Nilpotent Orbits in Bad Characteristic and the Springer Correspondence

by

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Abstract

Let G be a connected reductive algebraic group over an algebraically closed field of characteristic p, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. This thesis is concerned with nilpotent orbits in $\mathfrak{g}$ and $\mathfrak{g}^*$ and the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ when p is a bad prime.

Denote $\hat{W}$ the set of isomorphism classes of irreducible representations of the Weyl group W of G. Fix a prime number $l \neq p$. We denote $\mathfrak{A}_{\mathfrak{g}}$ (resp. $\mathfrak{A}_{\mathfrak{g}^*}$) the set of all pairs $(\mathrm{c}, \mathcal{F})$, where c is a nilpotent G-orbit in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) and $\mathcal{F}$ is an irreducible G-equivariant $\bar{\mathbb{Q}}_l$-local system on c (up to isomorphism).

In chapter 1, we study the Springer correspondence for $\mathfrak{g}$ when G is of type B, C or D ($p = 2$). The correspondence is a bijective map from $\hat{W}$ to $\mathfrak{A}_{\mathfrak{g}}$. In particular, we classify nilpotent G-orbits in $\mathfrak{g}$ (type B, D) over finite fields of characteristic 2.

In chapter 2, we study the Springer correspondence for $\mathfrak{g}^*$ when G is of type B, C or D ($p = 2$). The correspondence is a bijective map from $\hat{W}$ to $\mathfrak{A}_{\mathfrak{g}^*}$. In particular, we classify nilpotent G-orbits in $\mathfrak{g}^*$ over algebraically closed and finite fields of characteristic 2.

In chapter 3, we give a combinatorial description of the Springer correspondence constructed in chapter 1 and chapter 2 for $\mathfrak{g}$ and $\mathfrak{g}^*$.

In chapter 4, we study the nilpotent orbits in $\mathfrak{g}^*$ and the Weyl group representations that correspond to the pairs $(\mathrm{c}, \bar{\mathbb{Q}}_l) \in \mathfrak{A}_{\mathfrak{g}^*}$ under Springer correspondence when G is of an exceptional type.

Chapters 1, 2 and 3 are based on the papers [X1, X2, X3]. Chapter 4 is based on some unpublished work.

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Contents

Chapter 1

Classical Lie Algebras

Throughout this chapter, $\mathbf{k}$ denotes an algebraically closed field of characteristic 2, $\mathbf{F}_q$ denotes a finite field of characteristic 2 and $\bar{\mathbf{F}}_q$ denotes an algebraic closure of $\mathbf{F}_q$.

1.1 Introduction

In [H], Hesselink determines the nilpotent orbits in classical Lie algebras under the adjoint action of classical Lie groups over $\mathbf{k}$. In [Spa1], Spaltenstein gives a parametrization of these nilpotent orbits by pairs of partitions. We extend Hesselink's method to study the nilpotent orbits in the Lie algebras of orthogonal groups over $\mathbf{F}_q$. Using this extension and Spaltenstein's parametrization we classify the nilpotent orbits over $\mathbf{F}_q$. We determine the structure of the component groups of centralizers of nilpotent elements. In particular, we obtain the number of nilpotent orbits over $\mathbf{F}_q$.

Let G be a connected reductive algebraic group over an algebraically closed field of characteristic p and $\mathfrak{g}$ the Lie algebra of G. When the characteristic p is large enough, Springer [Sp1] constructs representations of the Weyl group of G which are related to the nilpotent G-orbits in $\mathfrak{g}$. Lusztig [L3] constructs the generalized Springer correspondence in all characteristic p which is related to the unipotent conjugacy classes in G. Assume G is of type B, C or D and the characteristic p is 2. We use a similar construction as in [L3, L5] to give the Springer correspondence for $\mathfrak{g}$. Let $\mathfrak{A}_\mathfrak{g}$ be the set of all pairs $(\mathrm{c}, \mathcal{F})$ where c is a nilpotent G-orbit in $\mathfrak{g}$ and $\mathcal{F}$ is an irreducible

G-equivariant local system on c (up to isomorphism). We construct a bijective map from the set of isomorphism classes of irreducible representations of the Weyl group of G to the set $\mathfrak{A}_{\mathfrak{g}}$. In the case of symplectic group a Springer correspondence (with a different definition than ours) has been established in [Ka]; in that case centralizers of the nilpotent elements are connected [Spa1]. A complicating feature in the orthogonal case is the existence of non-trivial equivariant local systems on a nilpotent orbit.

1.2 Hesselink's classification of nilpotent orbits over an algebraically closed field

We recall the results of Hesselink on nilpotent orbits in orthogonal Lie algebras in this section (see [H]). Let $\mathbb{K}$ be a field of characteristic 2, not necessarily algebraically closed.

1.2.1 A form space V is a finite dimensional vector space over $\mathbb{K}$ equipped with a quadratic form $Q : V \to \mathbb{K}$. Let $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{K}$ be the bilinear form $\langle v, w\rangle = Q(v+w)+Q(v)+Q(w)$. Let $V^{\perp} = \{v \in V | \langle v, w\rangle = 0, \forall\ w \in V\}$. Then V is called non-defective if $V^{\perp} = \{0\}$, otherwise, it is called defective; V is called non-degenerate if $\dim(V^{\perp}) \leq 1$ and $Q(v) \neq 0$ for all non-zero $v \in V^{\perp}$.

Let V be a non-degenerate form space of dimension N over $\mathbb{K}$. Define the orthogonal group $O(V) = O(V,Q)$ to be $\{g \in \mathrm{GL}(V) | Q(gv) = Q(v),\ \forall\ v \in V\}$. The orthogonal Lie algebra $\mathfrak{o}(V) = \mathfrak{o}(V,Q)$ is $\{x \in \mathrm{End}(V) | \langle xv, v\rangle = 0,\ \forall\ v \in V \text{ and } \mathrm{tr}(x) = 0\}$. In the case where $\mathbb{K}$ is algebraically closed, let $SO(V) = SO(V,Q)$ be the identity component of $O(V)$ and we define $O_N(\mathbb{K})$, $\mathfrak{o}_N(\mathbb{K})$ and $SO_N(\mathbb{K})$ to be $O(V)$, $\mathfrak{o}(V)$ and $SO(V)$ respectively.

1.2.2 A form module is defined to be a pair (V,T) where V is a non-degenerate form space and T is a nilpotent element in $\mathfrak{o}(V)$. Classifying nilpotent orbits in $\mathfrak{o}(V)$ is equivalent to classifying form modules (V,T) on the form space V.

Let $A = \mathbb{K}[[t]]$ be the ring of formal power series in the indeterminate t. The form module $V = (V,T)$ is considered as an A-module by $(\sum_{n\geq 0} a_n t^n)v = \sum_{n\geq 0} a_n T^n v$.

Let E be the vector space spanned by the linear functionals $t^{-n} : A \to \mathbb{K}$, $\sum a_i t^i \mapsto a_n$, $n \geq 0$. Let E_0 be the subspace $\sum_{n\geq 0} \mathbb{K}t^{-2n}$ and $\pi_0 : E \to E_0$ the natural projection. The space E is considered as an A-module by $(au)(b) = u(ab)$, for $a, b \in A, u \in E$.

An abstract form module is defined to be an A-module V with $\dim(V) < \infty$, which is equipped with mappings $\varphi : V \times V \to E$ and $\psi : V \to E_0$ satisfying the following axioms:

(a) The map $\varphi(\cdot, w)$ is A-linear for every $w \in V$.

(b) $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

(c) $\varphi(v, v) = 0$ for all $v \in V$.

(d) $\psi(v + w) = \psi(v) + \psi(w) + \pi_0(\varphi(v, w))$ for all $v, w \in V$.

(e) $\psi(av) = a^2\psi(v)$ for $v \in V$, $a \in A$.

The following proposition identifies a form module (V, T) with a corresponding abstract form module $V = (V, \varphi, \psi)$.

Propsition ([H]). *If* (V, φ, ψ) *is an abstract form module, then* $(V, \langle\cdot,\cdot\rangle, Q)$ *given by* (i) *is a form module. If* $(V, \langle\cdot,\cdot\rangle, Q)$ *is a form module, there is a unique abstract form module* (V, φ, ψ) *such that* (i) *holds; it is given by* (ii).

(i) $\langle v, w\rangle = \varphi(v, w)(1)$, $Q(v) = \psi(v)(1)$.

(ii) $\varphi(v, w) = \sum_{n\geq 0} \langle t^n v, w\rangle t^{-n}$, $\psi(v) = \sum_{n\geq 0} Q(t^n v)t^{-2n}$.

