Modules Over Affine Lie Algebras at Critical Level and Quantum Groups

by

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Abstract

There are two algebras associated to a reductive Lie algebra $\mathfrak{g}$: the De Concini-Kac quantum algebra and the Kac-Moody Lie algebra. Recent results show that the principle block of De Concini -Kac quantum algebra at an odd root of unity with (some) fixed central character is equivalent to the core of a certain t-structure on the derived category of coherent sheaves on certain Springer Fiber. Meanwhile, a certain category of representation of Kac-Moody Lie algebra at critical level with (some) fixed central character is also equivalent to a core of certain t-structure on the same triangulated category.

Based on several geometric results developed by Bezurkvanikov et. al. these two abelian categories turn out to be equivalent. i.e. the two t-structures coincide.

Thesis Supervisor: Roman Bezrukavnikov
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Chapter 1

Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, $\hat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ and $Rep(\mathfrak{U}_k\hat{\mathfrak{g}})$ the category of $\mathfrak{g}$ integral highest weight $\hat{\mathfrak{g}}$-modules of level k-h, where h denotes the dual Coxeter number of the Lie algebra $\mathfrak{g}$. Kazhdan and Lusztig used a 'fusion type' product to make $Rep(\mathfrak{U}_k\mathfrak{g})$ a tensor category. On the other hand,let U be the quantized enveloping algebra with parameter $q = exp(\pi\sqrt{-1}/d \cdot k)$, where d =1 if $\mathfrak{g}$ is a simple algebra of types A,D,E,d=2 for types B,C,F, and d=3 for type G. Let Rep(U) be the tensor category of finite dimensional U-modules. In [KL2] the authors have established an equivalence of tensor categories $Rep(U) \cong Rep(\mathfrak{U}_k\hat{\mathfrak{g}})$. The subcategory $block(U) \subset Rep(U)$ goes under the equivalence to the corresponding principal block $block(\mathfrak{U}_k\hat{\mathfrak{g}}) \subset Rep(\mathfrak{U}_k\hat{\mathfrak{g}})$

Let Gr^L be the affine Grassmanian of the dual loop group, $\widetilde{\mathcal{N}}$ be the Springer resolution of the nilpotent cone of G, in [ABG], the authors proved:

$$D^b(block^{mix}(U)) \cong D^{G^L \times G_m}(coh(\mathcal{N})) \cong D^b(Perv^{mix}_{I_0}(Gr^L))$$

and by comparing the t-structure, they proved:

$$block(U) \cong Perv_{I_0}(Gr^L)$$

In their proof, the geometry of Springer fiber plays an important role.

In a sequence papers of Lusztig [L1], there are various relations between representations of quantum group at root of unity, representations of affine Lie algebra at certain level and representations of Lie algebra over characteristic p. There are various groups of people proved many results in this direction. The main results of this paper is that the principle block of De Concini -Kac quantum algebra at an odd root of unity with (some) fixed central character is equivalent a certain category of representation of Kac-Moody Lie algebra at critical level with (some) fixed central character. More precisely, the principle block of De Concini -Kac quantum algebra at an odd root of unity with (some) fixed central character is equivalent to the core of a certain t-structure on the derived category of coherent sheaves on certain Springer Fiber. Meanwhile, a certain category of representation of Kac-Moody Lie algebra at critical level with (some) fixed central character is also equivalent to a core of certain t-structure on the same triangulated category. And in this paper, we will prove that these two t-structures coincide.

Chapter 2

Geometric Langlands Duality

2.1 Triangulated Category, t-structure and Perverse Sheaves

In this section we will take a brief review of some basic notations and basic results in perverse sheaves.

Definition 1. *A triangulated category is a category with*

a) a shift functor $[1] : \mathcal{C} \mapsto \mathcal{C}$ *(which is an autoequivalence)*

b) a family of triangles $X \longrightarrow Y \longrightarrow Z \xrightarrow{[1]} X[1]$ *(which are called distinguished triangles)*

satisfying the following axioms:

-1) Every triangle isomorphic to a distinguished triangle is a distinguished triangle.

0) Every morphism $f : X \mapsto T$ *can be completed to a distinguished triangle.*

1) $X \overset{Id}{\mapsto} X \mapsto 0 \mapsto X[1]$ *is a distinguished triangle.*

2) If $X \mapsto Y \mapsto Z \overset{[1]}{\mapsto}$ *is a distinguished triangle, then* $X[1] \mapsto Y[1] \mapsto Z[1] \mapsto$ *is a distinguished triangle.*

3) Given two triangles $X \mapsto Y \mapsto Z \mapsto X[1]$ *and* $X' \mapsto Y' \mapsto Z' \mapsto X'[1]$*, and morphism* $\phi : X \mapsto X'$, $\psi : Y \mapsto Y'$ *such that the first square in the following diagram is commutative, then there is (not unique) a morphism* $\xi : Z \mapsto Z'$ *such that the whole diagram is commutative:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{[1]} & X[1] \\ \phi\downarrow & & \psi\downarrow & & \xi\downarrow & & \phi[1]\downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{[1]} & X'[1] \end{array}$$

4) (Octahedral). Suppose we have the first three triangls, then we have the fourth one.

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ X & \longrightarrow & M & \longrightarrow & N & \longrightarrow & X[1] \\ Y & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & Y[1] \\ Z & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & Z[1] \end{array}$$

A typical example of triangulated category is the derived category of an abelian category.

Definition 2. *A t-structure* τ *in a triangulated category* $\mathcal{C}$ *is a pair of full subcategoies* $\mathcal{C}^{\leq 0}$ *and* $\mathcal{C}^{\geq 0}$ *such that: (for any* n*, we denoted* $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[-n]$ *and* $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[-n]$*)*

1) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}, \mathcal{C}^{\geq 1} \subset \mathcal{C}^{\geq 0}$,

2) If $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$, $Hom(A, B) = 0$

3) For any $X \in C$, there is $A \in \mathcal{C}^{\leq 0}, B \in \mathcal{C}^{\geq 1}$,, such that there is a distinguished triangle:

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

If furthermore we have the following:

$$\bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\leq n} = \emptyset, \bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\geq n} = \emptyset.$$

We will call this t-structure non-degenerated. This is the only case we will consider in this article.

Remark 1. *There is an easy fact. To give a t-structure we only need to specify $D^{\leq 0}$(resp. $D^{\geq 0}$), since we have $D^{\geq 1} = {}^{\perp}(D^{\leq 0})$ (resp. $D^{\leq -1} = (D^{\geq 0})^{\perp}$).*

Given a t-structure on $\mathcal{C}$, it is easy to see that the natural embedding functor $i_{\leq 0} : \mathcal{C}^{\leq 0} \mapsto \mathcal{C}$ (resp. $i_{\geq 0} : \mathcal{C}^{\geq 0} \mapsto \mathcal{C}$) has a natural right (resp. left)adjoint functor $\tau_{\leq 0} : \mathcal{C} \mapsto \mathcal{C}^{\leq 0}$. (resp. $\tau_{\geq 0} : \mathcal{C} \mapsto \mathcal{C}^{\geq 0}$). The core of this t-structure $C_{\tau}^{\leq 0} \bigcap C_{\tau}^{\geq 0}$ is an abelian category, and the functor ${}^{\tau}H^n : \mathcal{C} \mapsto C_{\tau}^{\leq n} \bigcap C_{\tau}^{\geq n} \overset{[n]}{\cong} C_{\tau}^{\leq 0} \bigcap C_{\tau}^{\geq 0}$ is called the n-th τ-cohomology functor.

Definition 3. *Let $(\mathcal{C}_1, \tau_1), (\mathcal{C}_2, \tau_2)$ be two triangulated categories with t-structure, a triangulated functor $F : \mathcal{C}_1 \mapsto \mathcal{C}_2$ is left t-exact(resp. right t-exact) if $F(\mathcal{C}_{\tau_1}^{\geq 0}) \subset \mathcal{C}_{\tau_2}^{\geq 0}$(resp. $F(\mathcal{C}_{\tau_1}^{\leq 0}) \subset \mathcal{C}_{\tau_2}^{\leq 0}$)*

Let X be a (reasonable) topological space, $D_c^b(X) = D_c^b(sh(X))$ be the bounded derived category of constructible sheaves on X,[1] we introduce the perverse t-structure

[1]In the following sections, we will take X to be a scheme over field of non zero characteristic and

by:

$$D^{b,\leq 0}_{perv} = \{\mathcal{F} \in D^b | \dim \mathrm{supp} \mathcal{H}^i(\mathcal{F}) \leq -i, \text{for any } i\}$$

$$D^{b,\geq 0}_{perv} = \{\mathcal{F} \in D^b | \mathbb{D}_X(\mathcal{F}) \in D^{b,\leq 0}_{perv}\}$$

where $\mathbb{D}_X$ is the Verdier duality functor on $D^b_c(X)$.

