathcal{H}$ be natural transformations. There is a natural transformation $\epsilon \circ \eta : \mathcal{F} \to \mathcal{H}$ with

$$(\epsilon \circ \eta)_A = \epsilon_A * \eta_A,$$

and the natural transformation $(\epsilon \circ \eta)^2$ given by the composition

$$
\begin{tikzcd}
\mathcal{H}f * (\epsilon_A * \eta_A) \arrow[r, Rightarrow, "\alpha"] & (\mathcal{H}f * \epsilon_A) * \eta_A \arrow[r, Rightarrow, "\epsilon^2 * 1"] & (\epsilon_B * \mathcal{G}f) * \eta_A \arrow[dl, Rightarrow, "\alpha^{-1}"'] & \\
& \epsilon_B * (\mathcal{G}f * \eta_A) \arrow[r, Rightarrow, "1*\eta^2"'] & \epsilon_B * (\eta_B * \mathcal{F}f) \arrow[r, Rightarrow, "\alpha"'] & (\epsilon_B * \eta_B) * \mathcal{F}f.
\end{tikzcd}
$$

Definition 2.11. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ be functors between 2-categories, and $\eta, \epsilon : \mathcal{F} \to \mathcal{G}$ natural transformations. A *modification* $\Gamma : \eta \to \epsilon$ consists of a 2-morphism $\Gamma_A : \eta_A \Rightarrow \epsilon_A$ for every $A \in Ob(\mathcal{C})$, such that for all $F : A \to B$, the diagram below commutes

$$
\begin{tikzcd}
\mathcal{G}f * \eta_A \arrow[r, Rightarrow, "\eta^2"] \arrow[d, Rightarrow, "1*\Gamma_A"'] & \eta_B * \mathcal{F}f \arrow[d, Rightarrow, "\Gamma_B * 1"] \\
\mathcal{G}f * \epsilon_A \arrow[r, Rightarrow, "\epsilon^2"] & \epsilon_B * \mathcal{F}f.
\end{tikzcd}
$$

We can compose modifications by composing the corresponding 2-morphisms. We will denote this composition by juxtaposition.

Remark 2.12. A modification $\Gamma : \eta \to \epsilon$ is called *invertible* if there exists a modification $\Gamma^{-1} : \epsilon \to \eta$ such that $\Gamma^{-1}\Gamma = id_\eta$. Note that this also implies that $\Gamma\Gamma^{-1} = id_\epsilon$. It is straightforward to prove that Γ is invertible if an only if the 2-morphism Γ_A is invertible for all A. The transformations η and ϵ are then said to be *isomorphic*.

Definition 2.13. A natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is called a *natural equivalence* if there exists a natural transformation $\epsilon : \mathcal{G} \to \mathcal{F}$ such that

$$id_{\mathcal{F}} \cong \epsilon \circ \eta \quad \text{and} \quad id_{\mathcal{G}} \cong \eta \circ \epsilon.$$

The functors $\mathcal{F}$ and $\mathcal{G}$ are then called *equivalent*.

Proposition 2.14. *The natural transformation η is an equivalence if and only if for all $A \in Ob(\mathcal{C})$, the 1-morphism η_A is a 1-equivalence.*

2.2 Equivalence of 2-categories

Definition 2.15. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an *equivalence of 2-categories* if there exists a functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ and natural equivalences

$$id_{\mathcal{C}} \simeq \mathcal{G} \circ \mathcal{F} \quad \text{and} \quad id_{\mathcal{D}} \simeq \mathcal{G} \circ \mathcal{F}.$$

The 2-categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent.*

Remark 2.16. Some authors use the term *biequivalence* for the definition above.

Theorem 2.17. *A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an equivalence of 2-categories if and only if*

1. *F is essentially surjective on objects, that is, every $D \in Ob(\mathcal{D})$ is equivalent to $\mathcal{F}C$ for some $C \in Ob(\mathcal{C})$;*

2. *for all objects A, B in $\mathcal{C}$, the functor $\mathcal{F}_{A,B} : \mathcal{C}(A, CB) \to \mathcal{D}(\mathcal{F}A, \mathcal{F}B)$ is an equivalence of categories.*

2.3 Pasting diagrams

Since 2-categories inherently have a 2-dimensional structure, we use 2-dimensional diagrams to represent them.

Let $\mathcal{C}$ be a 2-category, A, B objects, $f, f' : A \to B$ 1-morphisms and $\varphi : f \Rightarrow f'$ a 2-morphism. We depict this 2-morphism as

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\varphi"] & B. \end{tikzcd}$$

Horizontal and vertical composition are, respectively represented by the pasting diagrams

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\psi"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, bend left=30, "\Downarrow\varphi"] \arrow[r, phantom, bend right=30, "\Downarrow\varphi'"] & B. \end{tikzcd}$$

The functoriality of composition implies the interchange law:

$$(\psi' \circ \psi) * (\varphi' \circ \varphi) = (\psi' * \varphi') \circ (\psi * \varphi).$$

This law gives then a unique meaning to the diagram

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\varphi"] \arrow[r, phantom, shift right=3, "\Downarrow\varphi'"] & B \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow\psi"] \arrow[r, phantom, shift right=3, "\Downarrow\psi'"] & C. \end{tikzcd}$$

In particular, the law implies that we can read the diagram first vertically and then horizontally, or vice-versa.

A pasting diagram is a polygonal arrangement on the plane, that is a generalization of the diagrams shown above. The vertices correspond to objects, the directed edges correspond to 1-morphisms and the faces are usually filled with double arrows corresponding to 2-morphisms. For example, the diagram

$$\begin{tikzcd} & {} \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\sigma"] & \\ {} \arrow[ur, "f"] \arrow[rr, "h"'] & {} & {} \end{tikzcd}$$

indicates that σ is a 2-morphism from $g * f$ to h.

The pasting

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow id_f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\sigma"] & C \end{tikzcd}$$

will be represented by

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\sigma"] & C \end{tikzcd}$$

and denoted $\sigma * f$. We use a similar convention for post-composition.

The two pasting operations described above can be combined to give the general notion of pasting. All instances of pasting diagrams can be obtained from the following two:

$$\begin{tikzcd} {} \arrow[rr, "f"] \arrow[dr, "g"'] & {} \arrow[d, phantom, "\Downarrow\varphi"] & {} \arrow[dr, "k"] \arrow[d, phantom, "\Downarrow\psi"] & \\ & {} \arrow[ur, "h"] \arrow[rr, "l"'] & {} & {} \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[rr, "l"] \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\sigma"] & {} \arrow[d, phantom, "\Downarrow\tau"] & {} \\ {} \arrow[ur, "f"] \arrow[rr, "h"'] & {} & {} \arrow[ur, "k"'] & \end{tikzcd}$$

The first diagram indicates the 2-morphism given by the composition

$$k * f \xRightarrow{k*\varphi} k * (h * g) \xRightarrow{\alpha} (k * h) * g \xRightarrow{\psi * g} l * g,$$

while the second diagram is representing the composition

$$l * f \xRightarrow{\tau * f} (k * g) * f \xRightarrow{\alpha^{-1}} k * (g * f) \xRightarrow{k*\sigma} k * h.$$

We note here, in the simplest examples, that the 2-morphisms are not actually composable; we need to use the associativity isomorphism. We can use these basic situations to make sense of larger diagrams. In these larger diagrams we will find that the sources and targets of 2-morphisms differ by their bracketing. By the Coherence Theorem (Theorem 2.4) we know that there is a unique canonical associativity isomorphism, so we use this to make sense of the diagram.

Furthermore, a diagram does not make sense unless we specify a bracketing of the outside 1-morphisms. Once we do that, the diagram has a unique meaning, no matter what order we use to compose the 2-morphisms. This can be proved by induction using polygonal decompositions of the disk and the interchange law. For more information, we refer the reader to [KS]. As an example, below we show equation (2.9) as a pasting diagram:

$$\begin{tikzcd}
\mathcal{F}A \arrow[rr, "\eta_A"] \arrow[dr, "\mathcal{F}f"] \arrow[dd, "\mathcal{F}(g*f)"'] & & \mathcal{G}A \arrow[dr, "\mathcal{G}f"] \arrow[d, phantom, "\swarrow \eta^2_f"] & \\
\arrow[r, phantom, "\overset{\Leftarrow}{\mathcal{F}^2}"] & \mathcal{F}B \arrow[rr, "\eta_B"] \arrow[dl, "\mathcal{F}g"] & \arrow[d, phantom, "\swarrow \eta^2_g"] & \mathcal{G}B \arrow[dl, "\mathcal{G}g"] \\
\mathcal{F}C \arrow[rr, "\eta C"'] & & \mathcal{G}C &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F}A \arrow[rr, "\eta_A"] \arrow[dd, "\mathcal{F}(g*f)"'] & & \mathcal{G}A \arrow[dr, "\mathcal{G}f"] \arrow[dd, "\mathcal{G}(g*f)"] & \\
 & \swarrow \eta^2_{g*f} & \arrow[r, phantom, "\overset{\Leftarrow}{\mathcal{G}^2}"] & \mathcal{G}B \arrow[dl, "\mathcal{G}g"] \\
\mathcal{F}C \arrow[rr, "\eta C"'] & & \mathcal{G}C &
\end{tikzcd}$$

When we say "pasting diagram A is equal to pasting diagram B" we mean that with a given bracketing of the outside 1-morphisms, the given 2-morphisms that they both define are equal. Note that if this is true for a given bracketing, it is true for all bracketings.

2.4 Classifying spaces of 2-categories

Categories are closely related to spaces through the classifying space construction. To every category we can assign a space. This assignment gives a functor that is part of a Quillen equivalence between a given model structure on the category of small categories and the usual model structure in the category of topological spaces.

The same can be done with 2-categories. In fact, there are many distinct constructions of the classifying space of a 2-category ([CCG]). All these constructions give equivalent spaces, in the non-enriched case. Here we will describe the version of the nerve we will be using in subsequent chapters.

Lack and Paoli introduce a version of a nerve of 2-categories that gives rise to a simplicial object in *Cat*. This construction is closely related to the bar construction for monoidal categories defined in [BDR], as we will point out. This nerve is called *2-nerve* in [LP] and *Segal nerve* in [CCG]. We will use the notation of the latter.

Definition 2.18. Let $\mathcal{C}$ be a 2-category. The *Segal nerve* $S\mathcal{C}$ is the simplicial object in Cat given by normal functors, that is,

$$S_n\mathcal{C} = \underline{NorFunc}([n], \mathcal{C}),$$

where the objects are normal functors and the morphisms are oplax natural transformations relative to objects.

Oplax natural transformations are similar to the natural transformations described above, but instead of requiring the 2-morphism η^2 to be an invertible, we require it to be a map in the opposite direction. Being relative to objects means that the map $\eta_A : \mathcal{F}A \to \mathcal{G}A$ is the identity, so in particular, we require that $\mathcal{F}A = \mathcal{G}A$.