1.2.3 An element in $\mathfrak{o}(V)$ is nilpotent if and only if it is nilpotent in $\mathrm{End}(V)$. Let T be a nilpotent element in $\mathfrak{o}(V)$. There exists a unique sequence of integers $p_1 \geq \cdots \geq p_s \geq 1$ and a family of vectors $v_1, \ldots, v_s$ such that $T^{p_i} v_i = 0$ and the vectors $T^{q_i} v_i$, $0 \leq q_i \leq p_i - 1$ form a basis of V. We write $p(V, T) = (p_1, \ldots, p_s)$. Define the index function $\chi(V, T) : \mathbb{N} \to \mathbb{Z}$ by $\chi(V, T)(m) = \min\{k \geq 0 | T^m v = 0 \Rightarrow Q(T^k v) = 0\}$. Define $\mu(V)$ to be the minimal integer $m \geq 0$ such that $t^m V = 0$. For $v \in V$ (or E), we define $\mu(v) = \min\{m \geq 0 | t^m v = 0\}$.

1.2.4 Let V be a form module. An orthogonal decomposition of V is an expression of V as a direct sum $V = \sum_{i=1}^r V_i$ of mutually orthogonal submodules V_i. The form module V is called indecomposable if $V \neq 0$ and for every orthogonal decomposition

$V = V_1 \oplus V_2$ we have $V_1 = 0$ or $V_2 = 0$. Every form module V has some orthogonal decomposition $V = \sum_{i=1}^{r} V_i$ in indecomposable submodules $V_1, V_2, \ldots, V_r$. The indecomposable modules are classified as follows.

Proposition 1.2.1 ([H]). *Let V be a non-degenerate indecomposable form module. There exist $v_1, v_2 \in V$ such that $V = Av_1 \oplus Av_2$ and $\mu(v_1) \geq \mu(v_2)$. For any such pair we put $m = \mu(v_1), m' = \mu(v_2), \Phi = \varphi(v_1, v_2)$ and $\Psi_i = \psi(v_i)$. One of the following conditions holds:*

(i) $m' = \mu(\Phi) = m,\ \mu(\Psi_i) \leq 2m - 1.$

(ii) $m' = \mu(\Phi) = m - 1,\ \mu(\Psi_1) = 2m - 1 > \mu(\Psi_2).$

Conversely, let $m \in \mathbb{N}$, $m' \in \mathbb{N} \cup \{0\}$, $\Phi \in E$, $\Psi_1, \Psi_2 \in E_0$ be given satisfying (i) *or* (ii). *Up to a canonical isomorphism there exists a unique form module $V = Av_1 \oplus Av_2$ with $m = \mu(v_1), m' = \mu(v_2), \Phi = \varphi(v_1, v_2)$ and $\Psi_i = \psi(v_i)$. This form module is indecomposable. In case* (i) *it is non-defective. In case* (ii) *it is defective and non-degenerate.*

From now on assume $\mathbb{K}$ is algebraically closed. The indecomposable modules in Proposition 1.2.1 are normalized in [H, 3.4 and 3.5] as follows.

Proposition 1.2.2 ([H]). *The indecomposable non-degenerate form modules over $\mathbb{K}$ are*

(i) $W_l(m) = Av_1 \oplus Av_2, [\frac{m+1}{2}] \leq l \leq m,\ \mu(v_1) = \mu(v_2) = m,\ \psi(v_1) = t^{2-2l}$, $\psi(v_2) = 0$ *and* $\varphi(v_1, v_2) = t^{1-m}$; *([a] means the integer part of a.)*

(ii) $D(m) = Av_1 \oplus Av_2,\ \mu(v_1) = m, \mu(v_2) = m - 1,\ \psi(v_1) = t^{2-2m},\ \psi(v_2) = 0$ *and* $\varphi(v_1, v_2) = t^{2-m}$.

We have $\chi_{W_l(m)} = [m; l]$ and $\chi_{D(m)} = [m; m]$, where $[m; l] : \mathbb{N} \to \mathbb{Z}$ is defined by $[m; l](n) = \max\{0, \min\{n - m + l, l\}\}$. Among these types of indecomposable modules only the types $D(m)$ are defective.

Remark. *The notation we use here is slightly different from that of [H]. The form module $W_{[\frac{m+1}{2}]}(m)$ in* (i) *is isomorphic to the form module $W(m)$ in [H].*

Finally this normalization of indecomposable modules is used to classify all form modules. Let (V, T) be a non-degenerate from module with $p(V, T) = (\lambda_1, \ldots, \lambda_1, \ldots,$

$\lambda_k, \ldots, \lambda_k)$ where $\lambda_1 > \cdots > \lambda_k \geq 1$ and index function $\chi = \chi(V, T)$. Let $m_i \in \mathbb{N}$ be the multiplicity of λ_i in $p(V, T)$. The isomorphism class of (V, T) is determined by the symbol

$$S(V,T) = (\lambda_1)^{m_1}_{\chi(\lambda_1)} (\lambda_2)^{m_2}_{\chi(\lambda_2)} \cdots (\lambda_k)^{m_k}_{\chi(\lambda_k)}.$$

A symbol S of the above form is the symbol of an isomorphism class of non-degenerate form modules if and only if it satisfies the following conditions

(i) $\chi(\lambda_i) \geq \chi(\lambda_{i+1})$ and $\lambda_i - \chi(\lambda_i) \geq \lambda_{i+1} - \chi(\lambda_{i+1})$, for $i = 1, \ldots, k-1$;

(ii) $\frac{\lambda_i}{2} \leq \chi(\lambda_i) \leq \lambda_i$, for $i = 1, \ldots, k$;

(iii) $\chi(\lambda_i) = \lambda_i$ if m_i is odd, for $i = 1, \ldots, k$;

(iv) $\{\lambda_i | m_i \text{ odd}\} = \{m, m-1\} \cap \mathbb{N}$ for some $m \in \mathbb{Z}$.

In the following we denote by a symbol either a form module in the isomorphism class or the corresponding nilpotent orbit.

1.3 Indecomposable modules over $\mathbf{F}_q$

In this section, we study the non-degenerate indecomposable form modules over $\mathbf{F}_q$. Note that the classification of the indecomposable modules (Proposition 1.2.1) is valid over any field. Similar to [H, 3.5], the non-degenerate indecomposable form modules over $\mathbf{F}_q$ are normalized as follows. Fix an element $\delta \in \mathbf{F}_q \backslash \{x^2 + x | x \in \mathbf{F}_q\}$.

Proposition 1.3.1. *The non-degenerate indecomposable form modules over $\mathbf{F}_q$ are*

(i) $W_l^0(m) = Av_1 \oplus Av_2$, $[\frac{m+1}{2}] \leq l \leq m$, with $\mu(v_1) = \mu(v_2) = m$, $\psi(v_1) = t^{2-2l}, \psi(v_2) = 0$ and $\varphi(v_1, v_2) = t^{1-m}$;

(ii) $W_l^\delta(m) = Av_1 \oplus Av_2$, $\frac{m+1}{2} \leq l \leq m$, with $\mu(v_1) = \mu(v_2) = m$, $\psi(v_1) = t^{2-2l}$, $\psi(v_2) = \delta t^{2l-2m}$ and $\varphi(v_1, v_2) = t^{1-m}$;

(iii) $D(m) = Av_1 \oplus Av_2$ with $\mu(v_1) = m, \mu(v_2) = m-1$, $\psi(v_1) = t^{2-2m}$, $\psi(v_2) = 0$ and $\varphi(v_1, v_2) = t^{2-m}$.

We have $\chi_{W_l^0(m)} = \chi_{W_l^\delta(m)} = [m; l]$ and $\chi_{D(m)} = [m; m]$. Among these types only the types $D(m)$ are defective.

Proof. As pointed out in [H], the form modules in Proposition 1.2.1 (ii) over $\mathbf{F}_q$ can be normalized the same as in Proposition 1.2.2 (ii). Namely, there exist v_1 and v_2 such that the above modules have the form (iii).

Now let $U(m) = Av_1 \oplus Av_2$ be a form module as in Proposition 1.2.1 (i) with $\mu(v_1) = \mu(v_2) = m$. We can assume $\mu(\Psi_1) \geq \mu(\Psi_2)$. There are the following cases:

Case 1: $\Psi_1 = \Psi_2 = 0$. We can assume $\Phi = t^{1-m}$. Let $\tilde{v}_1 = v_1 + t^{m+1-2[\frac{m+1}{2}]}v_2$ and $\tilde{v}_2 = v_2$. We have $\psi(\tilde{v}_1) = t^{2-2[\frac{m+1}{2}]}$, $\psi(\tilde{v}_2) = 0$, $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 2: $\Psi_1 \neq 0$, $\Psi_2 = 0$. We can assume $\Phi = t^{1-m}$ and $\Psi_1 = t^{-2l}$, where $l \leq m-1$. If $l < [\frac{m-1}{2}]$, let $\tilde{v}_1 = v_1 + t^{m+1-2[\frac{m+1}{2}]}v_2 + t^{m-2l-1}v_2$, $\tilde{v}_2 = v_2$; otherwise, let $\tilde{v}_1 = v_1$, $\tilde{v}_2 = v_2$. Then we get $\psi(\tilde{v}_1) = t^{-2l}$, $[\frac{m-1}{2}] \leq l \leq m-1$, $\psi(\tilde{v}_2) = 0$, $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 3: $\Psi_1 \neq 0$, $\Psi_2 \neq 0$. We can assume $\Psi_1 = t^{-2l_1}$, $\Phi = t^{1-m}$, $\Psi_2 = \sum_{i=0}^{l_2} a_i t^{-2i}$ with $l_2 \leq l_1 \leq m-1$.