The core of the perverse t-structure is the abelian category of perverse sheaves and the n-th cohomological functor (which we denoted by) ${}^pH^n$ is the perverse cohomological functor.

When we work in the perverse t-structure, the six functors formulism is a useful tool and we give a brief recollection. Suppose we have two (reasonable) spaces X and Y and a (reasonable) map f: $X \mapsto Y$. There are several standard functors defined between $D^b_c(X)$ and $D^b_c(Y)$. i.e. $f_!, f_*, f^!, f^*, \mathbb{D}_X$ and $\mathbb{D}_Y$. Besides the usual adjoint properties, there are several isomorphisms of functors:

$$f^* \cong \mathbb{D}_X f^! \mathbb{D}_Y,$$

$$f_* \cong \mathbb{D}_X f_! \mathbb{D}_Y,$$

$$\mathbb{D}_X \mathbb{D}_X \cong id$$

Supposed that we have a devissage

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

consider the derived category of l-adic sheaves, these are some standard setting, we do not repeat them here.cf.[KW] where i is a closed embedding and j is an open embedding and $U = X\backslash Z$, there are more properties involved with the functors $i_!, i^*, i^!, j_!, j_*, j^*$ $\mathbb{D}_Z$, $\mathbb{D}_X$ and $\mathbb{D}_U$. Especially, if we consider the perverse t-structure on the corresponding derived categories of constructible sheaves, some properties of these functors are summarized in the following theorem. (cf [BBD])

Theorem 1. *Some formal properties of functors:*

1) $i_!$, j^* are t-exact, $i^!$, j_* are left t-exact, i^*, $j_!$ are right t-exact.

2) if f is affine then f_* is right t-exact and $f_!$ is left t-exact.

3) if j is an affine embedding, then j_*,$j_!$ are t-exact. especially, ${}^pH(j_*) = j_*$ and ${}^pH(j_!) = j_!$.

In case we have a devissage, there is one more convenient way to characterize the perverse t-structure on $D^b_c(X)$:

$$\begin{aligned} D^{b,\leq 0}_{perv}(X) &= \{\mathcal{F} \in D^b_{perv}(X) | i^*\mathcal{F} \in D^{b,\leq 0}_{perv}(Z), j^*\mathcal{F} \in D^{b,\leq 0}_{perv}(U)\} \\ D^{b,\geq 0}_{perv}(X) &= \{\mathcal{F} \in D^b_{perv}(X) | i^!\mathcal{F} \in D^{b,\geq 0}_{perv}(Z), j^!\mathcal{F} \in D^{b,\geq 0}_{perv}(U)\} \end{aligned}$$

The above characterization is very useful; especially, in case we have a stratification consisting of affine strata.

Example 1. *Let G be a simple algebraic group, B be a Borel subgroup, G/B be the flag variety. The B-orbits on G/B are affine and parameterized by the Weyl group W.i.e. $G/B = \bigcup_{w \in W} A_w$. Moreover, $A_w \cong \mathbb{A}^{l(w)}$ is the affine space, where $l(w)$ is the* *length of w. Let* $j_w : A_w \mapsto G/B$ *be the natural embedding, then it is an affine map. By Theorem 1,* $j_{w!}(\mathbb{C}[l(w)])$*,*$j_{w*}(\mathbb{C}[l(w)])$ *are indecomposable perverse sheaves, there is a natural morphism* $j_{w!}(\mathbb{C}[l(w)]) \mapsto j_{w*}(\mathbb{C}[l(w)])$*, the image is irreducible which will be denoted by* $j_{w!*}(\mathbb{C}[l(w)])$*. These are the standard objects, costandard objects and irreducible objects in* $Per_N(G/B)$*.*

2.2 Mirkovic-Vilonen Duality

Let G be a reductive algebraic group over some algebraic closed field, B be a Borel subgroup, T be a maximal torus. We have two lattices: the character lattice $\Lambda = Hom(T, G_m)$ and the cocharacter lattice $\check{\Lambda} = Hom(G_m, T)$. Let $R \subset \Lambda$ be the root system and $\check{R} \subset \check{\Lambda}$ be the coroot system. The reductive algebraic groups G were determined by the root datum $(\Lambda, R; \check{\Lambda}, \check{R})$. Associated to a root datum $(\Lambda, R; \check{\Lambda}, \check{R})$, there is another root datum $(\check{\Lambda}, \check{R}; \Lambda, R)$ which determins another reductive algebraic group–the Langlands dual group G^L of G.

We first review some basic notations and results about the irreducible finite dimensional representations of G. They are parameterized by Λ^+–the dominant weights and we have a decomposition of tensor product.

$$V_{\lambda_1} \otimes V_{\lambda_2} \cong \sum_{\lambda_3} m_{\lambda_3}^{\lambda_1,\lambda_2} V_{\lambda_3}$$

where $m_{\lambda_3}^{\lambda_1,\lambda_2} \in \mathbb{Z}^{\geq 0}$. Let χ_λ be the character of the representations V_λ, then the equation gives us:

$$\chi_{\lambda_1} \cdot \chi_{\lambda_2} = \chi_{V_{\lambda_1} \otimes V_{\lambda_2}} = \sum_{\lambda_3} m_{\lambda_3}^{\lambda_1,\lambda_2} \chi_{\lambda_3}$$

These equations gives a multiplication on class functions on G.

Let $\mathcal{K}$ denote a p-adic field or $\mathbb{C}((t))$, $\mathcal{O}$ denote the p-adic integer or $\mathbb{C}[[t]]$. The well known Satake isomorphism states an algebraic isomorphim:

$$\mathbb{C}(G^L(\mathcal{K})//G^L(\mathcal{O})) \cong \text{Class Functions on} G$$

where the multiplication on L.H.S. is convolution, on the R.H.S. is the multiplication we introduced before.

On the L.H.S, due to the sheaves-functions principle, every $G^L(\mathcal{O})$-equivariant perverse sheaf produce a spherical function on $G^L(\mathcal{K})$, while on the R.H.S., the characters of finite dimensional irreducible representations of $G(\mathbb{C})$ form a basis of class functions. The Satake isomorphism now can be understood as :

$$K(Perv_{G^L(\mathcal{O})}(Gr^L)) \cong K(Rep(G)).$$

In [MV], they provided a categorical version of the above results, i.e. the following categories are equivalent as tensor categories.

$$Perv_{G^L(\mathcal{O})}(Gr^L) \cong Rep(G) \cong coh([pt/G]).$$

where $[pt/G]$ is the algebraic stack[2] classifies G -torsors. Noticed that the irreducible objects on $Perv_{G_{\mathcal{O}}}(Gr)$ are parameterized by the same index. We denote them by IC_λ, for any $\lambda \in \Lambda^+$.

[2]Here we do not really need the notion of algebraic stack, the only thing we need to keep in mind is $coh[X/G] \cong coh^G(X)$

Chapter 3

Affine Hecke Algebra

3.1 Kazhdan-Lusztig Results

Let G be an adjoint reductive group, F be a local field, the affine Hecke algebra by definition is $\mathcal{H}_{aff} \cong C(G(F)//I)$, where I is Iwahori-subgroup. In this subsection, we will review some well known results.

Let W_{aff} be the affine Weyl group with generators $s_i, i = 0, 1, ...n.$ and relations

$$\begin{aligned} {s_i}^2 &= 1 \\ \underbrace{s_i s_j \ldots}_{a_{i,j}\text{copies}} &= \underbrace{s_j s_i \ldots}_{a_{i,j}\text{copies}} \end{aligned}$$

where $(a_{i,j})$ is the affine Cartan matrix associated to G.

Associated to this Coxeter group, we can define the affine Hecke algebra $\mathcal{H}_{aff}$ with parameter q as following: it has generators $T_i; i = 0, 1...n$ over $\mathbb{C}[q^{1/2}, q^{-1/2}]$ such that

$$(T_i + 1)(T_i - q) = 0$$

$$\underbrace{T_i T_j ...}_{a_{i,j}\text{copies}} = \underbrace{T_j T_i ...}_{a_{i,j}\text{copies}}$$

It is well known that the admissible representations of $G(F)$ are in one-one correspondence with the finite dimensional representations of some Hecke algebras. Understanding the representations of $G(F)$ is thus reduced to understanding the finite dimensional modules of Hecke algebra. Especially, the admissible representations with non-zero I-fixed vectors are in 1-1 correspondence with the finite dimensional modules of the affine Hecke algebra. Deligne and Langlands proposed a classification of such representations which was proved by Kazhdan and Lusztig in [KL1].