An object of $S_n\mathcal{C}$ is given then by a collection of diagrams

$$\begin{tikzcd} & C_j \arrow[dr, "f_{jk}"] \arrow[d, phantom, "\Downarrow\varphi_{ijk}"] & \\ C_i \arrow[ur, "f_{ij}"] \arrow[rr, "f_{ik}"'] & {} & C_k, \end{tikzcd} \qquad \text{for all } 0 \leqslant i < j < k \leqslant n, \tag{2.19}$$

where φ_{ijk} is an invertible 2-morphism. This collection must satisfy the following coherence condition for all $0 \leqslant i < j < k < l \leqslant n$:

$$
\begin{tikzcd}
 & C_j \arrow[dr] \arrow[d, phantom, "\Downarrow\varphi_{ijk}"] & \\
C_i \arrow[ur] \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\swarrow\varphi_{ikl}"] & C_k \arrow[dl] \\
 & C_l &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & C_j \arrow[dr] \arrow[dd] & \\
C_i \arrow[ur] \arrow[dr] \arrow[r, phantom, "\swarrow\varphi_{ijl}"] & \arrow[r, phantom, "\Leftarrow\varphi_{jkl}"] & C_k . \arrow[dl] \\
 & C_l &
\end{tikzcd}
\tag{2.20}
$$

Given objects $\{C_i, f_{ij}, \varphi_{ijk}\}$ and $\{C_i, f'_{ij}, \varphi'_{ijk}\}$ (note that the collections of objects are equal), a morphism between them is given by a collection of 2-morphisms $\eta_{ij} : f_{ij} \Rightarrow f'_{ij}$ for $i \leqslant j$, such that some coherence conditions ([CCG, Eq. (44)]) are satisfied.

Remark 2.21. We note that the bar construction for monoidal categories of [BDR] is equal to the Segal nerve. More precisely, if $\mathcal{M}$ is a monoidal category, then the simplicial category $B\mathcal{M}$ of [BDR] is equal to $S\Sigma\mathcal{M}$ (with a possible reordering of the indices).

The Segal nerve is functorial with respect to normal functors. It is the case that any functor can be normalized ([LP, Prop. 5.2]). More precisely, there is a functor

$$2Cat \to 2Cat^u$$

from the category of 2-categories to the category of 2-categories and normal functors that is an equivalence.

The Segal nerve is then a functor

$$S : 2Cat \to [\Delta^{op}, Cat].$$

It is important to note that this functor preserves products.

Definition 2.22. Let $\mathcal{C}$ be a 2-category. The *classifying space* of $\mathcal{C}$ is the realization $|S\mathcal{C}|$.

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ be functors, and $\eta : \mathcal{F} \to \mathcal{G}$ a natural transformation. As pointed out in the proof of [CCG, Prop. 7.1], these data gives rise to a functor

$$\mathcal{H} : \mathcal{C} \times \mathbf{1} \to \mathcal{D}$$

that restricts to $\mathcal{F}$ and $\mathcal{G}$ at 0 and 1. This functor can be normalized, yielding the following result:

Proposition 2.23. *A natural transformation between functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ gives rise to a homotopy between the maps*

$$|S\mathcal{F}|, |S\mathcal{G}| : |S\mathcal{C}| \to |S\mathcal{D}|.$$

Chapter 3

K-theory of bimonoidal categories

In [BDR], N. Baas, B. Dundas and J. Rognes introduce the notions of $2K$-theory and 2-vector bundles as a way to categorify topological K-theory and vector bundles. One of their objectives is to define a cohomology theory of a geometric nature that has chromatic level 2.

In general they define the K-theory of (symmetric) bimonoidal categories. In Section 3.1 we recall the definition of bimonoidal and symmetric bimonoidal categories, and their stricter analogues. This is again a topic in the literature where authors have used the same name to refer to different concepts, and also have given different names to the same concept. We refer the reader to [May2, Section 12] for a discussion on the different accounts on this topic. It is important to note that we will require the distributivity maps to be isomorphisms.

In Section 3.2, we define K-theory of a bimonoidal category, following [BDR]. We explain in Section 3.3 how the K-theory of a bimonoidal category $\mathcal{R}$ is related to the algebraic K-theory of the ring spectrum $K\mathcal{R}$. In Section 3.4, we identify the K-theory space of $\mathcal{R}$ with an appropriate group completion of the classifying space of the 2-category of modules over $\mathcal{R}$.

3.1 Bimonoidal and Symmetric Bimonoidal Categories

Definition 3.1. A *bimonoidal category* $(\mathcal{R}, \oplus, \otimes)$ is a category $\mathcal{R}$ endowed with a symmetric monoidal structure ($\oplus$, 0) and a monoidal structure ($\otimes$, 1), together with natural isomorphisms

$$\delta_r : (a \oplus b) \otimes c \to (a \otimes c) \oplus (b \otimes c),$$

$$\delta_l : a \otimes (b \oplus c) \to (a \otimes b) \oplus (a \otimes c),$$

$$0 \otimes a \to 0 \leftarrow a \otimes 0, \tag{3.2}$$

that are subject to certain coherence axioms.

These coherence axioms are partially spelled out in [EM, Def. 3.3], although there the isomorphisms in (3.2) are required to be identities. The axioms are also packed in the definition of **SMC**-category in [Gui]. We note that our definition of bimonoidal category coincides with that of an **SMC**-category with a single object.

In [EM] these categories are called *ring* categories, although the authors do not require δ_r and δ_l to be isomorphisms. In [BDRR1, BDRR2] the term used is *rig* categories, in order to emphasize the possible lack of negatives, that is, additive inverses.

Definition 3.3. A *strict bimonoidal category* $(\mathcal{R}, \oplus, 0, \gamma_\oplus, \otimes, 1, \delta)$ is a permutative category $(\mathcal{R}, \oplus, 0, \gamma_\oplus)$, together with a strict monoidal structure $(\otimes, 1)$, such that right distributivity and nullity of zero hold strictly, and there is a left distributivity natural isomorphism

$$\delta : a \otimes (b \oplus c) \to (a \otimes b) \oplus (a \otimes c).$$

These satisfy the coherence axioms spelled out in [Gui, Definition 3.1].

A strict bimonoidal category is the same as a **PC**-category with a single object, as defined in [Gui]. By [Gui, Thm. 1.2] we know that any bimonoidal category is equivalent to a strict bimonoidal category, via a map of bimonoidal categories. Thus, without loss of generality we can work with strict bimonoidal categories. The definition of K-theory below would also work for the non-strict case, with a few changes.

One can also require the multiplication to be commutative up to coherent isomoprhism. We call these categories *symmetric bimonoidal.* The symmetry requires extra coherence conditions relating the left and right distributivity isomorphisms. Laplaza [Lap] made a careful study of these conditions.

May [May3] introduces the notion of *bipermutative* category, which corresponds to the strict version of symmetric bimonoidal. We refer the reader to his work for the much shorter list of coherence diagrams. By [May3, Prop. 3.5] we know that any symmetric bimonoidal category is a equivalent to a bipermutative category, so again, we can work with bipermutative categories instead.

It is important to note that bipermutative categories are examples of strict bimonoidal categories.

3.2 Definition of K-theory of a bimonoidal category

Let $(\mathcal{R}, \oplus, 0, c_{\oplus}, \otimes, 1, \delta)$ be a strict bimonoidal category. Then, as in [BDR], we can define $M_n(\mathcal{R})$, the category of $n \times n$ matrices over $\mathcal{R}$. Its objects are matrices $V = (V_{i,j})_{i,j=1}^n$ whose entries are objects of $\mathcal{R}$. The morphisms are matrices $\phi = (\phi_{i,j})_{i,j=1}^n$ of isomorphisms in $\mathcal{R}$, such that the source (resp, target) of $\phi_{i,j}$ is the $(i,j)-$entry of the source (target) of ϕ. As a category, $M_n(\mathcal{R})$ is isomorphic to $\mathcal{R}^{n \times n}$.

Moreover, $M_n(\mathcal{R})$ is a monoidal category, with multiplication

$$M_n(\mathcal{R}) \times M_n(\mathcal{R}) \xrightarrow{\cdot} M_n(\mathcal{R})$$

given by sending the pair (U, V) to

$$W_{ik} = \bigoplus_{j=1}^{n} U_{ij} \otimes V_{jk}.$$

Since $\oplus$ is strictly associative, there is no ambiguity.

This multiplication has a unit object I_n, given by the matrix with 1 in the diagonal and 0 elsewhere. Both 0 and 1 are strict units for $\oplus$ and $\otimes$ respectively, and the nullity of 0 holds strictly, so I_n is a strict unit as well.

Proposition 3.4. *[BDR, 3.3] Matrix multiplication makes $(M_n(\mathcal{R}), \cdot, I_n)$ into a monoidal category.*

The natural associativity isomorphism

$$\alpha : U \cdot (V \cdot W) \to (U \cdot V) \cdot W$$

is given by entry-wise use of $c_{\oplus}$ and δ.

Recall that if R is a semi-ring, $GL_n(R)$ is the subgroup of $M_n(R)$ that contains all the matrices whose determinant is a unit in $Gr_+(R)$. The following definition is also taken from [BDR].

Definition 3.5. Let $GL_n(\mathcal{R}) \subset M_n(\mathcal{R})$ be the full subcategory of matrices $V = (V_{i,j})_{i,j=1}^n$ whose matrix of path components lies in $GL_n(\pi_0(\mathcal{R}))$. We call $GL_n(\mathcal{R})$ the category of *weakly invertible matrices.* By convention we will let $GL_0(\mathcal{R}) = \underline{1}$ be the unit category, with one object and one morphism.

Note that $GL_n(\mathcal{R})$ inherits the monoidal structure from $M_n(\mathcal{R})$.

Given a monoidal category $\mathcal{M}$, the authors in [BDR] define a bar construction for monoidal categories, $B\mathcal{M}$, which is a simplicial object in Cat. As pointed out in Remark 2.21, this definition coincides with $S\Sigma\mathcal{M}$.

We note that block sum of matrices in $\mathcal{R}$ makes

$$\coprod_{n\geqslant 0} |BGL_n(\mathcal{R})|$$

into an H-space, and hence we define the K-theory of $\mathcal{R}$, as the group completion

$$\mathcal{K}(\mathcal{R}) := \Omega B\big(\coprod_{n\geqslant 0} |BGL_n(\mathcal{R})|\big).$$

The motivation behind the definition of K-theory for bimonoidal categories comes from the categorification of complex K-theory. As we know well, the complex K-theory space classifies virtual vector bundles.

A 2-vector space, as defined in [KV], is a category equivalent to $(Vect_{\mathbb{C}})^n$ for some n. Heuristically, this should be thought of as a module category over $Vect_{\mathbb{C}}$. In [BDR], the authors introduce the notion of a complex 2-vector bundle over a topological space and construct a classifying space for these bundles. A 2-vector bundle is roughly a bundle of 2-vector spaces over X, defined in terms of transition functions, i.e., matrices of vector spaces. For the precise definition we refer the reader to [BDR, Section 2].

One of the main results in [BDR] is that the stable equivalence classes of virtual 2-vector bundles over a space X are in one-to-one correspondence with homotopy classes of maps from X to $\mathcal{K}(Vect_{\mathbb{C}})$, where $Vect_{\mathbb{C}}$ is a considered as a bipermutative category using direct sum and tensor product.