(1) $l_1 < [\frac{m}{2}]$, let $\tilde{v}_2 = v_2 + \sum_{i=0}^{m-1} x_i t^i v_1$, then $\psi(\tilde{v}_2) = 0$ has a solution for x_i's and we get to Case 2.

(2) $l_1 \geq [\frac{m}{2}]$, let $\tilde{v}_2 = v_2 + \sum_{i=0}^{m-1} x_i t^i v_1$. If $a_{m-l_1-1} \in \{x^2 + x | x \in \mathbf{F}_q\}$, then $\psi(\tilde{v}_2) = 0$ has a solution for x_i's and we get to Case 2. If $a_{m-l_1-1} \notin \{x^2 + x | x \in \mathbf{F}_q\}$, then $\psi(\tilde{v}_2) = \delta t^{-2(m-l_1-1)}$ has a solution for x_i's.

Summarizing Cases 1-3, we have normalized $U(m) = Av_1 \oplus Av_2$ with $\mu(v_1) = \mu(v_2) = m$ as follows:

(i) $[\frac{m+1}{2}] \leq \chi(m) = l \leq m$, $\psi(v_1) = t^{2-2l}$, $\psi(v_2) = 0$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $W_l^0(m)$.

(ii) $\frac{m+1}{2} \leq \chi(m) = l \leq m$, $\psi(v_1) = t^{2-2l}$, $\psi(v_2) = \delta t^{-2(m-l)}$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $W_l^\delta(m)$.

One can verify that these form modules are not isomorphic to each other. □

Remark 1.3.2. It follows that the isomorphism class of the form module $W_l(m)$ over $\bar{\mathbf{F}}_q$ remains as one isomorphism class over $\mathbf{F}_q$ when $l = \frac{m}{2}$ and decomposes into two isomorphism classes $W_l^0(m)$ and $W_l^\delta(m)$ over $\mathbf{F}_q$ when $l \neq \frac{m}{2}$. The isomorphism class of the form module $D(m)$ over $\bar{\mathbf{F}}_q$ remains as one isomorphism class over $\mathbf{F}_q$.

1.4 Nilpotent orbits over $\mathbf{F}_q$

In this section we study the nilpotent orbits in the orthogonal Lie algebras over $\mathbf{F}_q$ by extending the method in [H]. Let V be a non-degenerate form space over $\bar{\mathbf{F}}_q$. An isomorphism class of form modules on V over $\bar{\mathbf{F}}_q$ may decompose into several isomorphism classes over $\mathbf{F}_q$.

Proposition 1.4.1. *Let W be a form module $(\lambda_1)^{m_1}_{\chi(\lambda_1)}(\lambda_2)^{m_2}_{\chi(\lambda_2)}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$ on the form space V.*

(i) *Assume V is defective. The isomorphism class of W over $\bar{\boldsymbol{F}}_q$ decomposes into at most 2^{n_1} isomorphism classes over $\boldsymbol{F}_q$, where n_1 is the cardinality of $\{1 \le i \le s-1 | \chi(\lambda_i)+\chi(\lambda_{i+1}) \le \lambda_i,\ \chi(\lambda_i) \ne \lambda_i/2\}$.*

(ii) *Assume V is non-defective. The isomorphism class of W over $\bar{\boldsymbol{F}}_q$ decomposes into at most 2^{n_2} isomorphism classes over $\boldsymbol{F}_q$, where n_2 is the cardinality of $\{1 \le i \le s | \chi(\lambda_i)+\chi(\lambda_{i+1}) \le \lambda_i,\ \chi(\lambda_i) \ne \lambda_i/2\}$ (here define $\chi(\lambda_{s+1}) = 0$).*

Note that we have two types of non-defective form spaces of dimension $2n$ over $\mathbf{F}_q$, $V^+$and V^- with a quadratic form of Witt index n and $n-1$ respectively. We define $O^+_{2n}(\mathbf{F}_q)$ (resp. $O^-_{2n}(\mathbf{F}_q)$) to be $O(V^+)$ (resp. $O(V^-)$) and $\mathfrak{o}^+_{2n}(\mathbf{F}_q)$ (resp. $\mathfrak{o}^-_{2n}(\mathbf{F}_q)$) to be $\mathfrak{o}(V^+)$ (resp. $\mathfrak{o}(V^-)$). Let $SO^+_{2n}(\mathbf{F}_q) = O^+_{2n}(\mathbf{F}_q) \cap SO_{2n}(\bar{\mathbf{F}}_q)$. A form module on V^+ (resp. V^-) has an orthogonal decomposition $W^{\epsilon_1}_{l_1}(\lambda_1) \oplus \cdots \oplus W^{\epsilon_k}_{l_k}(\lambda_k)$ with $\#\{1 \le i \le k | \epsilon_i = \delta\}$ being even (resp. odd).

Corollary 1.4.2. (i) *The nilpotent $O_{2n+1}(\bar{\boldsymbol{F}}_q)$-orbit $(\lambda_1)^{m_1}_{\chi(\lambda_1)}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$ in $\mathfrak{o}_{2n+1}(\bar{\boldsymbol{F}}_q)$ decomposes into at most 2^{n_1} $O_{2n+1}(\boldsymbol{F}_q)$-orbits in $\mathfrak{o}_{2n+1}(\boldsymbol{F}_q)$.*

(ii) *If $\chi(\lambda_i) = \lambda_i/2$, $i = 1,\ldots,s$, the nilpotent $O_{2n}(\bar{\boldsymbol{F}}_q)$-orbit $(\lambda_1)^{m_1}_{\chi(\lambda_1)}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$ in $\mathfrak{o}_{2n}(\bar{\boldsymbol{F}}_q)$ remains one $O^+_{2n}(\boldsymbol{F}_q)$-orbit in $\mathfrak{o}^+_{2n}(\boldsymbol{F}_q)$; otherwise, it decomposes into at most 2^{n_2-1} $O^+_{2n}(\boldsymbol{F}_q)$-orbits in $\mathfrak{o}^+_{2n}(\boldsymbol{F}_q)$ and at most 2^{n_2-1} $O^-_{2n}(\boldsymbol{F}_q)$-orbits in $\mathfrak{o}^-_{2n}(\boldsymbol{F}_q)$.*

Here n_1, n_2 are as in Proposition 1.4.1.

Remark. *In Corollary 1.4.2* (ii), *if $\chi(\lambda_i) = \lambda_i/2$, $i = 1,\ldots,s$, then n is even; if $\chi(\lambda_i) \ne \lambda_i/2$ for some i, then $n_2 \ge 1$.*

Before we prove Proposition 1.4.1, we need the following lemma.

Lemma 1.4.3. (i) *Assume $k \geq m$ and $l \geq m$. We have $W_l^0(k) \oplus D(m) \cong W_l^\delta(k) \oplus D(m)$ if and only if $l+m > k$.*

(ii) *Assume $m > k$. We have $D(m) \oplus W_k^0(k) \cong D(m) \oplus W_k^\delta(k)$.*

(iii) *Assume $l_1 \geq l_2, \lambda_1 - l_1 \geq \lambda_2 - l_2$. If $l_1 + l_2 > \lambda_1$, then $W_{l_1}^0(\lambda_1) \oplus W_{l_2}^0(\lambda_2) \cong W_{l_1}^\delta(\lambda_1) \oplus W_{l_2}^\delta(\lambda_2)$ and $W_{l_1}^0(\lambda_1) \oplus W_{l_2}^\delta(\lambda_2) \cong W_{l_1}^\delta(\lambda_1) \oplus W_{l_2}^0(\lambda_2)$.*

(iv) *Assume $l_1 \geq l_2, \lambda_1 - l_1 \geq \lambda_2 - l_2$. If $l_1 + l_2 \leq \lambda_1$, then $W_{l_1}^{\epsilon_1}(\lambda_1) \oplus W_{l_2}^{\epsilon_2}(\lambda_2) \cong W_{l_1}^{\epsilon_1'}(\lambda_1) \oplus W_{l_2}^{\epsilon_2'}(\lambda_2)$ if and only if $\epsilon_1 = \epsilon_1', \epsilon_2 = \epsilon_2'$, where $\epsilon_i, \epsilon_i' = 0$ or δ, $i = 1, 2$.*