To make our statement more precisely, we would like to fix some notations. Let G be an algebraic group, B be a Borel subgroup, N be a unipotent radical of B, B^- be opposite Borel subgroup of B and N^- be the unipotent radical. Let $\mathfrak{g}^L$ be the Lie algebra of G^L, $\mathcal{N}^L$ be the nilpotent cone, $\pi_{\mathcal{N}^L} : \widetilde{\mathcal{N}^L} \mapsto \mathcal{N}^L$ be the Springer resolution, $\pi_{\mathfrak{g}^L} : \widetilde{\mathfrak{g}^L} \mapsto \mathfrak{g}^L$ be the Grothendieck simultaneous resolution, $St^L = \widetilde{\mathfrak{g}^L} \underset{\mathfrak{g}^L}{\times} \widetilde{\mathcal{N}^L}$ be the Steinberg variety, $p_L : St^L \mapsto \widetilde{\mathfrak{g}^L}$ and $p_R : St^L \mapsto \widetilde{\mathcal{N}^L}$ All these varieties have natural G^L action, so we can consider the G^L-equivariant coherent sheaves on these vareities.

The Deligne Langlands conjecture states that these representations are parameterized by (s, u, ρ), where s is semisimple, u is unipotent such that $sus^{-1} = u^q$ and ρ is a representation of some finite groups. In this section, we will quickly review several important steps of the proof.

In [L], Lusztig started studying the K theory of the Springer fiber. Especially, he proved the following:

Theorem 2. *$K(coh([\widetilde{\mathcal{N}^L}/(G^L \times G_m)]))$ has an $\mathcal{H}_{aff}$ module structure, which is isomorphic to the anti-spherical module $M_{asp} = Ind(Sgn)$, where Sgn stands for the sign representation of $\mathcal{H}$ (the Hecke algebra associated to Weyl group).*

In the sequence papers[CG][KL1], they provided a K-theoretical interpretation of $\mathcal{H}_{aff}$.i.e. They proved:

Theorem 3. *$K(coh([St^L/(G^L \times G_m]))$ has an algebra structure such that it is isomorphic to the affine Hecke algebra.*

3.2 Categorical Version:Bezrukavnikov Correspondence

The Satake isomorphism states that we have the algebra isomorphism $C(G(\mathcal{K})//\mathcal{O}) \cong K(Rep(G^L))$, while the geometric Langlands duality upgraded this isomorphism to categorical level:

$$Perv_{G(\mathcal{O})}(Gr) \cong Rep(G^L) \cong coh([pt/G^L])$$

In the previous subsection we have

$$\mathcal{H}_{aff} \cong C(G(F)//I) \cong K(coh([St^L/(G^L \times G_m)])).$$

Motivated by the geometric Langlands duality, in [CG], Ginzburg proposed a categorical version of the above isomorphism, which was proved by Bezrukavnikov and claimed in [B]. We will recall this and several related results. For saving notation in future, we would like to fix some conventions which are abstract nonsense.

Convention 1. *Suppose that we have several pairwise equivalent categories $\mathcal{A}_j, j = 1, .., k.$, we can take an abstract category $\mathcal{A}$ which is equivalent to these categories(and we will fix the corresponding equivalence F_j), and call it the abstract model of these categories and these $\mathcal{A}_j$ are realizations of $\mathcal{A}$. If furthermore, these $\mathcal{A}_j, j = 1, .., k.$ are triangulated categories with t-structure $t_j, j = 1, ..k.$ respectively, we will denote $\mathcal{A}_{t_j}$ the abstract model $\mathcal{A}$ with t-structure translated by the equivalence F_j. In more concrete contexts, i.e. supposed that we have two realizations of $\mathcal{A}$ in coherent sheaves context and perverse sheaves context, we will use ${}_{coh}\mathcal{A}$ and ${}_{perv}\mathcal{A}$ to remind reader which realization we are thinking about.*

Let $Fl_{aff} = G(\mathcal{K})/I$ be the affine flag variety,(For more precise description of affine flag variety and related topics, cf [B]) the I-orbits and I_0(the unipotent radical of I)-orbits on Fl_{aff} are parameterized by W_{aff}. There are I_0 equivariant perverse sheaves category $Perv_{I_0}(Fl_{aff})$ and I-equivariant perverse sheaves category $Perv_I(Fl_{aff})$. The irreducible objects of these two categories are both parameterized by W_{aff}. Since results concerning perverse sheaves on the affine flag variety will be intensively used below, we briefly review them in this subsection.

For each orbit $I_0 wI/I$, we denote by j_w the natural embedding. Let $J_{w*} = j_{w*}(\underline{\overline{\mathbb{Q}_l}})$, $J_{w!} = j_{w!}(\underline{\overline{\mathbb{Q}_l}})$,and $J_{w!*} = j_{w!*}(\underline{\overline{\mathbb{Q}_l}})$ be the costandard, standard and irreducible objects in the category $Per_{I_0}(Fl)$(by slightly abusing the notations, the corresponding objects in $Perv_I(Fl)$). The basic property of these objects summarized in the following theorem:

Theorem 4. *The following results are proved in [2]:*

1) Let $M \in {}_{perv}\mathfrak{T}^{\leq 0}_{perv}$ (resp. ${}_{perv}\mathfrak{T}^{\geq 0}_{perv}$), then for any $w \in W, M \star J_{w*}$ (resp. $M \star J_{w!}$) $\in {}_{perv}\mathfrak{T}^{\leq 0}_{perv}$ (resp. ${}_{perv}\mathfrak{T}^{\geq 0}_{perv}$)

2) $J_{w*} \star J_{w!} = J_e$;

3) $J_{w!} \star J_{w'!} = J_{ww'!}, J_{w*} \star J_{w'*} = J_{ww'*}$ if l(ww')=l(w)+l(w').

where $\star$ stands for the convolution.

Fix a generic character $\psi : N \mapsto G_a$, it defines a character of I_0. In [ArBe], they introduced the Iwahori-Whittaker(i.e. (I_0, ψ)-equivriant) perverse sheaves category $Perv_{IW}$, which turns out to be a quotient category of the I-equivariant perverse sheaves category $Perv_I(Fl_{aff})$. More precisely, irreducible objects in $Perv_{IW}(Fl_{aff})$ are parameterized by $\lambda \in \Lambda$, which can be identified with the minimal length representatives in $W\backslash W_{aff}$. Let $\triangle_v, \nabla_v$ $v \in W\backslash W_{aff}$ be the standard, costandard obejcts in the Iwahori-Whittaker perverse sheaves category. Especially, there is a special object $\triangle_0 \in Perv_{IW}(Fl_{aff})$, such that the following convolution functor is a quotient functor:

$$conv : Perv_I(Fl_{aff}) \mapsto Perv_{IW}(Fl_{aff}); \mathfrak{F} \mapsto \triangle_0 \star \mathfrak{F} \tag{3.1}$$

More precisely, in [AB], they proved the following lemma

Lemma 1. *a) We have*

$$\triangle_0 \star J_{w!*} = 0 \Leftrightarrow w \notin W \backslash W_{aff}$$

b) We have

$$\triangle_0 \cong \nabla_0$$

c) For $w = uv, u \in W, v \in W\backslash W_{aff}$, we have

$$\triangle_0 \star J_{w!} \cong \triangle_v$$

$$\triangle_0 \star J_{w*} \cong \nabla_v$$

Slightly modifying the argument in [AB], we can prove the following result.

Theorem 5. *The convolution functor:*

$$conv' : D^b_{I_0}(Fl_{aff}) \mapsto D^b_{IW}(Fl_{aff}); \mathcal{F} \mapsto \triangle_0 \star \mathcal{F} \tag{3.2}$$

is t-exact. (both sides with the tautological t-structures.)