3.3 Relationship with algebraic K-theory

The relevance of the definition of K-theory of bimonoidal categories, aside from the classification of 2-vector bundles, comes from its relationship with algebraic K-theory of ring spectra.

Given a strict bimonoidal category $\mathcal{R}$, by forgetting the multiplicative structure we can construct the K-theory spectrum $K\mathcal{R}$ corresponding to the permutative category $(\mathcal{R}, \oplus)$. The results of [EM] and [May2] show that the multiplicative structure of $\mathcal{R}$ makes $K\mathcal{R}$ into a ring spectrum, and furthermore, if $\mathcal{R}$ is bipermutative, $K\mathcal{R}$ is an E_∞ ring spectrum. Note that in [BDR, BDRR2] the authors denote $K\mathcal{R}$ by $H\mathcal{R}$, pointing out the analogy to the Eilenberg-MacLane spectrum of a ring.

The natural inclusion $B\mathcal{R} \to K\mathcal{R}_0$ extends to a map

$$\mathcal{K}(\mathcal{R}) \xrightarrow{\sim} K(K\mathcal{R}), \tag{3.6}$$

where the left-hand side corresponds to the algebraic K-theory of the ring spectrum $K\mathcal{R}$.

The main result of [BDRR2] is that the map in equation (3.6) is an equivalence of spaces.

3.4 K-theory as a classifying space of a 2-category

Definition 3.7. Let $Mod_{\mathcal{R}}$ be the 2-category of finite dimensional free modules over $\mathcal{R}$, defined as follows. The objects are labeled by the natural numbers $\mathbf{n} \geqslant 0$. Given objects $\mathbf{n}, \mathbf{m}$, the category of morphisms is

$$Mod_{\mathcal{R}}(\mathbf{n}, \mathbf{m}) = \begin{cases} GL_n(\mathcal{R}) & \text{if} \quad n = m \\ \varnothing & \text{if} \quad n \neq m. \end{cases}$$

and the composition is given by matrix multiplication. In other words,

$$Mod_{\mathcal{R}} = \coprod_{n \geqslant 0} \Sigma GL_n(\mathcal{R}).$$

Example 3.8. Let $Vect_k$ be the bipermutative category of vector spaces over the field k. Then Mod_{Vect_k} is a sub-2-category of the 2-category of 2-vector spaces defined by Kapranov and Voevodsky [KV]. The 1-morphisms are matrices of vector spaces such that their matrix of dimensions has determinant ± 1.

We can use the 2-category $Mod_{\mathcal{R}}$ to give an alternative definition of the K-theory of $\mathcal{R}$. We have that

$$\coprod_{n\geqslant 0} S\Sigma GL_n(\mathcal{R}) = S\big(\coprod_{n\geqslant 0} \Sigma GL_n(\mathcal{R})\big) = SMod_{\mathcal{R}}.$$

Hence, we can describe K-theory as $\Omega B|SMod_{\mathcal{R}}|$.

This is the definition we will use in the following chapters. Furthermore, we will show that the H-space structure comes from a functor

$$Mod_{\mathcal{R}} \times Mod_{\mathcal{R}} \to Mod_{\mathcal{R}},$$

which will give $Mod_{\mathcal{R}}$ the structure of a symmetric monoidal 2-category.

Chapter 4

Symmetric monoidal structure on $Mod_{\mathcal{R}}$

In this chapter we will show that the 2-category $Mod_{\mathcal{R}}$ is symmetric monoidal. In Section 4.1 we give an overview of symmetric monoidal 2-categories. In Section 4.2 we show that we can provide $Mod_{\mathcal{R}}$ with a symmetric monoidal structure.

4.1 Symmetric monoidal 2-categories

In broad terms, a symmetric monoidal 2-category is a 2-category with a product functor that is associative, unitary, and commutative up to coherent natural equivalences. The precise definition is quite involved, as one would imagine. To arrive to the current definition with the appropriate axioms took several papers by several mathematicians. The main complication comes from finding the correct minimal sets of coherence axioms. To be more precise, the natural equivalences satisfy coherence diagrams only up to invertible modification, and those modifications in turn have to be coherent.

In his Ph.D. dissertation [SP], Schommer-Pries gives a historical account and a concise definition of symmetric monoidal 2-categories. The symmetric monoidal structure we will place on $Mod_{\mathcal{R}}$ will be of a strict nature, in the sense that most of the natural equivalences and invertible modifications will be identities. We will thus refrain from giving the full definition and refer the reader to [SP, Chapter 3]. In what follows we will use the notation there to identify the symmetric monoidal structure on $Mod_{\mathcal{R}}$.

4.2 Structure on $Mod_{\mathcal{R}}$

Theorem 4.1. *The 2-category $Mod_{\mathcal{R}}$ is symmetric monoidal with the monoidal operation given by block sum of matrices:*

$$\begin{aligned}\boxplus : Mod_{\mathcal{R}} \times Mod_{\mathcal{R}} &\to Mod_{\mathcal{R}}\\ (\boldsymbol{n}, \boldsymbol{m}) &\mapsto \boldsymbol{n}{+}\boldsymbol{m}\\ (U, V) &\mapsto \left[\begin{array}{c|c} U & 0 \\ \hline 0 & V \end{array}\right]\\ (\varphi, \psi) &\mapsto \left[\begin{array}{c|c} \varphi & 0 \\ \hline 0 & \psi \end{array}\right].\end{aligned}$$

The matrix $[0]$ is the matrix with all entries equal to 0, the unit of $\oplus$ in $\mathcal{R}$.

Proof. We first note that the operation described above gives a strict functor of 2-categories, since it preserves the identity and the composition:

$$I_n \boxplus I_m = I_{n+m},$$

$$\left[\begin{array}{c|c} U' & 0 \\ \hline 0 & V' \end{array}\right] * \left[\begin{array}{c|c} U & 0 \\ \hline 0 & V \end{array}\right] = \left[\begin{array}{c|c} U' * U & 0 \\ \hline 0 & V' * V \end{array}\right].$$

The second equation holds because of the strict nullity and unity of 0 in $\mathcal{R}$.

The unit of $\boxplus$ is $\mathbf{0}$. The natural equivalences α, l, and r of [SP] can be taken to be the identity since for $U \in GL_n(\mathcal{R})$, $V \in GL_m(\mathcal{R})$, and $W \in GL_p(\mathcal{R})$:

$$(U \boxplus V) \boxplus W = U \boxplus (V \boxplus W),$$

$$I_0 \boxplus U = U = U \boxplus I_0.$$

The modifications π, μ, λ, ρ of [SP] are the identity modification. We can thus say that the monoidal structure is strict.

The natural equivalence $\beta_{n,m} : \mathbf{n} \boxplus \mathbf{m} \to \mathbf{m} \boxplus \mathbf{n}$ is given by the block matrix

$$\left[\begin{array}{c|c} 0 & I_m \\ \hline I_n & 0 \end{array}\right].$$

Since 0 and 1 are strict units in $\mathcal{R}$, for $U \in GL_n(\mathcal{R})$ and $V \in GL_m(\mathcal{R})$,

$$\beta_{n,m} * (U \boxplus V) = \left[\begin{array}{c|c} 0 & V \\ \hline U & 0 \end{array} \right] = (V \boxplus U) * \beta_{n,m},$$

so β is a strict natural transformation.

We note that $\beta_{m,n} * \beta_{n,m} = I_{n+m}$ which both implies that β is a natural isomorphism and that we can take the modification σ to be the identity. Similar arguments can be used to show that the modifications R and S can also be taken to be the identity.

Since all the modifications are the identity, all the coherence diagrams are satisfied. We conclude that $Mod_{\mathcal{R}}$ is a symmetric monoidal 2-category. $\square$

Chapter 5

Infinite loop space structure on $\mathcal{K}(\mathcal{R})$

In this chapter we will show that the symmetric monoidal structure on $Mod_{\mathcal{R}}$ endows $\mathcal{K}(\mathcal{R})$ with an infinite loop space structure. To this end, we will use Segal's Γ-space machinery which we will explain in Section 5.1. In Section 5.2 we show that the symmetric monoidal structure on $Mod_{\mathcal{R}}$ gives rise to a Γ-(2-category), thus giving an infinite delooping of the space $\mathcal{K}(\mathcal{R})$.

5.1 Γ-spaces

There is a close connection between cohomology theories and symmetric monoidal categories. This relationship can be made precise thanks to the work of G. Segal [Seg], J. P. May [May1], and others. In [Seg], it is shown that a symmetric monoidal category $\mathcal{C}$ gives rise to a connective spectrum, whose zero space is the group completion of the classifying space of $\mathcal{C}$. This in turn implies that any symmetric monoidal category gives rise to a generalized cohomology theory. As examples of this, we have

- the category of finite sets under disjoint union, which gives the sphere spectrum S, which in turn represents stable cohomotopy;
- the category of finite dimensional complex vector spaces under direct sum, which gives the connective K-theory spectrum ku and represents (connective) K-theory

- for any ring R, the category of finite rank projective modules over R under direct sum which gives the algebraic K theory spectrum $K(R)$.

In order to produce spectra from symmetric monoidal categories, Segal constructs an infinite loop space machine using Γ-spaces.

Let Fin_* denote that (skeletal) category of finite pointed sets and pointed maps. The skeletal version has as objects the sets $\underline{n} = \{0, 1, \ldots, n\}$, for $n \geqslant 0$. Here 0 is the basepoint.

For $1 \leqslant k \leqslant n$, we define $i_k : \underline{n} \to \underline{1}$ as:

$$i_k(j) = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k. \end{cases}$$

Definition 5.1. A Γ-*space* X is a functor $X : Fin_* \to Top$. We say X is *special* if the map

$$P_n : X(\underline{n}) \to X(\underline{1})^{\times n},$$

obtained by assembling the maps i_k, is a weak equivalence for all $n \geqslant 0$.

The conditions in the definition above roughly imply that the space $X(\underline{1})$ has a multiplication that is associative and commutative up to coherent higher homotopies. The precise statement in given by the following theorem:

Theorem 5.2. *[Seg, Prop. 1.4] Let X be a special Γ-space. Then $X(\underline{1})$ is an H-space and its group completion, $\Omega BX(\underline{1})$ is an infinite loop space.*

Segal and May show how to construct a Γ-category from a symmetric monoidal category, thus getting an infinite delooping of the classifying space of a symmetric monoidal category. We will imitate this approach in the context of 2-categories.

5.2 Γ-(2-category) structure on $Mod_{\mathcal{R}}$

Definition 5.3. A Γ-*(2-category)* $\mathcal{A}$ is a functor $\mathcal{A} : Fin_* \to 2Cat$. We say A is *special* if the map

$$P_n : \mathcal{A}(\underline{n}) \to \mathcal{A}(\underline{1})^{\times n}$$

is an equivalence of 2-categories for all $n \geqslant 0$.

This definition is analogous to that of a special Γ-space, with the connection made clear by Lemma 5.5.