Proof. We only prove (i). (ii)-(iv) are proved similarly. We take v_1, w_1 and v_2, w_2 such that $W_l^0(k) \oplus D(m) = Av_1 \oplus Aw_1 \oplus Av_2 \oplus Aw_2$ and $\psi(v_1) = t^{2-2l}, \psi(w_1) = 0, \varphi(v_1, w_1) = t^{1-k}, \psi(v_2) = t^{2-2m}, \psi(w_2) = 0, \varphi(v_2, w_2) = t^{2-m}, \varphi(v_1, v_2) = \varphi(v_1, w_2) = \varphi(w_1, v_2) = \varphi(w_1, w_2) = 0$, and take v_1', w_1' and v_2', w_2' such that $W_l^\delta(k) \oplus D(m) = Av_1' \oplus Aw_1' \oplus Av_2' \oplus Aw_2'$ and $\psi(v_1') = t^{2-2l}, \psi(w_1') = \delta t^{2l-2k}, \varphi(v_1', w_1') = t^{1-k}, \psi(v_2') = t^{2-2m}, \psi(w_2') = 0, \varphi(v_2', w_2') = t^{2-m}, \varphi(v_1', v_2') = \varphi(v_1', w_2') = \varphi(w_1', v_2') = \varphi(w_1', w_2') = 0$.

We have $W_l^0(k) \oplus D(m) \cong W_l^\delta(k) \oplus D(m)$ if and only if there exists an A-module isomorphism $g : V \to V$ such that $\psi(gv) = \psi(v)$ and $\varphi(gv, gw) = \varphi(v, w)$ for any $v, w \in V$. Assume $gv_j = \sum_{i=0}^{k-1}(a_{j,i}t^i v_1' + b_{j,i}t^i w_1') + \sum_{i=0}^{m-1} c_{j,i}t^i v_2' + \sum_{i=0}^{m-2} d_{j,i}t^i w_2', gw_j = \sum_{i=0}^{k-1}(e_{j,i}t^i v_1' + f_{j,i}t^i w_1') + \sum_{i=0}^{m-1} g_{j,i}t^i v_2' + \sum_{i=0}^{m-2} h_{j,i}t^i w_2', j = 1, 2$. Then $W_l^0(k) \oplus D(m) \cong W_l^\delta(k) \oplus D(m)$ if and only if the equations $\psi(gv_i) = \psi(v_i), \psi(gw_i) = \psi(w_i), \varphi(gv_i, gv_j) = \varphi(v_i, v_j), \varphi(gv_i, gw_j) = \varphi(v_i, w_j), \varphi(gw_i, gw_j) = \varphi(w_i, w_j)$ have solutions.

If $l+m \leq k$, some equations are $e_{1,2l-k-1}^2 + e_{1,2l-k-1} = \delta$ $(l \neq \frac{k+1}{2})$ or $a_{1,0}^2 + a_{1,0}b_{1,0} = 1, e_{1,0}^2 + e_{1,0}f_{1,0} = \delta, a_{1,0}f_{1,0} + b_{1,0}e_{1,0} = 1$ $(l = \frac{k+1}{2})$. In each case we get an "Artin-Schreier" equation $x^2 + x = \delta$, which has no solution over $\mathbf{F}_q$. This implies that $W_l^0(k) \oplus D(m) \ncong W_l^\delta(k) \oplus D(m)$.

If $l + m > k$, let $gv_1 = v_1', gw_1 = w_1' + \sqrt{\delta} t^{l+m-k-1} v_2', gv_2 = v_2', gw_2 = w_2' + \sqrt{\delta} t^l v_1'$. This is a solution for the equations. It follows that $W_l^0(k) \oplus D(m) \cong W_l^\delta(k) \oplus D(m)$. □

Proof of Proposition 1.4.1. We prove (i). One can prove (ii) similarly. By the classification in subsection 1.2.4, we can rewrite the symbol as

$$(\lambda_1)_{\chi(\lambda_1)}^2 (\lambda_2)_{\chi(\lambda_2)}^2 \cdots (\lambda_k)_{\chi(\lambda_k)}^2 (\lambda_{k+1})_{\lambda_{k+1}} (\lambda_{k+1} - 1)_{(\lambda_{k+1}-1)} (\lambda_{k+2})_{\lambda_{k+2}}^2 \cdots (\lambda_{k+l})_{\lambda_{k+l}}^2,$$

where $\lambda_1 \geq \cdots \geq \lambda_{k+1} > \lambda_{k+2} \geq \cdots \geq \lambda_{k+l}$ (by abuse of notation, we still use λ) and $l \geq 1$. A representative W for this isomorphism class over $\bar{\mathbf{F}}_q$ is $W_{\chi(\lambda_1)}(\lambda_1) \oplus \cdots \oplus W_{\chi(\lambda_k)}(\lambda_k) \oplus D(\lambda_{k+1}) \oplus W_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W_{\lambda_{k+l}}(\lambda_{k+l})$. By Proposition 1.3.1 and Remark 1.3.2, in order to study the isomorphism classes into which the isomorphism class of W over $\bar{\mathbf{F}}_q$ decomposes over $\mathbf{F}_q$, it is enough to study the isomorphism classes of form modules of the form $W^{\epsilon_1}_{\chi(\lambda_1)}(\lambda_1) \oplus \cdots \oplus W^{\epsilon_k}_{\chi(\lambda_k)}(\lambda_k) \oplus D(\lambda_{k+1}) \oplus W^{\epsilon_{k+2}}_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W^{\epsilon_{k+l}}_{\lambda_{k+l}}(\lambda_{k+l})$, where $\epsilon_i = 0$ or δ. Thus it suffices to show that modules of the above form have at most 2^{n_1} isomorphism classes.

We have $n_1 = \#\{1 \leq i \leq k | \chi(\lambda_i) + \chi(\lambda_{i+1}) \leq \lambda_i,\ \chi(\lambda_i) \neq \lambda_i/2\}$. Suppose $i_1, i_2, \ldots, i_{n_1}$ are such that $1 \leq i_j \leq k, \chi(\lambda_{i_j}) + \chi(\lambda_{i_j+1}) \leq \lambda_{i_j},\ \chi(\lambda_{i_j}) \neq \lambda_{i_j}/2, j = 1, \ldots, n_1$. Then using Lemma 1.4.3 one can easily show that a module of the above form is isomorphic to one of the following modules: $V_1^{\epsilon_1} \oplus \cdots \oplus V_{n_1}^{\epsilon_{n_1}} \oplus V_{n_1+1}$, where $V_t^{\epsilon_t} = W^0_{\chi(\lambda_{i_{t-1}+1})}(\lambda_{i_{t-1}+1}) \oplus \cdots \oplus W^0_{\chi(\lambda_{i_t-1})}(\lambda_{i_t-1}) \oplus W^{\epsilon_t}_{\chi(\lambda_{i_t})}(\lambda_{i_t})$, $t = 1, \ldots, n_1$, $i_0 = 0$, $\epsilon_t = 0$ or δ and $V_{n_1+1} = W^0_{\chi(\lambda_{i_{n_1}+1})}(\lambda_{i_{n_1}+1}) \oplus \cdots \oplus W^0_{\chi(\lambda_k)}(\lambda_k) \oplus D(\lambda_{k+1}) \oplus W^0_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W^0_{\lambda_{k+l}}(\lambda_{k+l})$. Thus (i) is proved. □

1.5 Number of nilpotent orbits over $\mathbf{F}_q$

1.5.1 In this subsection we recall Spaltenstein's parametrization of nilpotent orbits by pairs of partitions in $\mathfrak{o}(\bar{\mathbf{F}}_q)$ (see [Spa1]).

For $\mathfrak{o}_{2n+1}(\bar{\mathbf{F}}_q)$, the orbit $(\lambda_1)^2_{\chi(\lambda_1)} \cdots (\lambda_k)^2_{\chi(\lambda_k)} (\lambda_{k+1})_{\lambda_{k+1}} (\lambda_{k+1}-1)_{(\lambda_{k+1}-1)} (\lambda_{k+2})^2_{\lambda_{k+2}} \cdots (\lambda_{k+l})^2_{\lambda_{k+l}}$ is written as $(\alpha_1+\beta_1)^2_{(\alpha_1+1)} \cdots (\alpha_k+\beta_k)^2_{(\alpha_k+1)} (\alpha_{k+1}+1)_{(\alpha_{k+1}+1)} (\alpha_{k+1})_{\alpha_{k+1}} (\alpha_{k+2})^2_{\alpha_{k+2}} \cdots (\alpha_{k+l})^2_{\alpha_{k+l}}$ and the corresponding pair of partitions is (α, β), where $\alpha = (\alpha_1 \ldots, \alpha_{k+l})$ and $\beta = (\beta_1, \ldots, \beta_k)$ satisfy $\alpha_1 \geq \cdots \geq \alpha_{k+l} \geq 0$, $\beta_1 \geq \cdots \geq \beta_k \geq 1$ and $|\alpha| + |\beta| = n$.