As the subsection title suggested, there are some categorical results corresponding to the K-theory results, which are due to Bezrukavnikov et.al. The main results we will recall is the following[AB][B1]:

Theorem 6. *In the following equivalences, the superscript ^{mix} standards for the mixed version. (cf. [ABG].)*

1) $\mathbb{B}_{asp} : D^b(coh([\widetilde{\mathcal{N}^L}/G^L])) \cong D^b(Perv_{IW}(Fl)) \cong D^b({}^fP)$

$\mathbb{B}^{mix}_{asp} : D^b(coh([\widetilde{\mathcal{N}^L}/(G^L \times G_m)])) \cong D^b(Perv^{mix}_{IW}(Fl)) \cong D^b({}^fP^{mix})$

2) $\mathbb{B}_{af} : D^b(coh([St^L/G^L])) \cong D^b(Perv_{I_0}(Fl))$

$\mathbb{B}^{mix}_{af} : D^b(coh([St^L/(G^L \times G_m)])) \cong D^b(Perv^{mix}_{I_0}(Fl))$

3) $\mathbb{B}_{aa} : D^b(coh([\mathcal{N}^L/G^L])) \cong D^b({}^fP^f)$

$\mathbb{B}^{mix}_{aa} : D^b(coh([\mathcal{N}^L/(G^L \times G_m)])) \cong D^b({}^fP^{mix,f})$

where ${}^fP$ *and* ${}^fP^f$ *will be explaining in following.*

We will denote the abstract model of the triangulated category in the first line by $\mathcal{S}$ (resp. $\mathcal{S}^{mix}$), the triangulated categories in the second line by $\mathcal{T}$ (resp.$\mathcal{T}^{mix}$) and the triangulated category in the third line by $\mathcal{K}$ (resp. $\mathcal{K}^{mix}$). Besides the coherent realization and (Iwahori-Whittaker) perverse sheaves realization, we would like to emphasize that $\mathcal{S}$ has another realization which we call the quotient realization and denote by ${}_{quo}\mathcal{S} \cong D^b({}^fP)$. This is a result in [AB], where ${}^fP$ is a Serre quotient category of $perv_I(Fl)$. Furthermore, in [B1], there is a Serre quotient category of ${}^fP$, followed their notation, which we denoted by ${}^fP^f$. This category provides us a quotient realization of the triangulated category $\mathcal{K}$.

To help reader keep the previous convention in mind, we will introduce two commutative diagram, which are theorems due to Bezrukavnikov et. al.:(and we wrote out the corresponding realizations to help reader familiar with those convention)

Theorem 7. *We have following commutative diagrams[4]:*

1)
$$\begin{array}{ccc}
{}_{coh}\mathcal{T}(\cong D^b(coh([St^L/G^L]))) & \xleftrightarrow{\mathbb{B}_{af}} & {}_{perv}\mathcal{T}(\cong D^b(Perv_{I_0}(Fl))) \\
\downarrow{\scriptstyle Rp_!} & & \downarrow{\scriptstyle \triangle_0\star} \\
{}_{coh}\mathcal{S}(\cong D^b(coh([\widetilde{\mathcal{N}^L}/G^L]))) & \xleftrightarrow{\mathbb{B}_{sp}} & {}_{perv}\mathcal{S}(\cong D^b(Perv_{IW}(Fl)))
\end{array}$$

2)
$$\begin{array}{ccccc}
{}_{coh}\mathcal{S}(\cong D^b(coh([\widetilde{\mathcal{N}^L}/G^L]))) & \longleftrightarrow & {}_{quo}\mathcal{S}(\cong D^b({}^fP)) & \longleftrightarrow & {}_{perv}\mathcal{S}(\cong D^b(Perv_{IW}(Fl))) \\
\downarrow{\scriptstyle R\pi_{\mathcal{N}^L!}} & & \downarrow{\scriptstyle quotient} & & \\
{}_{coh}\mathcal{K}(\cong D^b(coh([\mathcal{N}^L/G^L]))) & \xleftrightarrow{\mathbb{B}_{aa}} & {}_{quot}\mathcal{K}(\cong D^b({}^fP^f)) & &
\end{array}$$

Chapter 4

Affine Braid Group Action on Triangulated Categories

4.1 Affine Braid Group

Given a Coxeter group $\mathbb{G} =< s_i; i = 1, 2, ...n > / < s_i^2 = 1, \underbrace{s_i s_j ...}_{a_{i,j}\text{copies}} = \underbrace{s_j s_i ...}_{a_{i,j}\text{copies}} >$, where $a_{i,j}$ are some fixed integers (may include ∞), there is a braid group $\mathcal{B}$ associated to it which is generated by the symbol $T_i, i = 1, ..., n.$ with the same relations except $T_i^2 = 1$. For any element $w \in \mathbb{G}$, fix a minimal length representative of $w = s_{i_1} ....s_{i_p}$, there is a well defined element $T_w = T_{i_1} T_{i_2} ...T_{i_p} \in \mathcal{B}$. This assignment gives an embedding $\iota : \mathbb{G} \mapsto \mathcal{B}$ (not a group morphism), and we will denote by $\mathcal{B}^+$ (resp $\mathcal{B}^-$)the semi-group generated by $\iota(\mathbb{G})$ (resp. $\iota(\mathbb{G})^{-1}$).

Definition 4. *A weak action of $\mathcal{B}$ on a category $\mathcal{C}$ is an assignment $b \in \mathcal{B} \mapsto R_b \in Auto(\mathcal{C})$ such that $R_b \circ R_{b'} \cong R_{bb'}$*

The following lemma is obvious:

Lemma 2. *To define a weak braid group action, we only need the assignment $\iota(w) \mapsto R_{\iota(w)} \in Auto(\mathcal{C})$ satisfying:*

$$R_{\iota(w)} \circ R_{\iota(w')} \cong R_{\iota(ww')}, \text{ if } l(ww') = l(w) + l(w')$$

Given a rank n root system (Λ, R) with a set of fixed simple roots $\{\alpha_j \in \Lambda, j = 1, 2, ..n\}$, there are n simple reflections s_j. Besides these simple reflections, there is one more affine reflection s_0 which is the reflection defined by the affine root α_0. The group $W_{aff} =< s_i, i = 0, ..n >$ is a Coxeter group, hence there is an affine braid group $\mathcal{B}_{aff}$ associated to it.

Suppose we have two triangulated categories with t-structures $(\mathcal{C}_1, \tau_1), (\mathcal{C}_2, \tau_2)$ and a triangulated functor $F : \mathcal{C}_1 \mapsto \mathcal{C}_2$, furthermore, if we assume there is an (affine) braid group $\mathcal{B}_{aff}$ action $\mathcal{C}_1$.

Definition 5. *The t-structure τ_1 is called braid left t-(resp. right t exact) positive for any $M \in C_1{}^{\leq 0}_{\tau_1}, b \in \mathcal{B}^+_{aff}$(resp. $M \in C_1{}^{\geq 0}_{\tau_1}, b \in \mathcal{B}^-_{aff}$), we have $b.M \in C_1{}^{\leq 0}_{\tau_1}$ (resp. $b.M \in C_1{}^{\geq 0}_{\tau_1}$.) And it is called braid positive, if it is both left and right braid positive.*

Definition 6. *The t-structure τ_1 is called braid positive above τ_2 by F, if τ_1 is braid positive and:*

1) for any $M \in C_1{}^{\leq 0}_{\tau_1}, b \in \mathcal{B}^+_{aff}$, we have $F(b.M) \in C_2{}^{\leq 0}_{\tau_2}$.

2) for any $M \in C_1{}^{\geq 0}_{\tau_1}, b \in \mathcal{B}^-_{aff}$, we have $F(b.M) \in C_2{}^{\geq 0}_{\tau_2}$.

In next section, we will introduce some examples to help reader understand those definitions.

4.2 Affine Braid Group Action I: Perverse Sheaves Version

Let us consider the perverse sheaves realization ${}_{perv}\mathcal{T} \cong D^b(perv_{I_0}(Fl))$ of $\mathcal{T}$ and with the perverse t-structure τ_{perv}.

Due to Theorem 4, the functor $L_w : {}_{perv}\mathcal{T} \mapsto {}_{perv}\mathcal{T}$ defined by mapping $\mathcal{F} \in {}_{perv}\mathcal{T}$ to $J_{w*} \star \mathcal{F}$ is an equivalence and moreover we have

$$L_w \circ L_{w'} \cong L_{ww'}, \text{if } l(ww') = l(w) + l(w').$$

i.e. We have a weak (affine) braid group action of $\mathcal{B}_{aff}$ acts on ${}_{perv}\mathcal{T}$ (hence $\mathcal{T}$). Similarly ,we can define R_w by convolution from the R.H.S, which introduced another affine braid group action. From the obvious reason, we call them the left affine braid group action and the right affine braid group action.