The main theorem of this chapter will give us the infinite delooping of $\mathcal{K}(\mathcal{R})$.

Theorem 5.4. *The symmetric monoidal structure on $Mod_{\mathcal{R}}$ gives rise to a special Γ-(2-category) $\widehat{Mod_{\mathcal{R}}}$ such that*

$$\widehat{Mod_{\mathcal{R}}}(\underline{1}) \cong Mod_{\mathcal{R}}.$$

Lemma 5.5. *Let $\mathcal{A}$ be a special Γ-(2-category). Then $|S\mathcal{A}| : \Gamma \to Top$ is a special Γ-space.*

Proof. The classifying space functor $|S(-)| : 2Cat \to Top$ preserves products and sends equivalences of 2-categories to homotopy equivalences of spaces (Proposition 2.23). $\square$

This lemma, together with Theorem 5.4 and Theorem 5.2 give us the main theorem of this chapter.

Theorem 5.6. *The K-theory of the bimonoidal category $\mathcal{R}$ inherits an infinite loop structure from the symmetric monoidal structure on $Mod_{\mathcal{R}}$.*

Proof of Theorem 5.4. We will first construct the 2-category $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ for $n \geqslant 0$ as follows:

1. Objects are of the form $\{A_S, a_{S,T}\}_{S,T}$, where S runs over all the subsets of $\underline{n}$ that do not contain the basepoint 0; (S,T) runs over all pairs of such subsets such that $S \cap T = \varnothing$; $A_S \in Ob(Mod_{\mathcal{R}})$ and $a_{S,T} : A_{S\cup T} \to A_S \boxplus A_T$ is a 1-equivalence. We require further

   (a) $A_{\varnothing} = 0$;

   (b) $a_{\varnothing,S} = I_{A_S} = a_{S,\varnothing}$;

   (c) for every triple (S,T,U) of subsets such that $S \cap T = S \cap U = T \cap U = \varnothing$, the diagram

$$\begin{tikzcd} A_{S\cup T\cup U} \arrow[r, "a_{S,T\cup U}"] \arrow[d, "a_{S\cup T,U}"'] & A_S \boxplus A_{T\cup U} \arrow[d, "I_{A_S}\boxplus a_{T,U}"] \\ A_{S\cup T} \boxplus A_U \arrow[r, "a_{S,T}\boxplus I_{A_U}"'] & A_S \boxplus A_T \boxplus A_U \end{tikzcd} \tag{5.7}$$

   strictly commutes;

(d) for every pair of subsets (S,T), the diagram

$$\begin{tikzcd} A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, equal] & A_S \boxplus A_T \arrow[d, "\beta_{A_S,A_T}"] \\ A_{T\cup S} \arrow[r, "a_{T,S}"'] & A_T \boxplus A_S \end{tikzcd} \tag{5.8}$$

strictly commutes.

2. A 1-morphism between $\{A_S, a_{S,T}\}$ and $\{A'_S, a'_{S,T}\}$ is given by a system $\{f_S, \phi_{S,T}\}_{S,T}$, where S,T are as above; $f_S : A_S \to A'_S$ is a 1-morphism in $Mod_{\mathcal{R}}$ and $\phi_{S,T}$ is a 2-isomorphism:

$$\begin{tikzcd} A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, "f_{S\cup T}"'] \arrow[dr, phantom, "\Swarrow\phi_{S,T}"] & A_S \boxplus A_T \arrow[d, "f_S\boxplus f_T"] \\ A'_{S\cup T} \arrow[r, "a'_{S,T}"'] & A'_S \boxplus A'_T. \end{tikzcd}$$

We require:

(a) $\phi_{\varnothing,S} : f_S * I_{A_S} = f_S \Rightarrow f_S = I_{A_S} * f_S$ is the identity 2-morphism and similarly for $\phi_{S,\varnothing}$;

(b) for every pairwise disjoint S,T,U the following equation holds

$$\begin{tikzcd} A_{S\cup T\cup U} \arrow[r, "a_{S,T\cup U}"] \arrow[d, "f_{S\cup T\cup U}"'] \arrow[dr, phantom, "\Swarrow\phi_{S,T\cup U}"] & A_S \boxplus A_{T\cup U} \arrow[r, "I_{A_S}\boxplus a_{T,U}"] \arrow[d, "f_S\boxplus f_{T\cup U}"] \arrow[dr, phantom, "\Swarrow \mathrm{id}\boxplus\phi_{T,U}"] & A_S \boxplus A_T \boxplus A_U \arrow[d, "f_S\boxplus f_T\boxplus f_U"] \\ A'_{S\cup T\cup U} \arrow[r, "a'_{S,T\cup U}"'] & A'_S \boxplus A'_{T\cup U} \arrow[r, "I_{A'_S}\boxplus a'_{T,U}"'] & A'_S \boxplus A'_T \boxplus A'_U \end{tikzcd} \tag{5.9}$$

$$\|$$

$$\begin{tikzcd} A_{S\cup T\cup U} \arrow[r, "a_{S\cup T,U}"] \arrow[d, "f_{S\cup T\cup U}"'] \arrow[dr, phantom, "\Swarrow\phi_{S\cup T,U}"] & A_{S\cup T} \boxplus A_U \arrow[r, "a_{S,T}\boxplus I_{A_U}"] \arrow[d, "f_{S\cup T}\boxplus f_U"] \arrow[dr, phantom, "\Swarrow \phi_{S,T}\boxplus\mathrm{id}"] & A_S \boxplus A_T \boxplus A_U \arrow[d, "f_S\boxplus f_T\boxplus f_U"] \\ A'_{S\cup T\cup U} \arrow[r, "a'_{S\cup T,U}"'] & A'_{S\cup T} \boxplus A'_U \arrow[r, "a'_{S,T}\boxplus I_{A'_U}"'] & A'_S \boxplus A'_T \boxplus A'_U; \end{tikzcd}$$

(c) for every S, T the following equation holds:

$$\begin{tikzcd}
A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, "f_{S\cup T}"'] \arrow[dr, phantom, "\swArrow\phi_{S,T}"] & A_S \boxplus A_T \arrow[r, "\beta"] \arrow[d, "f_S\boxplus f_T"] & A_T \boxplus A_S \arrow[d, "f_T\boxplus f_S"] & = & A_{T\cup S} \arrow[r, "a_{T,S}"] \arrow[d, "f_{T\cup S}"'] \arrow[dr, phantom, "\swArrow\phi_{T,S}"] & A_T \boxplus A_S \arrow[d, "f_T\boxplus f_S"] \\
A'_{S\cup T} \arrow[r, "a'_{S,T}"'] & A'_S \boxplus A'_T \arrow[r, "\beta"'] & A'_T \boxplus A'_S & & A'_{T\cup S} \arrow[r, "a'_{T,S}"'] & A'_T \boxplus A'_S.
\end{tikzcd} \tag{5.10}$$

3. Given 1-morphisms $\{f_S, \phi_{S,T}\}, \{g_S, \gamma_{S,T}\} : \{A_S, a_{S,T}\} \to \{A'_S, a'_{S,T}\}$, a 2-morphism between them is given by a system $\{\psi_S\}$ of 2-morphisms in $Mod_{\mathcal{R}}$, $\psi_S : f_S \Rightarrow g_S$, such that for all S, T as above the following equation holds:

$$\begin{tikzcd}
A_{S\cup T} \arrow[rr, "a_{S,T}"] \arrow[d, bend right=50, "g_{S\cup T}"'] \arrow[d, bend left=50, "f_{S\cup T}"] \arrow[d, phantom, "\overset{\Leftarrow}{\psi_{S\cup T}}"] & \arrow[d, phantom, "\swArrow\phi_{S,T}"] & A_S \boxplus A_T \arrow[d, "f_S\boxplus f_T"] & = & A_{S\cup T} \arrow[rr, "a_{S,T}"] \arrow[d, "g_{S\cup T}"'] & \arrow[d, phantom, "\gamma_{S,T}\swArrow"] & A_S \boxplus A_T \arrow[d, bend right=50, "g_S\boxplus g_T"'] \arrow[d, bend left=50, "f_S\boxplus f_T"] \arrow[d, phantom, "\overset{\Leftarrow}{\psi_S\boxplus\psi_T}"] \\
A'_{S\cup T} \arrow[rr, "a'_{S,T}"'] & & A'_S \boxplus A'_T & & A'_{S\cup T} \arrow[rr, "a'_{S,T}"'] & & A'_S \boxplus A'_T
\end{tikzcd} \tag{5.11}$$

We now need to show that these data indeed defines a 2-category. We will first show that given objects $\{A_S, a_{S,T}\}$, $\{A'{}_S, a'{}_{S,T}\}$, the 1-morphisms and 2-morphisms form a category $\widehat{Mod_{\mathcal{R}}}(\underline{n})(\{A_S, a_{S,T}\}, \{A'{}_S, a'{}_{S,T}\})$.

Vertical composition of 2-morphisms $\{\psi_S\}, \{\psi'_S\}$ is defined componentwise. We show that this composition satisfies equation (5.11). Given

$$\begin{tikzcd}
\{A_S, a_{S,T}\} \arrow[r, bend left=50, "\{f,\phi\}"{name=F}] \arrow[r, "\{g,\gamma\}"'{name=G, near start}] \arrow[r, bend right=50, "\{h,\eta\}"'{name=H}] & \{A'_S, a'_{S,T}\}
\arrow[from=F, to=G, phantom, "\Downarrow\{\psi\}"] \arrow[from=G, to=H, phantom, "\Downarrow\{\psi'\}"]
\end{tikzcd}$$

we see that for all S, T:

$$
\begin{tikzcd}
A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, bend right=50, "h"'] \arrow[d, "g" near start] \arrow[d, bend left=50, "f"] \arrow[d, phantom, bend right=25, "\Leftarrow\psi'"] \arrow[d, phantom, bend left=25, "\Leftarrow\psi"] \arrow[dr, phantom, "\swarrow\phi"] & A_S \boxplus A_T \arrow[d, "f\boxplus f"] \\
A'_{S\cup T} \arrow[r, "a'_{S,T}"'] & A'_S \boxplus A'_T
\end{tikzcd}
=
\begin{tikzcd}
A_{S\cup T} \arrow[rr, "a_{S,T}"] \arrow[d, bend right=40, "h"'] \arrow[d, bend left=40, "g"] \arrow[d, phantom, "\Leftarrow\psi'"] \arrow[drr, phantom, "\swarrow\gamma"] & & A_S \boxplus A_T \arrow[d, bend right=40, "g\boxplus g"'] \arrow[d, bend left=40, "f\boxplus f"] \arrow[d, phantom, "\Leftarrow\psi\boxplus\psi"] \\
A'_{S\cup T} \arrow[rr, "a'_{S,T}"'] & & A'_S \boxplus A'_T
\end{tikzcd}
=
$$

$$
\begin{tikzcd}
A_{S\cup T} \arrow[rr, "a_{S,T}"] \arrow[d, "h"'] \arrow[drr, phantom, "\swarrow\eta"] & & A_S \boxplus A_T \arrow[d, bend right=50, "h\boxplus h"'] \arrow[d, "g\boxplus g" near start] \arrow[d, bend left=50, "f\boxplus f"] \arrow[d, phantom, bend right=25, "\Leftarrow\psi'\boxplus\psi'"] \arrow[d, phantom, bend left=25, "\Leftarrow\psi\boxplus\psi"] \\
A'_{S\cup T} \arrow[rr, "a'_{S,T}"'] & & A'_S \boxplus A'_T
\end{tikzcd}
$$

as wanted.