For $\mathfrak{o}_{2n}(\bar{\mathbf{F}}_q)$, the orbit $(\lambda_1)^2_{\chi(\lambda_1)} \cdots (\lambda_k)^2_{\chi(\lambda_k)}$ is written as $(\alpha_1 + \beta_1)^2_{\alpha_1} \cdots (\alpha_k + \beta_k)^2_{\alpha_k}$ and the corresponding pair of partitions is (α, β), where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ satisfy $\alpha_1 \geq \cdots \geq \alpha_k \geq 1$, $\beta_1 \geq \cdots \geq \beta_k \geq 0$ and $|\alpha| + |\beta| = n$.

1.5.2 In this subsection we study the number of nilpotent orbits over $\mathbf{F}_q$. Denote by $p_2(n)$ the cardinality of the set of pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$ and $p(k)$ the number of partitions of the integer k.

Proposition 1.5.1. (i) *The number of nilpotent* $O_{2n+1}(\boldsymbol{F}_q)$*-orbits in* $\mathfrak{o}_{2n+1}(\boldsymbol{F}_q)$ *is at most* $p_2(n)$.

(ii) *The number of nilpotent* $O^+_{2n}(\boldsymbol{F}_q)$*-orbits in* $\mathfrak{o}^+_{2n}(\boldsymbol{F}_q)$ *is at most* $\frac{1}{2}p_2(n)$ *if* n *is odd and is at most* $\frac{1}{2}(p_2(n)+p(\frac{n}{2}))$ *if* n *is even.*

Proof. (i) The set of nilpotent orbits in $\mathfrak{o}_{2n+1}(\bar{\mathbf{F}}_q)$ is mapped bijectively to the set $\{(\alpha,\beta)||\alpha|+|\beta|=n, \beta_i\leq\alpha_i+2\} := \Delta$ ([Spa1]). By Corollary 1.4.2 (i), a nilpotent orbit in $\mathfrak{o}_{2n+1}(\bar{\mathbf{F}}_q)$ corresponding to $(\alpha,\beta)\in\Delta, \alpha=(\alpha_1,\ldots,\alpha_s), \beta=(\beta_1,\ldots,\beta_t)$ splits into at most 2^{n_1} orbits in $\mathfrak{o}_{2n+1}(\mathbf{F}_q)$, where $n_1=\#\{1\leq i\leq t|\alpha_{i+1}+2\leq\beta_i<\alpha_i+2\}$. We associate to this orbit 2^{n_1} pairs of partitions as follows. Suppose $r_1,r_2,\ldots,r_{n_1}$ are such that $\alpha_{r_i+1}+2\leq\beta_{r_i}<\alpha_{r_i}+2, i=1,\ldots,n_1$. Let $\alpha^{1,i}=(\alpha_{r_{i-1}+1},\ldots,\alpha_{r_i}), \beta^{1,i}=(\beta_{r_{i-1}+1},\ldots,\beta_{r_i}), \alpha^{2,i}=(\beta_{r_{i-1}+1}-2,\ldots,\beta_{r_i}-2), \beta^{2,i}=(\alpha_{r_{i-1}+1}+2,\ldots,\alpha_{r_i}+2), i=1,\ldots,n_1, \alpha^{n_1+1}=(\alpha_{r_{n_1}+1},\ldots,\alpha_s), \beta^{n_1+1}=(\beta_{r_{n_1}+1},\ldots,\beta_t)$. We associate to (α,β) the pairs of partitions $(\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_1}},\tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_1}})$, $\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_1}}=(\alpha^{\epsilon_1,1},\ldots,\alpha^{\epsilon_{n_1},n_1},\alpha^{n_1+1}), \tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_1}}=(\beta^{\epsilon_1,1},\ldots,\beta^{\epsilon_{n_1},n_1},\beta^{n_1+1})$, where $\epsilon_i\in\{1,2\}, i=1,\ldots,n_1$.

Notice that the pairs of partitions $(\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_1}},\tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_1}})$ are distinct and among them only $(\alpha,\beta)=(\tilde{\alpha}^{1,\ldots,1},\tilde{\beta}^{1,\ldots,1})$ is in Δ. One can verify that the set of all pairs of partitions $(\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_1}},\tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_1}})$ constructed as above for $(\alpha,\beta)\in\Delta$ is equal to the set $\{(\alpha,\beta)||\alpha|+|\beta|=n\}$, which has cardinality $p_2(n)$. But the number of nilpotent orbits in $\mathfrak{o}_{2n+1}(\mathbf{F}_q)$ is no greater than the cardinality of the former set. Thus (i) is proved.

(ii) Similarly, the set of nilpotent orbits in $\mathfrak{o}_{2n}(\bar{\mathbf{F}}_q)$ is mapped bijectively to the set $\{(\alpha,\beta)||\alpha|+|\beta|=n, \beta_i\leq\alpha_i\} := \Delta'$ ([Spa1]). By Corollary 1.4.2 (ii), a nilpotent orbit in $\mathfrak{o}_{2n}(\bar{\mathbf{F}}_q)$ corresponding to $(\alpha,\beta)\in\Delta'$ with $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_s), \beta=(\beta_1,\beta_2,\ldots,\beta_s)$ and $\alpha\neq\beta$ splits into at most 2^{n_2-1} orbits in $\mathfrak{o}^+_{2n}(\mathbf{F}_q)$, where $n_2=\#\{i|\alpha_{i+1}\leq\beta_i<\alpha_i\}-1$. We associate to this orbit 2^{n_2-1} pairs of partitions as follows. We can assume $\alpha_s\neq 0$. Suppose $r_1,r_2,\ldots,r_{n_2}$ are such that $\alpha_{r_i+1}\leq\beta_{r_i}<\alpha_{r_i}, i=1,\ldots,n_2$. Let $\alpha^{1,i}=(\alpha_{r_{i-1}+1},\ldots,\alpha_{r_i}), \beta^{1,i}=(\beta_{r_{i-1}+1},\ldots,\beta_{r_i}), \alpha^{2,i}=(\beta_{r_{i-1}+1},\ldots,\beta_{r_i}), \beta^{2,i}=(\alpha_{r_{i-1}+1},\ldots,\alpha_{r_i}), i=1,\ldots,n_2, \alpha^{n_2+1}=(\alpha_{r_{n_2}+1},\ldots,\alpha_s), \beta^{n_2+1}=(\beta_{r_{n_2}+1},\ldots,\beta_s)$. We have 2^{n_2} distinct pairs of partitions $(\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_2}},\tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_2}}),\tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_2}}=(\alpha^{\epsilon_1,1},\ldots,\alpha^{\epsilon_{n_2},n_2}, \alpha^{n_2+1}), \tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_2}}=(\beta^{\epsilon_1,1},\ldots,\beta^{\epsilon_{n_2},n_2},\beta^{n_2+1})$, where $\epsilon_i\in\{1,2\}, i=1,\ldots,n_2$. We show that in these pairs of partitions (α',β') appears if and only if (β',α') appears. In fact we have $\alpha_i = \beta_i$, for $i > r_{n_2}$, which implies $\alpha^{n_2+1} = \beta^{n_2+1}$. Thus we have $(\tilde{\alpha}^{\epsilon_1+1(mod2),\ldots,\epsilon_{n_2}+1(mod2)}, \tilde{\beta}^{\epsilon_1+1(mod2),\ldots,\epsilon_{n_2}+1(mod2)}) = (\tilde{\beta}^{\epsilon_1,\ldots,\epsilon_{n_2}}, \tilde{\alpha}^{\epsilon_1,\ldots,\epsilon_{n_2}})$. Hence we can identify (α', β') with (β', α'), and then associate 2^{n_2-1} pairs of partitions to the nilpotent orbit corresponding to (α, β).