From Theorem 4, it is easy to see the perverse t-structure on $\mathcal{T}$ is braid positive (for both left and right action). We will provide various examples of the Definition 3 and Definition 4 which will be used in future.

Example 2. *The perverse t-structure τ_{perv} on ${}_{perv}\mathcal{T}$ is braid positive above the perverse t-structure ς_{perv} on ${}_{perv}\mathcal{S}$ for both left braid group action and right braid group action.*

Proof This is an almost trivial consequence of Theorem 4 and Theorem 5. i.e.

In the perverse sheaves realization, the functor from ${}_{perv}\mathcal{T}$ to ${}_{perv}\mathcal{S}$ is the convolution functor $conv'(--) = \triangle_0 \star --$ which is also exact with respect to the perverse t-structures. For any $M \in {}_{perv}\mathcal{T}^{\leq 0}$, we have $M \star J_{w*} \in_{perv} \mathcal{T}^{\leq 0}$, hence $\triangle_0 \star M \star J_{w*} \in_{perv} \mathcal{T}^{\leq 0}$. For any $M \in {}_{perv}\mathcal{T}^{\geq 0}$, we have $M \star J_{w*} \in_{perv} \mathcal{T}^{\geq 0}$, hence $\triangle_0 \star M \star J_{w*} \in_{perv} \mathcal{T}^{g\leq 0}$. i.e. τ_{perv} is right braid positive above ς_{perv}. The left braid positivity can be proved by similar argument.

Example 3. *There is a braid group action on $\mathcal{S}$ which inherited from the right braid group action on $\mathcal{T}$. The perverse t-structure ς_{perv} on ${}_{perv}\mathcal{S} \cong D^b(Perv_{IW}(Fl_{aff}))$ is braid positive w.r.p.t. this braid group action.*

Proof This follows from the proof of Example 2.

Example 4. *With the (affine) braid group action on $\mathcal{S}$, the perverse t-structure ς_{perv} is braid positive above the perverse t-structure κ_{perv} on $\mathcal{K}$.*

Proof It is almost trivial in the quotient realization. i.e.In this realization, we have ${}_{quot}\mathcal{S} \cong D^b({}^fP)$, ${}_{coh}\mathcal{K} \cong D^b({}^fP^f)$. By Theorem 4, this statement is trivial.

Now, we would like to introduce another easy example of braid positive t-structure. There is a special maximal commutative subalgebra in $\mathcal{H}_{aff}$, which is generated by the characteristic functions $Ch_{I\lambda I}$ on $G(\mathcal{K})$ for $\lambda \in \Lambda$. On the categorical level, we also have a special subset of objects in $Perv_{I_0}(Fl)$, i.e. the Wakimoto sheaves $W_\lambda, \lambda \in \Lambda$. To define them, we need to take an (any) decomposition of $\lambda = \lambda_1 - \lambda_2$ where $\lambda_1, \lambda_2 \in \Lambda^+$, we define $W_\lambda = J_{\lambda_1 *} \star J_{-\lambda_2 !}$.

Lemma 3. *1) W_λ is a well defined perverse sheaf.*

2) $W_\lambda \star W_\mu = W_{\lambda+\mu}$

Proof

1) Let $\lambda = \lambda_1 - \lambda_2 = \nu_1 - \nu_2$ be two decompositions. (i.e. $\lambda_j, \nu_j \in \Lambda^+$). Since $J_{\lambda_1+\nu_2,!} = J_{\lambda_1 !} \star J_{\nu_2 !} = J_{\lambda_2 !} \star J_{\nu_1 !} = J_{\lambda_2+\nu_1 !}$, we get $J_{\lambda_1 !} \star J_{\lambda_2 *} = J_{\lambda_1 !} \star J_{\nu_2 !} \star J_{\nu_2 *} \star J_{\lambda_2,*} = J_{\lambda_1 !} \star J_{\nu_2 !} \star J_{\lambda_2+\nu_2,*} = J_{\lambda_2 !} \star J_{\nu_1 !} \star J_{\lambda_2+\nu_2,*} = J_{\nu_1 !} \star J_{\lambda_2 !} \star J_{\lambda_2 *} \star J_{\nu_2 *} = J_{\nu_1 !} \star J_{\nu_2 *}$

2) This follows from part 1).

After introducing the Wakimoto sheaves, we define new t-structure τ_{new} [1] on ${}_{perv}\mathcal{T}$ by

$${}_{perv}\mathcal{T}^{\leq 0}_{new} = \{M | M \star W_\lambda \in_{perv} \mathcal{T}^{\leq 0}_{perv} \text{ for any } \lambda \in \Lambda\}$$

and new t-structure ς_{new} [2] on ${}_{perv}\mathcal{S}$ by

$${}_{perv}\mathcal{S}^{\leq 0}_{new} = \{M | M \star W_\lambda \in_{perv} \mathcal{S}^{\leq 0}_{perv} \text{ for any } \lambda \in \Lambda\}$$

Example 5. *a) The t-structure τ_{new} on $\mathcal{T}$ is braid positive with respect to the left hand braid action.*

b) The t-structure τ_{new} is braid positive above the t-structure ς_{new} on $\mathcal{S}$.

Proof

a) Let $M \in {}_{perv}\mathcal{T}^{\leq 0}_{new}$, for any $\lambda \in \Lambda$ and $w \in W_{aff}$, we have $(J_{w*} \star M) \star W_\lambda = J_{w*} \star (M \star W_\lambda) \in {}_{perv}\mathcal{T}^{\leq 0}_{perv}$. i.e. τ_{new} is left braid positive.

[1] Here we secretly used the fact that it is a well defined t-structure. This was proved in [FG]
[2] The fact that it is a t-structure will be proved in section 4

For $M \in {}_{perv}\mathcal{T}^{\geq 0}_{new}$, we need prove $J_{w!} \star M \in {}_{perv}\mathcal{T}^{\geq 0}_{new}$, however for any $N \in {}_{perv}\mathcal{T}^{\leq -1}_{new}$ we have $Hom(J_{w*} \star N, M) = 0$ i.e. $Hom(N, J_{w!} \star M) = 0$ for any $N \in {}_{perv}\mathcal{T}^{\leq -1}_{new}$, which proves that τ_{new} is right braid positive.

b) The functor is $\triangle_0 \star (--) : {}_{perv}\mathcal{T} \mapsto {}_{perv}\mathcal{S}$ is t-exact with perverse t-structure on both sides, so for any $M \in {}_{perv}\mathcal{T}^{\leq 0}_{new}$, we have $(\triangle_0 \star M) \star W_\lambda = \triangle_0 \star (M \star W_\lambda) \in {}_{perv}\mathcal{S}^{\leq 0}_{perv}$, for any λ. i.e. $\triangle_0 \star M \in {}_{perv}\mathcal{S}^{\leq 0}_{new}$. In other words, $\triangle_0 \star (--)$ is left t-exact with respect to the new t-structures on $\mathcal{T}$ and $\mathcal{S}$.

To prove that $\triangle_0 \star (--)$ is right t-exact with respect to the new t-structures on $\mathcal{T}$ and $\mathcal{S}$, notice that there is a left adjoint functor F of $\triangle_0 \star (--)$ [3] such that $F(\mathcal{S}^{\leq 0}_{perv}) \subset \mathcal{T}^{\leq 0}_{perv}$, [4] we have that for any $N \in \mathcal{S}^{\leq 0}_{new} \Rightarrow F(N \star W_\lambda) \cong F(N) \star W_\lambda \in \mathcal{T}^{\leq 0}_{perv}$. i.e. $F(N) \in \mathcal{T}^{\leq 0}_{new}$. So for $M \in \mathcal{T}^{\geq 1}_{new}$, $N \in \mathcal{S}^{\leq 0}_{new}$, we have $Hom(N, \triangle_0 \star (M)) = Hom(F(N), M) = 0$. i.e. We have $\triangle_0 \star (--)$ is right t-exact.

We now recall some facts about perverse sheaves on stratified spaces.

For F $\in D_I(Fl_{aff})$, let $W^*_F = \{w \in W_{aff} | j^*_w F \neq 0\}$; $W^!_F = \{w \in W_{aff} | j^*_w F \neq 0\}$. The following result was proved in [AB]:

Theorem 8. *For $X \in D_I(Fl_{aff})$ there exists a finite subset $S \subset W_{aff}$, such that for all $w \in W_{aff}$ we have*

$$W^!_{j_{w*} \star X}, W^*_{j_{w!} \star X} \subset w \cdot S;$$

[3] This follows from the coherent realization of $\mathcal{T}$, since in this realization $\triangle_0$ is equivalent to the push down $R\pi_{\widetilde{\mathcal{N}L}_!}$.