We also note that $\{\mathrm{id}_{f_S}\}$ is a well-defined automorphism for $\{f_S, \phi_{S,T}\}$ and it is the identity of the componentwise composition.

The composition functor $*$ is given by:

$$
\begin{aligned}
(\{g_S, \gamma_{S,T}\}, \{f_S, \phi_{S,T}\}) &\mapsto \{g_S * f_S, (\gamma \diamond \phi)_{S,T}\} \\
(\{\psi'_S\}, \{\psi_S\}) &\mapsto \{\psi'_S * \psi_S\},
\end{aligned}
$$

where the 2-morphism $(\gamma \diamond \phi)_{S,T}$ is defined by the pasting diagram:

$$
\begin{tikzcd}
A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, "f_{S\cup T}"'] \arrow[dr, phantom, "\swarrow\phi_{S,T}"] & A_S \boxplus A_T \arrow[d, "f_S\boxplus f_T"] \\
A'_{S\cup T} \arrow[r, "a'_{S,T}"] \arrow[d, "g_{S\cup T}"'] \arrow[dr, phantom, "\swarrow\gamma_{S,T}"] & A'_S \boxplus A'_T \arrow[d, "g_S\boxplus g_T"] \\
A''_{S\cup T} \arrow[r, "a''_{S,T}"'] & A''_S \boxplus A''_T.
\end{tikzcd}
$$

Showing that $\{g_S * f_S, (\gamma \diamond \phi)_{S,T}\}$ is a well-defined 1-morphism (that is, it satisfies equations (5.9) and (5.10)) can be done again using pasting diagrams and the fact that both $\{f_S, \phi_{S,T}\}$ and $\{g_S, \gamma_{S,T}\}$ satisfy those same equations. Analogously we can show that $\{\psi'_S * \psi_S\}$ is a well-defined 2-morphism.

The natural associativity isomorphism in this 2-category is given by the component-wise associativity isomorphisms in $Mod_{\mathcal{R}}$. More precisely, given $\{f_S, \phi_{S,T}\}$, $\{g_S, \gamma_{S,T}\}$, and $\{h_S, \eta_{S,T}\}$ composable 1-morphisms, we define the 2-morphism $\{\alpha_S\}$, where

$$\alpha_S : h_S * (g_S * f_S) \Rightarrow (h_S * g_S) * f_S$$

is the associativity isomorphism in $Mod_{\mathcal{R}}$.

The fact that

$$\{(h_S * g_S) * f_S, ((\eta \diamond \gamma) \diamond \phi)_{S,T}\} \Rightarrow \{h_S * (g_S * f_S), (\eta \diamond (\gamma \diamond \phi))_{S,T}\}$$

is an allowed 2-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ will follow from the uniqueness of pasting diagrams. Naturality and the pentagonal axiom follow from those in $Mod_{\mathcal{R}}$.

Given and object $\{A_S, a_{S,T}\}$, the identity 1-morphism is given by $\{I_{A_S}, \mathrm{id}_{a_{S,T}}\}$. It is clear that this is an allowed 1-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ and $\{I_{A_S}, \mathrm{id}_{a_{S,T}}\}$ is a strict identity, giving that I_{A_S} is a strict identity in $Mod_{\mathcal{R}}$. We conclude thus that $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ is indeed a 2-category.

We now need to prove that this construction extends to a functor $\widehat{Mod_{\mathcal{R}}} : Fin_* \to 2Cat$. Given a morphism $\theta : \underline{n} \to \underline{m}$ in Fin_* we will define a functor

$$\theta_* : \widehat{Mod_{\mathcal{R}}}(\underline{n}) \to \widehat{Mod_{\mathcal{R}}}(\underline{m})$$

as follows:

$$\begin{aligned}
\{A_S, a_{S,T}\} &\longmapsto \{A^\theta_U, a^\theta_{U,V}\} = \{A_{\theta^{-1}(U)}, a_{\theta^{-1}(U),\theta^{-1}(V)}\} \\
\{f_S, \phi_{S,T}\} &\longmapsto \{f^\theta_U, \phi^\theta_{U,V}\} = \{f_{\theta^{-1}(U)}, \phi_{\theta^{-1}(U),\theta^{-1}(V)}\} \\
\{\psi_S\} &\longmapsto \{\psi^\theta_U\} = \{\psi_{\theta^{-1}(U)}\},
\end{aligned}$$

where U, V range over disjoint subsets of $\underline{m}$ that do not contain the basepoint. Since θ is basepoint preserving, $\theta^{-1}(U)$ does not contain the basepoint and it is an allowed indexing subset of $\underline{n}$. Also, since U and V are disjoint, their pre-images under θ are also disjoint.

This assignment commutes strictly with all the compositions and identities in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$

and $\widehat{Mod_{\mathcal{R}}}(\underline{m})$, giving a strict functor between these 2-categories.

It is clear from the construction that $\widehat{Mod_{\mathcal{R}}}(\underline{1})$ is isomorphic to $Mod_{\mathcal{R}}$.

We will end the proof by showing that for every $n \geqslant 0$, the functor

$$p_n : \widehat{Mod_{\mathcal{R}}}(\underline{n}) \to Mod_{\mathcal{R}}^{\times n}$$

is an equivalence of 2-categories. This will show that the Γ-(2-category) is special. For ease of notation we will denote the subset $\{i\} \in \underline{n}$ as i. The functor p_n takes

$$\begin{aligned} \{A_S, a_{S,T}\} &\longmapsto \{A_i\}_{i=1}^n \\ \{f_S, \phi_{S,T}\} &\longmapsto \{f_i\}_{i=1}^n \\ \{\psi_S\} &\longmapsto \{\psi_i\}_{i=1}^n. \end{aligned}$$

We will define an inverse functor $i_n : Mod_{\mathcal{R}}^{\times n} \to \widehat{Mod_{\mathcal{R}}}(\underline{n})$:

$$\begin{aligned} \{A_i\}_{i=1}^n &\longmapsto \{\boxplus_{i\in S} A_i, e_{S,T}\} \\ \{f_i\}_{i=1}^n &\longmapsto \{\boxplus_{i\in S} f_i, \mathrm{id}\} \\ \{\psi_i\}_{i=1}^n &\longmapsto \{\boxplus_{i\in S} \psi_i\}. \end{aligned}$$

Here, $\boxplus_{i\in S}$ denotes the iterated monoidal operation $\boxplus$ with the usual order of the indices in $S \subset \underline{n}$. Recall that $\boxplus$ is strictly associative.

The 1-morphism

$$e_{S,T} : \boxplus_{i\in S\cup T} A_i \longrightarrow \boxplus_{i\in S} A_i \boxplus \boxplus_{i\in T} A_i$$

is the unique composition of instances of the braiding β that reorders the summands. Note that since 0 and 1 are strict units for $\oplus$ and $\otimes$ in $\mathcal{R}$, this composition is truly unique, not just unique up to associativity isomorphisms. The matrix $e_{S,T}$ is a permutation matrix. It is clear that $\{\boxplus_{i\in S} A_i, e_{S,T}\}$ satisfies equations (5.7) and (5.8).

We also have that $(\boxplus_{i\in S} f_i \boxplus \boxplus_{i\in T} f_i) * e_{S,T} = e_{S,T} * (\boxplus_{i\in S\cup T} f_i)$, thus we can choose the 2-isomorphism to be the identity. The collection $\{\boxplus_{i\in S} f_i, \mathrm{id}\}$ satisfies automatically equations (5.9) and (5.10). It is also automatic that for any $\{\psi_i\} : \{f_i\} \Rightarrow \{g_i\}$, we get that $\{\boxplus_{i\in S} \psi_i\}$ is an allowed 2-morphism between $\{\boxplus_{i\in S} f_i, \mathrm{id}\}$ and $\{\boxplus_{i\in S} g_i, \mathrm{id}\}$.

This assignment gives a strict functor since

$$i_n(\{g_i * f_i\}) = \{\boxplus_{i\in S}(g_i * f_i), \mathrm{id}\} = \{(\boxplus_{i\in S} g_i) * (\boxplus_{i\in S} f_i), \mathrm{id}\diamond\mathrm{id}\} = i_n(\{g_i\}) * i_n(\{f_i\})$$

$$i_n(\{I_{A_i}\}) = \{\boxplus_{i\in S} I_{A_i}, \mathrm{id}\} = \{I_{\boxplus_{i\in S} A_i}, \mathrm{id}\} = \mathrm{id}_{i_n(\{A_i\})}.$$

Clearly $p_n \circ i_n = Id_{Mod_{\mathcal{R}}^{\times n}}$. We now construct a natural equivalence

$$\xi : Id_{\widehat{Mod_{\mathcal{R}}}(\underline{n})} \to i_n \circ p_n.$$

For every object $\{A_S, a_{S,T}\}$ in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$, we need a 1-morphism

$$\xi_{\{A_S,a_{S,T}\}} : \{A_S, a_{S,T}\} \to \{\boxplus_{i\in S} A_i, e_{S,T}\}.$$

Given the subset S, we define a^S inductively as the composition:

$$A_S \xrightarrow{a_{j,S-j}} A_j \boxplus A_{S-j} \xrightarrow{\mathrm{id}_{A_j}\boxplus a^{S-j}} A_j \boxplus \boxplus_{i\in S-j} A_i = \boxplus_{i\in S} A_i,$$

where j is the smallest index in S.

Note that by conditions (5.7) and (5.8) on the $a_{S,T}$, the two compositions in the diagram below differ by a unique associativity 2-isomorphism:

$$\begin{tikzcd} A_{S\cup T} \arrow[r, "a_{S,T}"] \arrow[d, "a^{S\cup T}"'] \arrow[dr, phantom, "\Swarrow\eta_{S,T}"] & A_S \boxplus A_T \arrow[d, "a^S\boxplus a^T"] \\ \boxplus_{i\in S\cup T} A_i \arrow[r, "e_{S,T}"'] & \boxplus_{i\in S} A_i \boxplus \boxplus_{i\in T} A_i. \end{tikzcd}$$

Since associativity isomorphisms are unique, $\{a^S, \eta_{S,T}\}$ is a well-defined 1-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$. This will be the corresponding 1-morphism of the natural transformation ξ.