One can verify that the set of all pairs of partitions associated to $(\alpha, \beta) \in \Delta'$ as above is in bijection with the set of pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$ with (α, β) identified with (β, α), which has cardinality $\frac{1}{2}p_2(n)$ if n is odd and $\frac{1}{2}(p_2(n) + p(\frac{n}{2}))$ if n is even. Thus (ii) follows. □

Corollary 1.5.2. *The number of nilpotent $SO_{2n}^+(\boldsymbol{F}_q)$-orbits in $\mathfrak{o}_{2n}^+(\boldsymbol{F}_q)$ is at most $\frac{1}{2}p_2(n)$ if n is odd and is at most $\frac{1}{2}(p_2(n) + 3p(\frac{n}{2}))$ if n is even.*

Proof. We show that the $O_{2n}^+(\mathbf{F}_q)$ orbits that split into two $SO_{2n}^+(\mathbf{F}_q)$-orbits are exactly the orbits corresponding to the pairs of partitions of the form (α, α). The number of these orbits is $p(\frac{n}{2})$. Let x be a nilpotent element in $\mathfrak{o}_{2n}^+(\mathbf{F}_q)$. The $O_{2n}^+(\mathbf{F}_q)$-orbit of x splits into two $SO_{2n}^+(\mathbf{F}_q)$-orbits if and only if the centralizer $Z_{O_{2n}^+(\mathbf{F}_q)}(x) \subset SO_{2n}^+(\mathbf{F}_q)$. It is enough to show that for an indecomposable module V, $Z_{O_{2n}^+(\mathbf{F}_q)}(V) \subset SO_{2n}^+(\mathbf{F}_q)$ if and only if $\chi(m) \leq \frac{1}{2}m$, for all $m \in \mathbb{N}$.

Assume $V = W_l^0(m)$ or $W_l^\delta(m)$, $l \geq \frac{m+1}{2}$. Let $\epsilon = 0$ or δ and $v_1^\epsilon, v_2^\epsilon$ be such that $W_l^\epsilon(m) = Av_1^\epsilon \oplus Av_2^\epsilon$ and $\psi(v_1^\epsilon) = t^{2-2l}$, $\psi(v_2^\epsilon) = \epsilon t^{-2m+2l}$, $\varphi(v_1^\epsilon, v_2^\epsilon) = t^{1-m}$. Let $w_2^\epsilon = v_2^\epsilon + t^{2l-1-m}v_1^\epsilon$. Define u^ϵ by $u^\epsilon(a_1v_1^\epsilon + a_2v_2^\epsilon) = a_1v_1^\epsilon + a_2w_2^\epsilon$. Then $u^\epsilon \in Z_{O_{2n}^+(\mathbf{F}_q)}(W_l^\epsilon(m))$, but $u^\epsilon \notin SO_{2n}^+(\mathbf{F}_q)$ (see [H]). This shows that $Z_{O_{2n}^+(\mathbf{F}_q)}(V) \not\subset SO_{2n}^+(\mathbf{F}_q)$.

Assume $V = W_l^0(m)$, $l = m/2$. Let v_1, v_2 be such that $W_l^0(m) = Av_1 \oplus Av_2$ and $\psi(v_1) = t^{2-m}$, $\psi(v_2) = 0$, $\varphi(v_1, v_2) = t^{1-m}$. Let W be the subspace of V spanned by t^iv_1, t^iv_2, $i = \frac{m}{2}, \frac{m}{2}+1, \ldots, m-1$. Then W is a maximal totally singular subspace in V and $Z_{O_{2n}^+(\mathbf{F}_q)}(V)$ stabilizes W. Hence $Z_{O_{2n}^+(\mathbf{F}_q)}(V) \subset SO_{2n}^+(\mathbf{F}_q)$. □

1.6 Springer corresopondence

Throughout subsections 1.6.1-1.6.5, let G be a connected adjoint algebraic group of type B_r, C_r or D_r over $\mathbf{k}$ and $\mathfrak{g}$ the Lie algebra of G. Fix a Borel subgroup B of G

with Levi decomposition $B = TU$. Let $\mathfrak{b}, \mathfrak{t}$ and $\mathfrak{n}$ be the Lie algebra of B, T and U respectively. Let $\mathcal{B}$ be the variety of Borel subgroups of G.

We construct in this section the Springer correspondence for the Lie algebra $\mathfrak{g}$ following [L3, L5]. The construction and proofs are essentially the same as (actually simpler than) those for the unipotent case in [L3, L5]. For completeness, we include the proofs here. We only point out that Lemma 1.6.2 is essential for the construction. The lemma is probably well-known for which we include an elementary proof.

1.6.1 Let Z be the variety $\{(x, B_1, B_2) \in \mathfrak{g}\times\mathcal{B}\times\mathcal{B} | x \in \mathfrak{b}_1\cap\mathfrak{b}_2\}$ and Z' the Steinberg variety [St2] $\{(x, B_1, B_2) \in \mathfrak{g}\times\mathcal{B}\times\mathcal{B} | x \in \mathfrak{n}_1\cap\mathfrak{n}_2\}$. Denote r the dimension of T. Let c be a nilpotent orbit in $\mathfrak{g}$. A stronger version of the following lemma in the group case is due to Springer, Steinberg and Spaltenstein (see for example [Spa2]). We include a proof for the Lie algebra case here.

Lemma 1.6.1. (i) *We have* $\dim(\mathrm{c}\cap\mathfrak{n}) \leq \frac{1}{2}\dim \mathrm{c}$.

(ii) *Given* $x \in \mathrm{c}$, *we have* $\dim\{B_1 \in \mathcal{B} | x \in \mathfrak{n}_1\} \leq (\dim G - r - \dim \mathrm{c})/2$.

(iii) *We have* $\dim Z = \dim G$ *and* $\dim Z' = \dim G - r$.

Proof. We have a partition $Z = \cup_{\mathcal{O}} Z_{\mathcal{O}}$ according to the G-orbits $\mathcal{O}$ on $\mathcal{B}\times\mathcal{B}$ where $Z_{\mathcal{O}} = \{(x, B_1, B_2) \in Z | (B_1, B_2) \in \mathcal{O}\}$. Define in the same way a partition $Z' = \cup_{\mathcal{O}} Z'_{\mathcal{O}}$. Consider the maps from $Z_{\mathcal{O}}$ and $Z'_{\mathcal{O}}$ to $\mathcal{O}$: $(x, B_1, B_2) \mapsto (B_1, B_2)$. We have $\dim Z_{\mathcal{O}} = \dim(\mathfrak{b}_1\cap\mathfrak{b}_2) + \dim\mathcal{O} = \dim G$ and $\dim Z'_{\mathcal{O}} = \dim(\mathfrak{n}_1\cap\mathfrak{n}_2) + \dim\mathcal{O} = \dim G - r$. Thus (iii) follows.

Let $Z'(\mathrm{c}) = \{(x, B_1, B_2) \in Z' | x \in \mathrm{c}\} \subset Z'$. From (iii), we have $\dim Z'(\mathrm{c}) \leq \dim G - r$. Consider the map $Z'(\mathrm{c}) \to \mathrm{c}$, $(x, B_1, B_2) \mapsto x$. We have $\dim Z'(\mathrm{c}) = \dim \mathrm{c} + 2\dim\{B_1 \in \mathcal{B} | x \in \mathfrak{n}_1\} \leq \dim G - r$. Thus (ii) follows.

Consider the variety $\{(x, B_1) \in \mathrm{c}\times\mathcal{B} | x \in \mathfrak{n}_1\}$. By projecting it to the first coordinate, and using (ii), we see that it has dimension $\leq (\dim G - r + \dim \mathrm{c})/2$. If we project it to the second coordinate, we get $\dim(\mathrm{c}\cap\mathfrak{n}) + \dim\mathcal{B} \leq (\dim G - r + \dim \mathrm{c})/2$ and (i) follows. □

1.6.2 Recall that a semisimple element x in $\mathfrak{g}$ is called regular if the connected centralizer $Z_G^0(x)$ in G is a maximal torus of G.

Lemma 1.6.2. *There exist regular semisimple elements in $\mathfrak{g}$ and they form an open dense subset in $\mathfrak{g}$.*

Proof. We first show that regular semisimple elements exist in $\mathfrak{g}$.

(i) $G = SO(2n+1)$. Let $V, Q, \langle , \rangle$ be as in 1.2.1 with $\dim V = 2n+1$ and Q nondegenerate. We take $G = SO(V)$. Since Q is non-degenerate, $\dim V^{\perp} = 1$. Assume $V^{\perp} = \text{span}\{v_0\}$. Then $gv_0 = v_0$, for any $g \in G$. Hence the Lie algebra is $\{x \in \text{End}(V) | \langle xv, v\rangle = 0, \forall\ v \in V; xv_0 = 0\}$. With respect to a suitable basis, we can assume $Q(v) = \sum_{i=1}^{n} v_i v_{n+i} + v_{2n+1}^2$, for $v = (v_i) \in V = \mathbf{k}^{2n+1}$. A maximal torus of G is $T = \{\text{diag}(t_1, t_2, \ldots, t_n, 1/t_1, 1/t_2, \ldots 1/t_n, 1) | t_i \in \mathbf{k}^*, i = 1, \ldots, n\}$. The Lie algebra of T is $\mathfrak{t} = \{\text{diag}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots x_n, 0) | x_i \in \mathbf{k}, i = 1, \ldots, n\}$. Since every semisimple element in $\mathfrak{g}$ is conjugate to an element in $\mathfrak{t}$ under the adjoint action of G, it is enough to consider elements in $\mathfrak{t}$.