[4] This is because we have adjoint pair $(F, \triangle_0 \star (--))$ and $\triangle_0 \star (--)$ is exact with respect to the perverse t-structures on $\mathcal{T}$ and $\mathcal{S}$

$$W^!_{X\star j_{w*}}, W^*_{X\star j_{w!}} \subset S\cdot w;$$

The above notation can be applied to $D_{IW}(Fl_{aff})$. Slightly modifying the proof of the above theorem, we can prove (and to save notation, we will identify Λ with $W\backslash W_{aff}$):

Theorem 9. *For $X \in D_{IW}(Fl_{aff})$ there exists a finite subset $S \subset \Lambda$, such that for all $w \in W_{aff}$ we have*

$$W^!_{X\star j_{w*}}, W^*_{X\star j_{w!}} \subset S\cdot w \subset \Lambda;$$

4.3 Affine Braid Group Action II: Coherent Sheaves Version

The variety $\widetilde{\mathfrak{g}^L}$ is smooth, and the map $\pi_{\mathfrak{g}^L}$ is proper and generically finite of degree $|W|$, where W is the Weyl group. It factors as a composition of a resolution of singularities $\pi'_{\mathfrak{g}^L}: \widetilde{\mathfrak{g}^L} \mapsto \mathfrak{g}^L \times_{\mathfrak{h}^L/W} \mathfrak{h}^L$ and the finite projection $\mathfrak{g}^L \times_{\mathfrak{h}^L/W} \mathfrak{h}^L \mapsto \mathfrak{g}^L$; here $\mathfrak{h}^L$ is the Cartan algebra of $\mathfrak{g}^L$. Let $\mathfrak{g}^{L,reg} \subset \mathfrak{g}^L$ denote the subspace of regular (not necessarily semi-simple) elements, and $\widetilde{\mathfrak{g}^{L,reg}}$ be the preimage of $\mathfrak{g}^{L,reg}$ in $\widetilde{\mathfrak{g}^{L,reg}}$; then $\pi'_{\mathfrak{g}^L}$ induces an isomorphism $\widetilde{\mathfrak{g}^{L,reg}} = \mathfrak{g}^L \times_{\mathfrak{h}^L/W} \mathfrak{h}^L$.

We now introduce the affine braid group action. Let $\check{\Lambda}$ be the root lattice of G^L. For $\lambda \in \check{\Lambda}$, we will write $\mathcal{O}(\lambda)$ for the corresponding G^L-equivariant line bundle on the flag variety B, and we set $F(\lambda) = F \otimes \mathcal{O}(\lambda)$ if $F \in D^b(coh(X))$ for some X with natural map mapping to B. Let W be the Weyl group, and set $W_{aff} = W \ltimes \check{\Lambda}$. Then W, W_{aff} are Coxeter groups. Notice that W_{aff} is the affine Weyl group of the group

G^L. It was mentioned above that $\widetilde{\mathfrak{g}^{L,reg}} = \mathfrak{g}^L \times_{\mathfrak{h}^L/W} \mathfrak{h}^L$; thus W acts on this space via its action on the second factor. The formulas $\check{\Lambda} \ni \lambda : F \mapsto F(\lambda), W \in w : F \mapsto w.(F)$ are easily shown to define an action of W_{aff} on the category of coherent sheaves on $\widetilde{\mathfrak{g}^{L,reg}}$.

Recall that to each Coxeter group one can associate an Artin braid group; Let B_{aff} denote the group corresponding to W_{aff} . It admits a topological interpretation, as the fundamental group of the space of regular semi-simple conjugacy classes in the universal cover of the dual group G. For w $\in W_{aff}$ consider the minimal decomposition of w as a product of simple reflection, and take the product of corresponding generators of B_{aff} . This product is well known to be independent on the choice of the decomposition of w, thus we get a map $W_{aff} \in B_{aff}$ which is one-sided inverse to the canonical surjection $B_{aff} \in W_{aff}$. We denote this map by $w \in T_w$. The map is not a homomorphism, however, we have $T_{uv} = T_u \cdot T_v$ for any u, v $\in W_{aff}$ such that $l(uv) = l(u) + l(v)$, where $l(w)$ denotes the length of the minimal decomposition of w. Let $B^+_{aff} \subset B_{aff}$ be the sub-monoid generated by T_w, $w \in W_{aff}$. For a simple reflection $s_i \in W$ let $S_i \in \widetilde{\mathfrak{g}^L}^2$ be the closure of the graph of s_i acting on $\widetilde{\mathfrak{g}^{L,reg}}$. We let $S_i^{'}$ denote the intersection of S_i with $\tilde{\mathcal{N}}^2$. Let $pr_l^i : S_i \mapsto \widetilde{\mathfrak{g}^L}, pr_l^{'i} : S_i \mapsto \tilde{\mathcal{N}}^L, pr_r^i : S_i \mapsto \widetilde{\mathfrak{g}^L}, pr_r^{'i} : S_i \mapsto \tilde{\mathcal{N}}^L$, where l(resp. r) standards for the the projections to the first factor(resp. second factor). Let $\check{\Lambda}^+ \subset \check{\Lambda}$ be the set of dominant weights in Λ. For a scheme Y over $\mathfrak{g}$ we set $\tilde{Y}' = \widetilde{\mathcal{N}^L} \times_{\mathfrak{g}} Y$, $\tilde{Y} = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} Y$. We have the following theorem which claimed in [B] and partially proved in [BM]

Theorem 10. *a) There exists an (obviously unique) action of B_{aff} on $D(coh(\widetilde{\mathfrak{g}}))$,(resp. $D(coh(\widetilde{\mathcal{N}^L}))$) such that for $\lambda \in \Lambda^+ \subset \Lambda \subset W_{aff}$ we have $\tilde{\lambda} : F \mapsto F(\lambda)$ and for a simple reflection $s_i \in W$ we have $\tilde{s}_i : F \mapsto (pr^i_{l*}).(pr^i_{r*}2)F$. (respectively, $\tilde{s'_i} : F \mapsto (pr'^{i}_{1*}).(pr'^{i}_{r*})F$).*

b) This action induces an action on $D(coh(\tilde{Y})), D(coh(\tilde{Y'}))$ for any scheme Y over g such that $Tor^i_{\mathcal{O}_\mathfrak{g}}(\mathcal{O}_{\tilde{\mathfrak{g}}}, \mathcal{O}_Y) = 0$, respectively $Tor^i_{\mathcal{O}_{\mathfrak{g}^L}}(\mathcal{O}_{\widetilde{\mathcal{N}^L}}, \mathcal{O}_Y) = 0$, for $i > 0$.

c) There is a unique t-structure on $D^b(coh(\tilde{Y}))$(resp. $D^b(coh(\tilde{Y'}))$) which is braid positive above the tautological t-structure on $coh(\tilde{Y})$(resp. $D^b(coh(Y'))$).

d) The above results have an equivariant version and dg-fiber version(i.e. Without the higher torsion vanishing condition).

Remark 2. *Though we will not replicate the proof of the above results, I would like to emphasize one result coming from in the proof, the braid positive t-structure on $D^b(coh(\widetilde{S}))$ above the tautological t-structure on $D^b(coh(S))$ is characterized by:*

$$D^{b,\leq 0}(coh(\widetilde{S})) = \{M | \pi_*(b'.M) \in D^{b,\leq 0}_{taut}(coh(S)), \text{ for any } b' \in B^+_{aff}\}$$

And moreover, the above results can be modified to the other t-structure on $D^b(coh(S))$.

Remark 3. *The induced action of B_{aff} on the Grothendieck group $K(\widetilde{\mathcal{N}^L})$ factors through W_{aff} . If one passes to the category of sheaves equivariant with respect to the multiplicative group, acting by dilations in the fibers of the projection $\tilde{\mathcal{N}^L} \mapsto B$, then the induced action factors through the affine Hecke algebra H. Furthermore, this construction yields an action of H on the Grothendieck group $K(\pi^{-1}(e))$ for each*

$e \in \mathcal{N}^L$; *these H modules are called the standard H-modules. Thus the Theorem provides a categorification of the standard modules for the affine Hecke algebra.*

Particularly, let us consider the coherent realization ${}_{coh}\mathcal{T}$ of $\mathcal{T}$. Notice that $St^L = \widetilde{\mathfrak{g}^L} \overset{L}{\times}_{\mathfrak{g}^L} \widetilde{\mathcal{N}^L} = \widetilde{\widetilde{\mathfrak{g}^L}}' = \widetilde{\widetilde{\mathcal{N}^L}}$, hence as the perverse sheaves realization, there are two braid group actions on this coherent realization. Moreover, the equivalence $\mathbb{B}_{af}$: ${}_{perv}\mathcal{T} \cong {}_{coh}\mathcal{T}$ is compatible with these braid group actions on both sides[B2].