To complete the data of the natural transformation, for every pair of objects $\{A_S, a_{S,T}\}$, $\{A'_S, a'_{S,T}\}$ in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ we need to provide a natural isomorphism ξ_2, which on the component $\{f_S, \phi_{S,T}\}$is given by a 2-morphism

$$\xi_2(\{f_S, \phi_{S,T}\}) : \{\boxplus_{i\in S} f_i, \mathrm{id}\} * \{a^S, \eta_{S,T}\} \Rightarrow \{a'^S, \eta'_{S,T}\} * \{f_S, \phi_{S,T}\}.$$

Given S, we define a 2-isomorphism in $Mod_{\mathcal{R}}$, $\phi^S : (\boxplus_{i\in S} f_i) * a^S \Rightarrow a'^S * f_S$, inductively as the pasting diagram:

$$\begin{tikzcd}
A_S \arrow[r, "a_{j,S-j}"] \arrow[d, "f_S"'] \arrow[dr, phantom, "\swarrow\phi_{j,S-j}"] & A_j \boxplus A_{S-j} \arrow[r, "I_{A_j}\boxplus a'^{S-j}"] \arrow[d, "f_j \boxplus f_{S-j}"] \arrow[dr, phantom, "\swarrow \mathrm{id}\boxplus\phi^{S-j}"] & \boxed{+}_{i\in S} A_i \arrow[d, "\boxplus_{i\in S} f_i"] \\
A'_S \arrow[r, "a'_{j,S-j}"'] & A'_j \boxplus A'_{S-j} \arrow[r, "I_{A'_j}\boxplus a'^{S-j}"'] & \boxed{+}_{i\in S} A'_i,
\end{tikzcd}$$

where j is the smallest index in S. We need to show that $\{\phi^S\}_S$ gives a 2-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$, that is, it satisfies equation (5.11). This is done by induction on $|S \cup T|$ using pasting diagrams. We let $\xi_2(\{f_S, \phi_{S,T}\}) = \{\phi^S\}$.

To show the naturality of ξ_2, we need to show that

$$\begin{tikzcd}
A_S \arrow[r, "a^S"] \arrow[d, bend right=60, "g_S"'] \arrow[d, bend left=60, "f_S"] \arrow[d, phantom, "\Leftarrow\psi_S"] \arrow[dr, phantom, "\swarrow\phi^S"] & \boxed{+}_{i\in S} A_i \arrow[d, "\boxplus_{i\in S} f_i"] \\
A'_S \arrow[r, "a'^S"'] & \boxed{+}_{i\in S} A'_i
\end{tikzcd}
=
\begin{tikzcd}
A_S \arrow[r, "a^S"] \arrow[d, "g_S"'] \arrow[dr, phantom, "\gamma^S \swarrow"] & \boxed{+}_{i\in S} A_i \arrow[d, bend right=60, "\boxplus_{i\in S} g_i"'] \arrow[d, bend left=60, "\boxplus_{i\in S} f_i"] \arrow[d, phantom, "\Leftarrow \boxplus_{i\in S}\psi_i"] \\
A'_S \arrow[r, "-a'^S"'] & \boxed{+}_{i\in S} A'_i.
\end{tikzcd}$$

This follows by induction on $|S|$, using the inductive definition of ϕ^S and equation (5.11). Since ϕ^S is invertible, we get a natural isomorphism as wanted.

For the axiom (2.9) of a natural transformation we need to show

$$\begin{tikzcd}
A_S \arrow[r, "a^S"] \arrow[d, "f_S"'] \arrow[dr, phantom, "\swarrow\phi^S"] & \boxed{+}_{i\in S} A_i \arrow[d, "\boxplus_{i\in S} f_i"] \\
A'_S \arrow[r, "a'^S"] \arrow[d, "g_S"'] \arrow[dr, phantom, "\swarrow\gamma^S"] & \boxed{+}_{i\in S} A'_i \arrow[d, "\boxplus_{i\in S} g_i"] \\
A''_S \arrow[r, "a''^S"'] & \boxed{+}_{i\in S} A''_i
\end{tikzcd}
=
\begin{tikzcd}
A_S \arrow[r, "a^S"] \arrow[d, "g_S * f_S"'] \arrow[dr, phantom, "\swarrow(\gamma\diamond\phi)^S"] & \boxed{+}_{i\in S} A_i \arrow[d, "\boxplus_{i\in S} g_i * f_i"] \\
A''_S \arrow[r, "a''^S"'] & \boxed{+}_{i\in S} A''_i.
\end{tikzcd}$$

This is straightforward using induction on $|S|$ and the definition of $\gamma \diamond \phi$.

Axiom (2.10) is true since all the 2-morphisms involved are the identity.

Hence we have a natural transformation between $Id_{\widehat{Mod_{\mathcal{R}}}(\underline{n})}$ and $i_n \circ p_n$. The 1-morphism of this transformation, $\{a^S, \eta_{S,T}\}$ is a 1-equivalence in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$, thus by Proposition 2.14, we get a natural equivalence between functors.

We conclude that the 2-categories $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ and $Mod_{\mathcal{R}}^{\times n}$ are 2-equivalent, making $\widehat{Mod_{\mathcal{R}}}$ into a special Γ-(2-category).

$\square$

Chapter 6

Constructing the infinite loop space map

The goal of the construction of two-vector bundles in [BDR] is to find a cohomology theory that is geometrically defined and has chromatic level 2. In order to show that $\mathcal{K}(\mathcal{R})$ actually defines a cohomology theory, the authors conjecture and later prove in [BDRR2] that the map of equation (3.6) is an equivalence of spaces. Hence $\mathcal{K}(\mathcal{R})$ inherits an infinite loop space structure from $K(K\mathcal{R})$.

In Chapter 5, we constructed an infinite loop space structure on $\mathcal{K}(\mathcal{R})$ that came directly from the symmetric monoidal structure on $Mod_{\mathcal{R}}$. We would like to know if these two infinite loop space structures on $\mathcal{K}(\mathcal{R})$ are compatible. More precisely, we will show in this chapter that the map in equation (3.6) is a map of infinite loop spaces. This is done in Section 6.2, where we give an alternate construction of the Γ-space associated to a symmetric monoidal 2-category. For this construction we use a special kind of bisimplicial sets, which we call *horizontally categorical*. These are discussed in Section 6.1.

6.1 Preliminaries

In this chapter we will be working with bisimplicial sets that have a filling condition for inner horns. This section contains some properties of these bisimplicial sets.

Definition 6.1. A bisimplicial set X is *horizontally categorical* if the horizontal simplicial sets $X_n = X_{(-,n)}$ have the unique filling condition with respect to all inner horns. That is, every map f has a unique extension as shown in the diagram:

$$\begin{tikzcd} \Lambda^m_k \arrow[r, "f"] \arrow[d, hook] & X_n \\ \Delta^m \arrow[ur, dashed, "\exists!"'] \end{tikzcd} \quad \text{for } 0 < k < m.$$

Remark 6.2. If $\mathcal{C}$ is a category enriched over simplicial sets, then $N\mathcal{C}$ is a horizontally categorical bisimplicial set. Indeed, we can think of $\mathcal{C}$ as a simplicial object in categories. Then the horizontal simplicial sets $N\mathcal{C}$ are nerves of categories and hence have the unique filling condition with respect to inner horns.

Let Δ^m, Λ^m_k denote also the vertically constant bisimplicial sets given by Δ^m, Λ^m_k respectively. The spine of Δ^m will be denoted by $Spine_m$. This is the simplicial set (and the vertically constant bisimplicial set) given by the edges $0 \to 1, 1 \to 2, \cdots, (m-1) \to m$.

Let $\underline{Hom}(X, Y)$ denote the internal hom in bisimplicial sets. If X is a horizontally categorical bisimplicial set, then there is a unique composition of paths:

$$\odot : \underline{Hom}(\Delta^1, X) \underset{X}{\times} \underline{Hom}(\Delta^1, X) \to \underline{Hom}(\Delta^1, X).$$

Lemma 6.3. *Composition of paths is associative. In particular, there is a unique map*

$$\underline{Hom}(\Delta^1, X) \underset{X}{\times} \cdots \underset{X}{\times} \underline{Hom}(\Delta^1, X) \to \underline{Hom}(\Delta^1, X).$$

Proof. Since X is horizontally categorical, any map $Spine_m \to X_n$ can be extended uniquely to a map $\Delta^m \to X_n$ and then projected onto the edge $0 \to m$. This in turn implies the uniqueness of the map above. $\square$

6.2 Proof of Main Result

The infinite loop space structure on $K(K\mathcal{R})$ comes from the Γ-space construction on the classifying space of the symmetric monoidal category $Mod_{K\mathcal{R}}$ of finitely generated free modules over $K\mathcal{R}$. In order to prove that the map in equation (3.6) is a map of infinite loop spaces we will prove that the map extends to a map of Γ-spaces. In particular, we will prove the following theorem.

Theorem 6.4. *There is a zigzag of maps of* Γ*-spaces*

$$|S\widehat{Mod_{\mathcal{R}}}| \longrightarrow |N\widetilde{Mod_{K\mathcal{R}}}| \xleftarrow{\sim} |N\widehat{Mod_{K\mathcal{R}}}|.$$

At level 1, the right-hand map is an equality, and the left-hand map corresponds to the map in equation (3.6).

Corollary 6.5. *There is a zigzag of equivalences of spectra*

$$K(\mathcal{R}) \xrightarrow{\sim} \widetilde{K}(K\mathcal{R}) \xleftarrow{\sim} K(K\mathcal{R}),$$

which at the level of zeroth spaces gives the maps

$$\mathcal{K}(\mathcal{R}) \underset{(3.6)}{\longrightarrow} \widetilde{K_0}(K\mathcal{R}) \underset{=}{\longleftarrow} K_0(K\mathcal{R}).$$

In the theorem above, $Mod_{K\mathcal{R}}$ refers to the category of modules over the ring spectrum $K\mathcal{R}$, $\widehat{Mod_{K\mathcal{R}}}$ is the standard Γ-category construction on a symmetric monoidal category, and $\widetilde{Mod_{K\mathcal{R}}}$ is an alternative construction which we will describe.

The model we are taking for $K\mathcal{R}$ is that of [EM]. We construct $Mod_{K\mathcal{R}}$ as follows.

Let $GL_n(K\mathcal{R})$ be the group-like monoid of weakly invertible matrices over $K\mathcal{R}$. It is defined by the pullback

$$\begin{tikzcd} GL_n(K\mathcal{R}) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\mathbf{m}\in I} \Omega^m M_n(K\mathcal{R}(m)) \arrow[d] \\ GL_n(\pi_0 K\mathcal{R}) \arrow[r] & M_n(\pi_0 K\mathcal{R}), \end{tikzcd}$$

The category $Mod_{K\mathcal{R}}$ has as objects the natural numbers $\mathbf{n}$. The space of morphisms is given by

$$Mod_{K\mathcal{R}}(\mathbf{n},\mathbf{m}) = \begin{cases} GL_n(K\mathcal{R}) & \text{if} \quad n = m \\ \varnothing & \text{if} \quad n \neq m. \end{cases}$$

Note that since $K\mathcal{R}(0) = N\mathcal{R}$, there is a map of spaces

$$NGL_n(\mathcal{R}) \longrightarrow GL_n(K\mathcal{R}),$$

which extends to the equivalence of [BDRR2]

$$\mathcal{K}(\mathcal{R}) \longrightarrow K(K\mathcal{R}).$$

Let $(\mathcal{D}, \oplus, 0, \tau)$ be a permutative category enriched over bisimplicial sets. We further require that for every objects A and B, the bisimplicial set $\mathcal{D}(A,B)$ is horizontally categorical. To ease the notation, we will think of these bisimplicial sets as simplicial spaces.