Let $x = \text{diag}(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots x_n, 0)$, where $x_i \neq x_j$, for any $i \neq j$ and $x_i \neq 0$ for any i (such x exists). It can be easily verified that $Z_G(x)$ consists of elements of the form $g = \begin{pmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $A_i = \text{diag}(a_i^1, a_i^2, \ldots, a_i^n), i = 1, 2, 3, 4$, satisfy $a_1^j a_3^j = a_2^j a_4^j = 0$ and $a_1^j a_4^j + a_2^j a_3^j = 1, j = 1, \ldots, n$. Hence we see that $Z_G^0(x) = T$ and x is regular.

(ii) G is the adjoint group of type C_n. We have the following construction of G. Let V be a $2n$ dimensional vector space equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle : V \times V \to \mathbf{k}$. Then G is defined as

$$\{(g, \lambda) \in GL(V) \times \mathbf{k}^* | \forall\ v, w \in V, \langle gv, gw\rangle = \lambda\langle v, w\rangle\}/\{(\mu I, \mu^2) | \mu \in \mathbf{k}^*\}.$$

Hence the Lie algebra $\mathfrak{g}$ is

$$\{(x, \lambda) \in \text{End}(V) \times \mathbf{k} | \forall\ v, w \in V, \langle xv, w\rangle + \langle v, xw\rangle = \lambda\langle v, w\rangle\}/\{(\mu I, 0) | \mu \in \mathbf{k}\}.$$

With respect to a suitable basis, we can assume $\langle v, w\rangle = \sum_{i=1}^{n} v_i w_{n+i}$ for $v = (v_i), w = (w_i) \in V = \mathbf{k}^{2n}$. A maximal torus of G is $T = \{\text{diag}(t_1, t_2, \ldots, t_n, \lambda/t_1, \lambda/t_2, \ldots \lambda/t_n) | t_i \in \mathbf{k}^*, i = 1, \ldots, n; \lambda \in \mathbf{k}^*\}/\{(\mu I, \mu^2) | \mu \in \mathbf{k}^*\}$. The Lie algebra of T is $\mathfrak{t} = \{\text{diag}(x_1, x_2, \ldots, x_n, \lambda + x_1, \lambda + x_2, \ldots \lambda + x_n) | x_i \in \mathbf{k}, i = 1, \ldots, n; \lambda \in \mathbf{k}\}/\{(\mu I, 0) | \mu \in \mathbf{k}\}$. Let $x = \overline{\text{diag}(x_1, x_2, \ldots, x_n, x_1 + \lambda, x_2 + \lambda, \ldots x_n + \lambda)} \in \mathfrak{t}$, where $x_i \neq x_j$, for any

$i \neq j$ and $\lambda \neq x_i + x_j$ for any i, j (such x exists). It can be verified that the elements in $Z_G(x)$ must have the form $g = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where each row and each column of G has only one nonzero entry and $(A_1)_{ij} \neq 0 \Leftrightarrow (A_4)_{ij} \neq 0, (A_2)_{ij} \neq 0 \Leftrightarrow (A_3)_{ij} \neq 0$. Hence we see that $Z_G^0(x) = T$ and x is regular.

(iii) G is the adjoint group of type D_n. We have the following construction of G. Let $V, Q, \langle , \rangle$ be as in 1.2.1 with $\dim V = 2n$ and Q nondegenerate. Then G is defined as $\{(g, \lambda) \in GL(V) \times \mathbf{k}^* | \forall\ v \in V, Q(gv) = \lambda Q(v)\}/\{(\mu I, \mu^2) | \mu \in \mathbf{k}^*\}$. Hence the Lie algebra $\mathfrak{g}$ is $\{(x, \lambda) \in \mathrm{End}(V) \times \mathbf{k} | \forall\ v \in V, \langle xv, v\rangle = \lambda Q(v)\}/\{(\mu I, 0) | \mu \in \mathbf{k}\}$. With respect to a suitable basis, we can assume $Q(v) = \sum_{i=1}^n v_i v_{n+i}$ for all $v = (v_i) \in V = \mathbf{k}^{2n}$. A maximal torus of G is $T = \{\mathrm{diag}(t_1, t_2, \ldots, t_n, \lambda/t_1, \lambda/t_2, \ldots \lambda/t_n) | t_i \in \mathbf{k}^*, i = 1, \ldots, n; \lambda \in \mathbf{k}^*\}/\{(\mu I, \mu^2) | \mu \in \mathbf{k}^*\}$. The Lie algebra $\mathfrak{t}$ of T is $\{\mathrm{diag}(x_1, x_2, \ldots, x_n, \lambda + x_1, \lambda + x_2, \ldots \lambda + x_n) | x_i \in \mathbf{k}, i = 1, \ldots, n; \lambda \in \mathbf{k}\}/\{(\mu I, 0) | \mu \in \mathbf{k}\}$.

Let $x = \overline{\mathrm{diag}(x_1, x_2, \ldots, x_n, x_1 + \lambda, x_2 + \lambda, \ldots x_n + \lambda)} \in \mathfrak{t}$ where $x_i \neq x_j$, for any $i \neq j$ and $\lambda \neq x_i + x_j$ for any $i \neq j$ (such x exists), then similarly one can show that $Z_G^0(x) = T$ and x is regular.

Now denote by $\mathfrak{t}_0$ the set of regular elements in $\mathfrak{t}$. From the above construction, one easily see that $\mathfrak{t}_0$ is a dense subset in $\mathfrak{t}$. Thus $\dim \mathfrak{t}_0 = \dim \mathfrak{t} = r$. Consider the map $\mathfrak{t}_0 \times G \to \mathfrak{g}$, $(x, g) \mapsto \mathrm{Ad}(g)x$. The fiber at y in the image of the map is $\{(x, g) \in \mathfrak{t}_0 \times G | \mathrm{Ad}(g)x = y\}$. We consider the projection of $\{(x, g) \in \mathfrak{t}_0 \times G | \mathrm{Ad}(g)x = y\}$ to the first coordinate. The fiber of this projection at $x \in \mathfrak{t}_0$ is isomorphic to $Z_G(x)$, which has dimension r, and the image of this projection is finite. Hence $\dim\{(x, g) \in \mathfrak{t}_0 \times G | \mathrm{Ad}(g)x = y\} = r$. It follows that the image of the map $\mathfrak{t}_0 \times G \to \mathfrak{g}$, $(x, g) \mapsto \mathrm{Ad}(g)x$ has dimension equal to $\dim(\mathfrak{t}_0 \times G) - r = \dim \mathfrak{g}$. This proves the lemma. □

Remark. *Lemma 1.6.2 is not always true when G is not adjoint.*

1.6.3 Let Y (resp. $\mathfrak{t}_0$) be the set of regular semisimple elements in $\mathfrak{g}$ (resp. $\mathfrak{t}$). By Lemma 1.6.2, $\dim Y = \dim G$. Let $\widetilde{Y} = \{(x, gT) \in Y \times G/T | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{t}_0\}$. Define $\pi : \widetilde{Y} \to Y$ by $\pi(x, gT) = x$. The Weyl group $W = NT/T$ acts (freely) on $\widetilde{Y}$ by $n : (x, gT) \mapsto (x, gn^{-1}T)$.

Lemma 1.6.3. *$\pi : \widetilde{Y} \to Y$ is a principal W-bundle.*

Proof. We show that if $x \in \mathfrak{g}, g, g' \in G$ are such that $\mathrm{Ad}(g^{-1})x \in \mathfrak{t}_0$ and $\mathrm{Ad}(g'^{-1})x \in \mathfrak{t}_0$, then $g' = gn^{-1}$ for some $n \in NT$. Let $\mathrm{Ad}(g^{-1})x = t_1 \in \mathfrak{t}_0, \mathrm{Ad}(g'^{-1})x = t_2 \in \mathfrak{t}_0$, then we have $Z_G^0(x) = Z_G^0(\mathrm{Ad}(g)t_1) = gZ_G^0(t_1)g^{-1} = gTg^{-1}$, similarly $Z_G^0(x) = Z_G^0(\mathrm{Ad}(g')t_2) = g'Z_G^0(t_2)g'^{-1} = g'Tg'^{-1}$, hence $g'^{-1}g \in NT$. $\square$

Let $X = \{(x, gB) \in \mathfrak{g} \times G/B | \mathrm{Ad}(g^{-1})x \in \mathfrak{b}\}$. Define $\varphi : X \to \mathfrak{g}$ by $\varphi(x, gB) = x$. The map φ is G-equivariant with G-action on X given by $g_0 : (x, gB) \mapsto (\mathrm{Ad}(g_0)x, g_0gB)$.

Lemma 1.6.4. (i) *X is an irreducible variety of dimension equal to* $\dim G$.