Similarly, in the coherent realization ${}_{coh}\mathcal{S}$ of $\mathcal{S}$, noticed that $\mathcal{N}^L = \mathfrak{g}^L \underset{\mathfrak{g}^L}{\times} \widetilde{\mathcal{N}^L}$ there is a braid group action, which compatible with the equivalence: $\mathbb{B}_{asp}$: ${}_{coh}\mathcal{S} \cong {}_{perv}\mathcal{S}$[B2]. For more convenience, we can take this braid group action as inherited from the left braid group action on $\mathcal{T}$, so that, the commutative diagram in Theorem 7 is compatible with this left braid group action.

Chapter 5

Two t-structures on Derived Category of Coherent Sheaves over Springer Fiber

5.1 Localization of De Concini-Kac Quantum group at Odd Root of Unity

In this section, we will review some results of [BK],[BK1], and we will follow [BK]'s assumption on q: if it is l-th root of unity then it is primitive of odd order and in case G has a component of type G_2 the order is also prime to 3.

Let $\mathfrak{U}_q$ (resp. U_q)be the De Concini-Kac (resp. Lusztig) integral form of the simply connected quantized enveloping algebra of $\mathfrak{g}^L/\mathbb{C}$. Let $\mathfrak{Z}$ be the center of $\mathfrak{U}_q$, in case q is root of unity, it contains the Harish-Chandra centher $\mathfrak{Z}^{HC}$ and the l-center

$\mathfrak{Z}^l$. The Harish-Chandra homomorphism provids an algebra isomorphism:

$$\mathfrak{Z} \cong \mathfrak{Z}^l \underset{\mathfrak{Z}^l \bigcap \mathfrak{Z}^{HC}}{\otimes} \mathfrak{Z}^{HC} \tag{5.1}$$

A cental character of $\mathfrak{U}_q$ consists of compatible pair (χ, λ) where χ is a character of l-center and λ is a character of Harishi-Chandra center, and we will denote by $\mathfrak{U}_\chi^\lambda - mod$ the category of modules with Harish-Chandra character λ and l-character χ.

For each q (root of unity), there is an algebra morphism $\mathfrak{U}_q \mapsto U_q$ whose image is u_q with algebraic kernel is the augmented ideal of $\mathfrak{Z}^{(l)}$.

In [KB2], they introduce a category of quantum D-module $\mathcal{D}_{B_q}^\lambda(G_q)$ and proved that:

$$D^b(\mathcal{D}_{B_q}^\lambda(G_q)) \cong D^b(\mathfrak{U}_q^\lambda - mod)$$

where λ is integral and regular. If furthermore, let us fix a χ in B_-B and is unipotent, we get:

$$D^b(\mathcal{D}_{B_q}^\lambda(G_q)_\chi) \cong D^b_{\hat{\chi}}(\mathfrak{U}_q^\lambda - mod)$$

where the L.H.S. denote those (complexes of) $\mathcal{D}_{B_q}^\lambda$-modules supported on the Springer fiber of (χ, λ) and the R.H.S. denote $\mathfrak{U}_q^\lambda$ modules which are locally annihilated by a power of the maximal ideal in $\mathfrak{Z}^l$ corresponding to χ (generalized central l-character χ).

By the Azumaya splitting we have

$$\mathcal{D}^{\lambda}_{B_q}(G_q) \cong Qcoh(T^*X)_{\chi}$$

where the latter category is the quasi coherent $\mathcal{O}$-modules on T^*X supported on the Springer fiber of χ (with respect to the usual Springer resolution).

By standard base change argument, the above equivalence can be rewritten as:

$$D^b(\mathfrak{U}^{\lambda}_{\chi} - mod) \cong D^b(Qcoh(Spr_{\chi}))$$

where $Spr'_{\chi} = \{\chi\} \overset{L}{\underset{\mathcal{N}^L}{\times}} \widetilde{\mathcal{N}^L}$.

Moreover, in [BK2], they proved that if the Harish-Chandra character is generalized character, the above equivalence of the form:

$$D^b(\mathfrak{U}^{\hat{\lambda}}_{\chi} - mod) \cong D^b(Qcoh(Spr_{\chi}))$$

where $Spr_{\chi} = \{\chi\} \overset{L}{\underset{\mathfrak{g}^L}{\times}} \widetilde{\mathfrak{g}^L}$ is the derived Springer fiber.

From Theorem 10, the R.H.S. has a affine braid group action, and in a forthcoming paper [K], the tautological t-structure on L.H.S. is braid positive above the tautological t-structure on $D^b(coh(\{\chi\}))$.

5.2 New t-structure Introduced by Frenkel-Gaitsgory

Given a simple algebra $\mathfrak{g}$, the category $\mathcal{O}$ is equivalent to the category of N-equivariant holonomic D-modules on the flag vareity. The Riemann-Hilbert correspondence tells us there is an equivalence:

$$Perv(X) \cong D - mod(X)_{H.R.S}$$

Thus, we have the following equivalence:

$$\mathcal{O} \cong D - mod(G/B)^N \cong Perv_N(G/B)$$

This is the well known localization theorem for $\mathfrak{g}mod$. Suggested by this theory, there are various localization theories in other context. We will briefly recall the results in [FG]. Let $\hat{\mathfrak{g}}$ be the Kac-Moody algebra associated to $\mathfrak{g}$, $\hat{\mathfrak{g}}_{crit} - mod$ be the category of continuous $\hat{\mathfrak{g}} - mod$ at the critcal level, $\hat{\mathfrak{g}}_{crit} - mod_{nilp}$ be the abelian category of $\hat{\mathfrak{g}}_{crit} - mod$, on which the center $\mathfrak{Z}_{\mathfrak{g}} = Z(\widetilde{U}(\hat{\mathfrak{g}})_{crit})$ acts through its quotient $\mathfrak{Z}_{\mathfrak{g}}^{nilp}$ (see [FG] for the precise definition). Unlike the classical finite dimensional Lie algebra case, we no longer have well behavior localization type results in the affine algebra case, however, we still have the following gloabl section functor:

$$\Gamma : D^b(\mathcal{D}(Fl_{aff})_{crit} - mod) \mapsto D^b(\hat{\mathfrak{g}}_{crit} - mod_{nilp})$$

This functor can not be essentially surjective, due to the existence of large center on the right hand side. Notice that there is an action of $D^b(coh(\widetilde{\mathcal{N}^L/G^L}))$ on the left hand side and we have a geometric realization of $\mathcal{Z}_{\mathfrak{g}} \cong \mathcal{O}(Op^{nilp})$ with residue map $Res : Op^{nilp} \mapsto \widetilde{\mathcal{N}^L}/G^L$, the above functor can be upgraded to

$$\Gamma_{nilp} : D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(\mathcal{D}(Fl_{aff})_{crit} - mod) \mapsto D^b(\hat{\mathfrak{g}}_{crit} - mod_{nilp})$$

which they conjectured to be equivalence.

This conjecture holds partially, i.e. when restricted to some subcategory, the functor is an equivalence. More precisely, considering the I_0-equivariant object in the right hand side, in [FG], they proved the following:

Theorem 11. *The functor*

$$\Gamma_{nilp} : D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(\mathcal{D}(Fl_{aff})_{crit} - mod)^{I_0} \cong D^b(\hat{\mathfrak{g}}_{crit} - mod_{nilp})^{I_0}$$

is an equivalence of categories.