We will build a Γ-category $\widetilde{\mathcal{D}}$ which will turn out to be equivalent to $\widehat{\mathcal{D}}$. For the construction of $\widetilde{\mathcal{D}}(\underline{n})$ we mimic that of 2-categories in Chapter 5.

The objects are given by $\{A_S, a_{S,T}\}$, with S and T as above; $A_S \in Ob\mathcal{D}$ and $a_{S,T}$: $A_{S\cup T} \to A_S \boxplus A_T$ is an invertible morphism, that is, a 0-simplex in $\mathcal{D}(A_{S\cup T}, A_S \boxplus A_T)$. We require conditions 1a-1d in the proof of Theorem 5.4 to hold. We note that the objects in $\widetilde{\mathcal{D}}(\underline{n})$ are the same as the objects in the usual Γ-category construction $\widehat{\mathcal{D}}(\underline{n})$.

Given two objects $\{A_S, a_{S,T}\}$ and $\{B_S, b_{S,T}\}$, the simplicial space of morphisms between them is defined as a subspace of

$$\prod_S \mathcal{D}(A_S, B_S) \times \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)).$$

Let X be defined by the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \arrow[d, "(d^{1*}, d^{0*})"] \\ \prod_S \mathcal{D}(A_S, B_S) \arrow[r, "\prod_{S,T}(a^*, b_*)"'] & \prod_{S,T} \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)^{\times 2}, \end{tikzcd} \tag{6.6}$$

where the lower horizontal map corresponds to $(a^*_{S,T}, (b_{S,T})_*)$ for the S, T component.

Let Y be the equalizer

$$Y \longrightarrow X \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \ , \tag{6.7}$$

where p_1 is just the projection and p_2 is the projection composed with τ_*. This reproduces condition (5.10) in this setting.

Let Z be the equalizer

$$Z \longrightarrow Y \underset{q_2}{\overset{q_1}{\rightrightarrows}} \prod_{S,T,U} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)) \ , \tag{6.8}$$

where q_1 and q_2 are as defined below. This condition mimics that of equation (5.9).

Let f be the following composition

$$\begin{array}{c}
\mathcal{D}(A_U, B_U) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\Big\downarrow{\scriptstyle \tilde{\oplus}\times\mathrm{id}} \\
\underline{Hom}(\mathcal{D}(A_{S\cup T}, B_S \oplus B_T), \mathcal{D}(A_{S\cup T} \oplus A_U, B_S \oplus B_T \oplus B_U)) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\Big\downarrow{\scriptstyle \circ} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T} \oplus A_U, B_S \oplus B_T \oplus B_U)) \\
\Big\downarrow{\scriptstyle a^*} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)),
\end{array}$$

where $\tilde{\oplus}, \circ, a$ denote the adjoint of $\oplus$, composition of function spaces, $a_{S\cup T,U}$, respectively.

We also have the map

$$\begin{array}{c}
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_{S\cup T} \oplus B_U)) \\
\Big\downarrow{\scriptstyle b_*} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)),
\end{array}$$

Note that the pullback conditions on X imply that $d^{0*}f$ is equal to $d^{1*}b_*$, thus, putting the two maps f and b_* together we get a map from Y to

$$\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)) \underset{\mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)}{\times} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)).$$

We can further compose with $\odot$, getting a map to $\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U))$.

Finally, we take the product over all (S, T, U) to get

$$q_1 : Y \longrightarrow \prod_{S,T,U} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T\cup U}, B_S \oplus B_T \oplus B_U)).$$

We define q_2 similarly, starting with

$$\mathcal{D}(A_S, B_S) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{T\cup U}, B_T \oplus B_U))$$

instead.

We define $\widetilde{\mathcal{D}}(\underline{n})(\{A_S, a_{S,T}\}, \{B_S, b_{S,T}\}) := Z$.

We now show that the collection of objects described above with the simplicial spaces of morphisms $\widetilde{\mathcal{D}}(\underline{n})$ form a category enriched over simplicial spaces.

Given objects $A = \{A_S, a_{S,T}\}$, $B = \{B_S, b_{S,T}\}$, $C = \{C_S, c_{S,T}\}$, we define a composition map

$$\widetilde{\mathcal{D}}(\underline{n})(B, C) \times \widetilde{\mathcal{D}}(\underline{n})(A, B) \to \widetilde{\mathcal{D}}(\underline{n})(A, C)$$

as follows: On one hand, we have a map given by the composition maps in $\mathcal{D}$:

$$\prod_S \mathcal{D}(A_S, B_S) \times \prod_S \mathcal{D}(B_S, C_S) \xrightarrow{\bullet} \prod_S \mathcal{D}(A_S, C_S).$$

Given (S, T), let g be the following composition

$$\begin{array}{c}
\mathcal{D}(B_S, C_S) \times \mathcal{D}(B_T, C_T) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\Big\downarrow {\scriptstyle \oplus \times \mathrm{id}} \\
\mathcal{D}(B_S \oplus B_T, C_S \oplus C_T) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\Big\downarrow {\scriptstyle \tilde{\bullet} \times \mathrm{id}} \\
\underline{Hom}(\mathcal{D}(A_{S\cup T}, B_S \oplus B_T), \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\Big\downarrow {\scriptstyle \circ} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)).
\end{array}$$

Similarly, we let h be

$$\begin{array}{c}
\mathcal{D}(A_{S\cup T}, B_{S\cup T}) \times \underline{Hom}(\Delta^1, \mathcal{D}(B_{S\cup T}, C_S \oplus C_T)) \\
\Big\downarrow {\scriptstyle \tilde{\bullet} \times \mathrm{id}} \\
\underline{Hom}(\mathcal{D}(B_{S\cup T}, C_S \oplus C_T), \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)) \times \underline{Hom}(\Delta^1, \mathcal{D}(B_{S\cup T}, C_S \oplus C_T)) \\
\Big\downarrow {\scriptstyle \circ} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)).
\end{array}$$

One can check that when we restrict the source of these maps to the pullback condition (6.6), the condition $d^{0*}g = d^{1*}h$ is satisfied. Hence we can put g and h together to get a map to

$$\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)) \underset{\mathcal{D}(A_{S\cup T}, C_S \oplus C_T)}{\times} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T))$$

that we can later compose with $\odot$ we get a map to

$$\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)).$$

Taking the product of these maps and the composition maps over S, T, we get a map

$$\widetilde{\mathcal{D}}(\underline{n})(B,C) \times \widetilde{\mathcal{D}}(\underline{n})(A,B) \xrightarrow{k} \prod_S \mathcal{D}(A_S, C_S) \times \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T)).$$

We want to show now that the image of k is contained in $\widetilde{\mathcal{D}}(\underline{n})(A,C)$.

On the $\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, C_S \oplus C_T))$-component, $d^{1*}k = d^{1*}g$, which is equal to a^* on $\mathcal{D}(A_S, C_S) \times \mathcal{D}(A_T, C_T)$ by the pullback condition (6.6) on the space $\widetilde{\mathcal{D}}(\underline{n})(A,B)$.

Similarly for $d^{0*}k$, we show it is the same as $c_{S,T*}$, thus showing we land in the pullback (6.6) for $\widetilde{\mathcal{D}}(\underline{n})(A,C)$.

To show that conditions (6.7) and (6.8) hold we use the fact that they hold for $\widetilde{\mathcal{D}}(\underline{n})(A,B)$, $\widetilde{\mathcal{D}}(\underline{n})(B,C)$ and Lemma 6.3. We conclude that the composition is well defined. The fact that composition in $\mathcal{D}$ is associative and Lemma 6.3 imply that the composition is associative.

The identity of $\{A_S, a_{S,T}\}$ is the 0-simplex given by the collections of id_{A_S} together with the constant path at $a_{S,T} \in \mathcal{D}(A_{S\cup T}, A_S \oplus A_T)$.

We thus get a category $\widetilde{\mathcal{D}}(\underline{n})$. We can extend this construction to a Γ-category in the usual way. Let $\theta : \underline{n} \to m$ be a morphism in Γ. We construct the functor

$$\theta_* \widetilde{\mathcal{D}}(\underline{n}) \longrightarrow \widetilde{\mathcal{D}}(\underline{m})$$

as follows.

We send the object $\{A_S, a_{S,T}\}$ of $\widetilde{\mathcal{D}}(\underline{n})$ to the object $\{A^\theta{}_S, a^\theta{}_{S,T}\}$ of $\widetilde{\mathcal{D}}(\underline{m})$, where $A^\theta_S = A_{\theta^{-1}S}$ and $a^\theta_{S,T} = a_{\theta^{-1}S, \theta^{-1}T}$.

For morphisms we construct a map

$$\widetilde{\mathcal{D}}(\underline{n})(A, B) \longrightarrow \widetilde{\mathcal{D}}(\underline{m})(A^\theta, B^\theta)$$

taking the projections of the $\theta^{-1}S$, $(\theta^{-1}S, \theta^{-1}T)$-components to the S, (S,T)-components, respectively.

Theorem 6.9. *The* Γ*-space* $N\widetilde{\mathcal{D}}$ *is special.*

This theorem is a consequence of the following proposition.

Consider the following diagram of functors

$$\begin{tikzcd}
\widetilde{\mathcal{D}}(\underline{n}) \arrow[rr, "q"] & & \mathcal{D}^{\times n} \arrow[dl, dotted, bend left, "j"] \\
& \widehat{\mathcal{D}}(\underline{n}) \arrow[ul, "i"] \arrow[ur, "p"'] &
\end{tikzcd}$$

In the diagram above q and p are the projections onto the $(\{1\}, \cdots, \{n\})$-components. It is clear that $qi = p$. On the other hand, j is the usual inverse equivalence for p, explained for example in [SS, Lemma 2.2]. We recall from [SS] that $pj = \mathrm{id}$ and there is a natural isomorphism from jp to the identity in $\widehat{\mathcal{D}}(\underline{n})$.

Proposition 6.10. *The functors* i *and* q *are a weak equivalence of categories (that is, they induce equivalence at the level of classifying spaces).*

Corollary 6.11. *The map* $i : \widehat{\mathcal{D}}(\underline{n}) \to \widetilde{\mathcal{D}}(\underline{n})$ *induces a levelwise equivalence of* Γ*-spaces*

$$N\widehat{\mathcal{D}} \longrightarrow N\widetilde{\mathcal{D}}.$$

Proof. It is clear that the functor i is compatible with the Γ-category structure. At the nth level, Ni is a weak homotopy equivalence, thus giving a levelwise equivalence of Γ-spaces. $\square$

Proof of Theorem 6.9. The result from Proposition 6.10 implies that Nq is a weak homotopy equivalence, thus proving that $N\widetilde{\mathcal{D}}$ is special. $\square$

Proof of Proposition 6.10. The proof will proceed as follows. Since i is the identity on objects and q is surjective on objects, it is enough to prove that both i and q induce weak equivalences for the simplicial spaces of morphisms.