However, with the tautological t-structures on both sides, the functor Γ_{nilp} is not t-exact. Due to the Riemann-Hilbert correspondence, we have

$$D^b(\mathcal{D}(Fl_{aff})_{crit} - mod)^{I_0} \cong D^b(perv_{I_0}(Fl_{aff}))$$

Hence, the special family of the perverse sheaves (i.e. Wakimoto Sheaves) has its D-module context realization which we still denoted by W_λ, $\lambda \in \Lambda$, and the previous category is equivalent to

$$D' = D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(perv_{I_0}(Fl_{aff}))$$

They introduced the new t-structure on $D = D^b(\mathcal{D}(Fl_{aff})_{crit} - mod)^{I_0}$ by

$$D^{\leq 0}_{new} = \{M \in D | W_\lambda \star M \in D^{\leq 0}_{taut} \text{ for any} \lambda \in \Lambda\}$$

where W_λ is Wakimoto modules. This new-t structure easily defines a new t-structure on D'. In Theorem 11, after introducing the new t-structure on L.H.S. , with the tautological t-structure on R.H.S, Γ_{nilp} is a t-exact functor.

Due to the Riemann-Hilbert correspondence and the previous Bezrukavnikov correspondence we have:

$$\begin{aligned} & D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(\mathcal{D}(Fl_{aff})_{crit} - mod)^{I_0} \\ \cong\ & D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(perv_{I_0}(Fl_{aff})) \\ \cong\ & D^b(coh(Op^{nilp})) \underset{D^b(coh(\widetilde{\mathcal{N}^L/G^L}))}{\times} D^b(coh([St^L/G^L])) \end{aligned}$$

Let us fix a nilpotent oper with the residue image , then we have the following equivalence:(after base change from the above results).

$$\Gamma \ : D^b(\hat{\mathfrak{g}}_{crit} - mod_{nilp,} \ )^{I_0} \cong D^b(Qcoh(\ \overset{L}{\underset{\mathfrak{g}^L}{\times}} \widetilde{\mathfrak{g}^L})) \tag{5.2}$$

where $\overset{L}{\underset{\mathfrak{g}^L}{\times}} \widetilde{\mathfrak{g}^L}$ which will be denoted by Spr is the derived Springer fiber. And on R.H.S, the t-structure inherited from the new t-structure on $\mathcal{T}$ makes Γ exact.(L.H.S with the tautological t-structure.) In next section, we will prove this t-structure is braid positive above the tautological t-structure on $D^b(Qcoh(\ ))$.

5.3 Comparing t-structure

In this section, we will prove that the new t structure τ_{new} on $\mathcal{T}$ is braid positive above the coherent t structure ς_{coh} on $\mathcal{S}$, and as an application, after a base change, the following equivalence

$$D^b(\mathfrak{U}^{\hat{0}} - mod) \cong D^b(Qcoh(Spr\ )) \cong D^b(\hat{\mathfrak{g}}_{crit} - mod^{I_0}_{nilp,}\ ).$$

can be upgraded to equivalence of two abelian categories:

$$\mathfrak{U}^{\hat{0}} - mod \cong \hat{\mathfrak{g}}_{crit} - mod^{I_0}_{nilp,}\ . \tag{5.3}$$

where $Spr = \overset{L}{\underset{\mathfrak{g}^L}{\times}} \widetilde{\mathfrak{g}^L}$ is the derived Springer fiber.

We start the proof with a lemma:

Lemma 4. *In coherent realization of $\mathcal{S}$, we have the following:*

1) $D^{b,\leq 0}_{coh}(coh([\widetilde{\mathcal{N}^L}/G^L])) = \{\mathcal{O}(\lambda)[i] | \lambda \in \Lambda, i \geq 0.\}$

2) $\pi_*(R_w\mathcal{O}(\lambda)) \in D^{b,\leq 0}_{perv}(coh([\mathcal{N}^L/G^L]))$

3) $D^{b,\leq 0}_{coh}(coh([\widetilde{\mathcal{N}^L}/G^L])) \subset D^{b,\leq 0}_{new}(coh([\widetilde{\mathcal{N}^L}/G^L]))$

Proof:

1) This is obvious.

2) The statement in coherent realization is not obvious, however,if we use the quotient realization, it follows from the Example 4.

3) Note that the quotient t-structure ς_{quot} on ${}_{quot}\mathcal{S}$ is braid positive above the quotient t-structure κ_{quot} on ${}_{quot}\mathcal{K}$. After a slightly modification of the proof of the result mentioned in Remark 2, we have:

$$_{quot}\mathcal{S}^{\leq 0}_{quot} = \{M \in \mathcal{S} | \pi_*(M.b) \in {}_{quot}\mathcal{K}^{\leq 0}_{quot}\}$$

Then 3) follows from 1) and 2) and the coherent realization of the above results.

Theorem 12. *1) The new t-structure ς_{new} on $\mathcal{S}$ coincide withe the tautological t-structure ς_{coh} on ${}_{coh}\mathcal{S}$.*[1]

2) The new t-structure τ_{new} on $\mathcal{T}$ is braid positive above ς_{new} on $\mathcal{S}$

Proof

1) The lemma implies that ${}_{coh}\mathcal{S}^{\leq 0}_{coh} \subset {}_{coh}\mathcal{S}^{\leq 0}_{new}$, we know these two categories are

[1]As a direct corollary, we proved that ς_{new} is a t-structure.

coincide if we can prove that ${}_{coh}\mathcal{S}^{\leq 0}_{new} \subset {}_{coh}\mathcal{S}^{\leq 0}_{coh}$: [2]

$$\begin{aligned} M \in {}_{coh}\mathcal{S}^{\leq 0}_{coh} &\Leftrightarrow Ext^i_{{}_{coh}\mathcal{S}}(M \otimes V, \mathcal{O}(\lambda)) = 0, \forall \lambda \in \Lambda, i > 0, V \in Rep(G^L) \\ &\Leftrightarrow Ext^i_{{}_{perv}\mathcal{S}}(M \star \mathcal{Z}_V, \triangle_0 \star W_\lambda), \forall \lambda \in \Lambda, i > 0, V \in Rep(G^L) \\ &\Leftrightarrow Ext^i_{{}_{perv}\mathcal{S}}(M \star \mathcal{Z}_V \star W_\nu, \triangle_0 \star W_\lambda \star W_\nu), \forall \lambda, \nu \in \Lambda, i > 0, V \in Rep(G^L) \end{aligned}$$

Due to the Theorem 9, the support of $M \star \mathcal{Z}_V$ consists of a finite S of the I-orbits on Fl_{aff}. Moreover, the theorem tell us the support of $W^*_{M \star \mathcal{Z}_V \star W_\nu} \subset S + \nu$, i.e. if $\nu \gg 0$, then the support of $M \star \mathcal{Z}_V \star W_\nu$ consists of finite I-orbits on Fl_{aff} which all parameterized by some positive weights. Noted that $\triangle_0 \star W_\lambda \star W_\nu = \triangle_{\nu+\lambda}$ is a standard object, for any $\lambda + \nu > 0$, hence we get $M \star \mathcal{Z}_V \star W_\nu \in {}_{perv}\mathcal{S}^{\leq 0}_{perv}$. However from $(--) \star \mathcal{Z}_V$ is an eaxtact functor, we know $M \star W_\nu \in {}_{perv}\mathcal{S}^{\leq 0}_{perv}$ for $\nu \gg 0$.

On the other hand, if $M \star W_\nu \in {}_{perv}\mathcal{S}^{\leq 0}_{perv}$ $\forall \nu \gg 0$, by inverse the above argument, we know $M \in {}_{coh}\mathcal{S}^{\leq 0}_{coh}$. i.e. $M \in {}_{coh}\mathcal{S}^{\leq 0}_{coh} \Leftrightarrow M \star W_\nu \in {}_{perv}\mathcal{S}^{0}_{perv}$ for any $\nu \gg 0$. i.e. we proved that $\mathcal{S}^{\leq 0}_{new} \subset_{coh} \mathcal{S}^{\leq 0}_{coh}$.

2) (This is the example 5)

Remark 4. *Since we proved that the new t-structure on $\mathcal{T}$ is braid positive above the new t-structure(hence coherent t-structure) on $\mathcal{S}$, after a base change argument, in the equivalence (4), the tautological t-structure on L.H.S introduced a braid positive t-*

[2] We want make a remark on the notation, in [G], there is a functor maps spherical perverse sheaves to the central sheaves in $perv_{I_0}(Gr)$, and by geometric Langlands duality, the category of spherical perverse sheaves $\cong Rep(G^L)$, we will denote $\mathcal{Z}_V$ be the corresponding central sheaves.

structure on the R.H.S (w.r.p.t the tautological t-structure on Qcoh(pt)), and Theorem 10 assert there are only one t-structure braid positive above the tautological t-structure on Qcoh(pt), hence it coincide with the one introduced by De Concini-Kac Quantum group. Hence we proved the equivalence (5).

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