We will first recall the definition of j in [SS, Lemma 2.2].

Given an object $(A_1, \cdots, A_n)$ in $\mathcal{D}^{\times n}$, we let

$$j(A_1, \cdots, A_n) = \{\bigoplus_{i \in S} A_i, e_{S,T}\}$$

where the sum is taken in the order of the indices of $S \subset \underline{n}$. The morphism $e_{S,T}$ is the uniquely determined isomorphism from $A_{S \cup T}$ to $A_S \oplus A_T$ given by composition of instances of τ.

For morphisms, we let

$$j(f_1, \cdots f_n) = \{\bigoplus_{i \in S} f_i\}.$$

We then have that $pj = \mathrm{id}$ and that there is a natural isomorphism $\lambda : \mathrm{id} \to jp$ given on the object $\{A_S, a_{S,T}\}$ by the composition

$$A_S \to A_{\{i_1\}} \oplus A_{S-i_1} \to \cdots A_{\{i_1\}} \oplus \cdots \oplus A_{\{i_k\}}$$

of the corresponding a's.

Given an object $A = \{A_S, a_{S,T}\}$ in $\widehat{\mathcal{D}}(\underline{n})$ (and thus, in $\widetilde{\mathcal{D}}(\underline{n})$), let $\dot{A} = \{\bigoplus_{i \in S} A_i, e^A_{S,T}\}$ denote its image under jq.

Given a pair of objects in $\widehat{\mathcal{D}}(\underline{n})$, (A, B), consider the following diagram:

$$\begin{tikzcd}
\widehat{\mathcal{D}}(\underline{n})(A,B) \arrow[r, "i"] & \widetilde{\mathcal{D}}(\underline{n})(A,B) \arrow[d, "q"] \\
\widehat{\mathcal{D}}(\underline{n})(\dot{A},\dot{B}) \arrow[u, "\lambda", "\cong"'] & \prod_i \mathcal{D}(A_i, B_i) \arrow[l, "\cong"', "j"]
\end{tikzcd}$$

The map λ above is gotten by pre- and post-composition with the natural isomorphism λ already defined. We note that $\lambda jqi = \mathrm{id}$ and $qi\lambda j = \mathrm{id}$.

Thus, if we show that $i\lambda jq$ is homotopic to the identity we will have shown that λjq and $i\lambda j$ are homotopy inverses for i and q respectively, giving us the result we want.

To build a homotopy from $f = i\lambda jq$ to the identity, we show that there exists a map ϕ making the diagram below commute. For ease of notation we let $Z = \widetilde{\mathcal{D}}(\underline{n})(A, B)$.

$$
\begin{tikzcd}
 & Z \\
Z \arrow[ur, "f"] \arrow[r, dashed, "\phi"] \arrow[dr, "\mathrm{id}"'] & \underline{Hom}(\Delta^1, Z) \arrow[u, "d^{1*}"'] \arrow[d, "d^{0*}"] \\
 & Z
\end{tikzcd}
$$

We construct a map

$$
\begin{array}{c}
\prod_S \mathcal{D}(A_S, B_S) \times \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)) \\
\downarrow \\
\prod_S \underline{Hom}(\Delta^1, \mathcal{D}(A_S, B_S)) \times \prod_{S,T} \underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)),
\end{array}
$$

and show that when restricted to Z it lands on $\underline{Hom}(\Delta^1, Z)$.

We will first construct the map ϕ_S to the component $\underline{Hom}(\Delta^1, \mathcal{D}(A_S, B_S))$. Say $S = \{i_1 < \cdots < i_k\}$. Let $S_j = \{i_j, \cdots, i_k\}$.

We consider the map g_j

$$
\begin{array}{c}
\prod_{l=1}^{j-1} \mathcal{D}(A_{i_l}, B_{i_l}) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S_j}, B_{i_j} \oplus B_{S_{j+1}})) \\
\Big\downarrow{\scriptstyle \oplus} \\
\mathcal{D}(\bigoplus_{l=1}^{k} A_{i_l}, \bigoplus_{l=1}^{k} B_{i_l}) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_{S_j}, B_{i_j} \oplus B_{S_{j+1}})) \\
\Big\downarrow{\scriptstyle \tilde{\oplus}} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{i_1} \oplus \cdots \oplus A_{i_{j-1}} \oplus A_{S_j}, B_{i_1} \oplus \cdots \oplus B_{i_j} \oplus B_{S_{j+1}})) \\
\Big\downarrow{\scriptstyle (a^*)_*((b^{-1})_*)_*} \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_S, B_S)).
\end{array}
$$

Here a is the composition of the instances of $a_{S,T}$ that take A_S to $A_{i_1} \oplus \cdots \oplus A_{i_{j-1}} \oplus A_{S_j}$, and similarly for b taking B_S into $B_{i_1} \oplus \cdots \oplus B_{i_j} \oplus B_{S_{j+1}}$.

When we restrict to Z, $d^1 g_j = d^0 g_{j+1}$ and thus we get a map into

$$
\underline{Hom}(Spine_{|S|}, \mathcal{D}(A_S, B_S)).
$$

By Lemma 6.3, this extends to $\underline{Hom}(\Delta^1, \mathcal{D}(A_S, B_S))$. The result is the map ϕ_S

From the construction it is clear that $d^1\phi_S = f$ and $d^0\phi_S = \text{id}$ on this component.

Now, we look at the component $\underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T))$.

Let g be the composition

$$\begin{tikzcd}
Z \arrow[d, "\phi_S \times \phi_T"'] \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_S, B_S)) \times \underline{Hom}(\Delta^1, \mathcal{D}(A_T, B_T)) \arrow[d, "\oplus_*"'] \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_S \oplus A_T, B_S \oplus B_T)) \arrow[d, "(a^*)_*"'] \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)).
\end{tikzcd}$$

Note that g the projection $Z \to \underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T))$ give composable paths. By Lemma 6.3 and given the conditions on Z, this map is equal to the map h:

$$\begin{tikzcd}
Z \arrow[d, "\phi_{S\cup T}"'] \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_{S\cup T})) \arrow[d, "b_*"'] \\
\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T)).
\end{tikzcd}$$

Therefore, we can think then of these maps together as giving a map into $\underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T))$, since the paths are equal:

$$\begin{tikzcd}
\bullet \arrow[r, "proj"] & \bullet \\
\bullet \arrow[u, "g"] \arrow[r, "id"'] & \bullet \arrow[u, "h"']
\end{tikzcd}$$

Using the conditions on Z, it is easy to check that this constructions yields a map

$$Z \to \underline{Hom}(\Delta^1, Z)$$

that restricts to the identity and f at each end.

We have thus constructed the desired homotopy. □

We end this chapter with the proof of Theorem 6.4.

Proof of Theorem 6.4. We first note that we can take a model for $Mod_{K\mathcal{R}}$ that satisfies the conditions imposed on $\mathcal{D}$ above. In the construction of $K\mathcal{R}$ in [EM], the spaces of the spectrum are nerves of simplicially enriched categories. Hence, the construction of $GL_n(K\mathcal{R})$ involves taking the homotopy colimit of nerves of categories, and we can use Thomason's machinery [Tho] to do this, thus having a model of $GL_n(K\mathcal{R})$ that is horizontally categorical. We can then apply the construction above to $Mod_{K\mathcal{R}}$, and Corollary 6.11 gives the right-hand equivalence of the theorem

$$N\widetilde{Mod_{K\mathcal{R}}} \xleftarrow{\sim} |N\widetilde{Mod_{K\mathcal{R}}}|.$$

To complete the proof we prove that there is a map of Γ-spaces

$$|S\widehat{Mod_{\mathcal{R}}}| \longrightarrow |N\widetilde{Mod_{K\mathcal{R}}}|$$

that extends the map at level 1 constructed in [BDRR2].

We will indeed build a map of bisimplicial spaces from the levelwise nerve of the simplicial category $S\widehat{Mod_{\mathcal{R}}}(\underline{n})$ to $N\widetilde{Mod_{K\mathcal{R}}}(\underline{n})$. Recall that $NGL_m(\mathcal{R})$ maps into $GL_m(K\mathcal{R})$, hence we can think of 1-morphisms in $Mod_{\mathcal{R}}$ as morphisms in $Mod_{K\mathcal{R}}$. Thus we can think about objects in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ as objects in $\widetilde{Mod_{K\mathcal{R}}}(\underline{n})$.

The $(0,-)$ simplicial spaces for $NS\widehat{Mod_{\mathcal{R}}}(\underline{n})$ and $N\widetilde{Mod_{K\mathcal{R}}}(\underline{n})$ are given by the objects of $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ and $\widetilde{Mod_{K\mathcal{R}}}(\underline{n})$, respectively. The map desired is obtained by the identification above.

Recall that a 1-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ is given by a collection $\{f_S, \phi_{S,T}\}$ of 1-morphisms and 2-morphisms in $Mod_{\mathcal{R}}$. When considering the maps $NGL_m(\mathcal{R}) \to GL_m(K\mathcal{R})$, we can then think of $\phi_{S,T}$ as a 0-simplex in $\underline{Hom}(\Delta^1, \mathcal{D}(A_{S\cup T}, B_S \oplus B_T))$. As noted above, the conditions on the construction of $\widetilde{Mod_{K\mathcal{R}}}(\underline{n})(A,B)$ reflect the coherence axioms for $\{f_S, \phi_{S,T}\}$, so in general, we can think of a 1-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ as a 0-simplex in $\widetilde{Mod_{K\mathcal{R}}}(\underline{n})(A,B)$. Similarly, we can think of a 2-morphism in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$ as a 1-simplex.

We can thus construct a map

$$NS\widehat{Mod_{\mathcal{R}}}(\underline{n})_{p,q} \to N\widetilde{Mod_{K\mathcal{R}}}(\underline{n})_{p,q}$$

as follows.

Recall that a (p,q)-simplex in $NS\widehat{Mod_{\mathcal{R}}}(\underline{n})$ is given by a collection $\{A_i\}_{i=0}^{p}$ of objects in $\widehat{Mod_{\mathcal{R}}}(\underline{n})$, together with diagrams of the form

$$\begin{tikzcd} & A_j \arrow[dr, "f^l_{jk}"] \arrow[d, phantom, "\Downarrow\varphi^l_{ijk}"] & \\ A_i \arrow[ur, "f^l_{ij}"] \arrow[rr, "f^l_{ik}"'] & {} & A_k, \end{tikzcd} \qquad \text{for all } 0 \leqslant i < j < k \leqslant p, 0 \leqslant l \leqslant q$$

subject to the coherence conditions in (2.20) and 2-morphisms $f^l_{ij} \Rightarrow f^{l+1}_{ij}$.

We can map this to

$$\coprod_{A_0,\dots,A_p} \widetilde{Mod_{K\mathcal{R}}}(\underline{n})(A_0,A_1) \times \cdots \times \widetilde{Mod_{K\mathcal{R}}}(\underline{n})(A_{p-1},A_p)$$

by projecting the $(i,i+1)$-entries and using the identification described above.

It is clear that this maps extends to a map of Γ-spaces, and that at level 0 it is the map of equation 3.6. □

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