(ii) *φ is proper and $\varphi(X) = \mathfrak{g} = \bar{Y}$.*

(iii) *$(x, gT) \to (x, gB)$ is an isomorphism $\gamma : \widetilde{Y} \xrightarrow{\sim} \varphi^{-1}(Y)$.*

Proof. (i) and (ii) are easy. For (iii), we only prove that γ is a bijection. First we show that γ is injective. Suppose $(x_1, g_1T), (x_2, g_2T) \in \widetilde{Y}$ are such that $(x_1, g_1B) = (x_2, g_2B)$, then we have $\mathrm{Ad}(g_1^{-1})(x_1) \in \mathfrak{t}_0, \mathrm{Ad}(g_2^{-1})(x_2) \in \mathfrak{t}_0$ and $x_1 = x_2, g_2^{-1}g_1 \in B$. Similar argument as in the proof of Lemma 1.6.3 shows $g_2^{-1}g_1 \in NT$, hence $g_2^{-1}g_1 \in B \cap NT = T$ and it follows that $g_1T = g_2T$. Now we show that γ is surjective. For $(x, gB) \in \varphi^{-1}(Y)$, we have $x \in Y, \mathrm{Ad}(g^{-1})(x) \in \mathfrak{b}$, hence there exists $b \in B, x_0 \in \mathfrak{t}_0$ such that $\mathrm{Ad}(g^{-1})(x) = \mathrm{Ad}(b)(x_0)$. Then $\gamma(x, gbT) = (x, gB)$. $\square$

Since $\pi : \widetilde{Y} \to Y$ is a finite covering, $\pi_!\bar{\mathbb{Q}}_{l\widetilde{Y}}$ is a well defined local system on Y. Thus the intersection cohomology complex $IC(\mathfrak{g}, \pi_!\bar{\mathbb{Q}}_{l\widetilde{Y}})$ is well defined.

Proposition 1.6.5. *$\varphi_!\bar{\mathbb{Q}}_{lX}$ is canonically isomorphic to $IC(\mathfrak{g}, \pi_!\bar{\mathbb{Q}}_{l\widetilde{Y}})$. Moreover, $End(\varphi_!\bar{\mathbb{Q}}_{lX}) = End(\pi_!\bar{\mathbb{Q}}_{l\widetilde{Y}}) = \bar{\mathbb{Q}}_l[W]$.*

Proof. By base change theorem, we have $\varphi_!\bar{\mathbb{Q}}_{lX}|_Y = \pi_!\bar{\mathbb{Q}}_{l\widetilde{Y}}$. Since φ is proper and X is smooth of dimension equal to $\dim Y$, we have that the Verdier dual $\mathfrak{D}(\varphi_!\bar{\mathbb{Q}}_{lX}) = \varphi_!(\mathfrak{D}\bar{\mathbb{Q}}_{lX}) \cong \varphi_!\bar{\mathbb{Q}}_{lX}[2\dim Y]$. Hence by the definition of intersection cohomology complex, it is enough to prove that $\forall\, i > 0, \dim \mathrm{supp}\mathcal{H}^i(\varphi_!\bar{\mathbb{Q}}_{lX}) < \dim Y - i$. For $x \in \mathfrak{g}$, the stalk $\mathcal{H}_x^i(\varphi_!\bar{\mathbb{Q}}_{lX}) = H_c^i(\varphi^{-1}(x), \bar{\mathbb{Q}}_l)$. Hence it is enough to show $\forall\, i > 0, \dim\{x \in$

$\mathfrak{g}|H^i_c(\varphi^{-1}(x),\bar{\mathbb{Q}}_l)\neq 0\} < \dim Y - i$. If $H^i_c(\varphi^{-1}(x),\bar{\mathbb{Q}}_l)\neq 0$, then $i\leq 2\dim\varphi^{-1}(x)$. Hence it is enough to show that $\forall\, i>0, \dim\{x\in\mathfrak{g}|\dim\varphi^{-1}(x)\geq i/2\} < \dim Y - i$. Suppose this is not true for some i, then $\dim\{x\in\mathfrak{g}|\dim\varphi^{-1}(x)\geq i/2\}\geq\dim Y - i$. Let $V=\{x\in\mathfrak{g}|\dim\varphi^{-1}(x)\geq i/2\}$, it is closed in $\mathfrak{g}$ but not equal to $\mathfrak{g}$. Consider the map $p: Z\to\mathfrak{g},\ (x,B_1,B_2)\mapsto x$. We have $\dim p^{-1}(V)=\dim V+2\dim\varphi^{-1}(x)\geq\dim V+i\geq\dim Y$ (for some $x\in V$). Thus by Lemma 1.6.1 (iii), $p^{-1}(V)$ contains some $Z_{\mathcal{O}}, \mathcal{O}=G$-orbit of (B,nBn^{-1}) in $\mathcal{B}\times\mathcal{B}$. If $x\in\mathfrak{t}_0$, then $(x,B,nBn^{-1})\in Z_{\mathcal{O}}$, hence x belongs to the projection of $p^{-1}(V)$ to $\mathfrak{g}$ which has dimension $\dim V<\dim Y$. But this projection is G-invariant hence contains all Y. We get a contradiction.

Since π is a principal W-bundle, we have $\mathrm{End}(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}})=\bar{\mathbb{Q}}_l[W]$. It follows that $\mathrm{End}(\varphi_!\bar{\mathbb{Q}}_{lX})=\bar{\mathbb{Q}}_l[W]$. □

1.6.4 In this subsection, we introduce some sheaves on the variety of semisimple G-conjugacy classes in $\mathfrak{g}$ similar to [L3, L5].

Let $\mathbf{A}$ be the set of closed G-conjugacy classes in $\mathfrak{g}$. These are precisely the semisimple classes in $\mathfrak{g}$ (for a proof in the group case see for example [St1], and one can prove for the Lie algebra case similarly). By geometric invariant theory, $\mathbf{A}$ has a natural structure of affine variety and there is a well-defined morphism $\sigma:\mathfrak{g}\to\mathbf{A}$ such that $\sigma(x)$ is the G-conjugacy class of x_s, where x_s is the semisimple part of x. There is a unique $\varsigma\in\mathbf{A}$ such that $\sigma^{-1}(\varsigma)=\{x\in\mathfrak{g}|x \text{ nilpotent}\}$.

Recall that $Z=\{(x,B_1,B_2)\in\mathfrak{g}\times\mathcal{B}\times\mathcal{B}|x\in\mathfrak{b}_1\cap\mathfrak{b}_2\}$. Define $\tilde{\sigma}:Z\to\mathbf{A}$ by $\tilde{\sigma}(x,B_1,B_2)=\sigma(x)$. For $a\in\mathbf{A}$, let $Z^a=\tilde{\sigma}^{-1}(a)$.

Lemma 1.6.6. *We have* $\dim Z^a\leq d_0$, *where* $d_0=\dim G-r$.

Proof. Define $m:Z^a\to\sigma^{-1}(a)$ by $(x,B_1,B_2)\mapsto x$. Let $\mathrm{c}\subset\sigma^{-1}(a)$ be a G-conjugacy class. Consider $m:m^{-1}(\mathrm{c})\to\mathrm{c}$. We have $\dim m^{-1}(\mathrm{c})\leq\dim\mathrm{c}+2(\dim G-r-\dim\mathrm{c})/2=\dim G-r$ (use Lemma 1.6.1 (ii)). Since $\sigma^{-1}(a)$ is a union of finitely many G-conjugacy classes, it follows that $\dim Z^a\leq d_0$. □

Let $\mathcal{T}=\mathcal{H}^{2d_0}\tilde{\sigma}_!\bar{\mathbb{Q}}_{lZ}$. Recall that we set $Z_{\mathcal{O}}=\{(x,B_1,B_2)\in Z|(B_1,B_2)\in\mathcal{O}\}$, where $\mathcal{O}$ is an orbit of G action on $\mathcal{B}\times\mathcal{B}$. Let $\mathcal{T}^{\mathcal{O}}=\mathcal{H}^{2d_0}\sigma^0_!\bar{\mathbb{Q}}_l$, where $\sigma^0: Z_{\mathcal{O}}\to\mathbf{A}$ is the restriction of $\tilde{\sigma}$ on $Z_{\mathcal{O}}$.

Lemma 1.6.7. *We have $\mathcal{T}^{\mathcal{O}} \cong \bar{\sigma}_!\bar{\mathbb{Q}}_l$, where $\bar{\sigma} : \mathfrak{t} \to \boldsymbol{A}$ is the restriction of σ.*

Proof. The fiber of the natural projection $pr_{23} : Z_{\mathcal{O}} \to \mathcal{O}$ at $(B, nBn^{-1}) \in \mathcal{O}$ can be identified with $V = \mathfrak{b} \cap n\mathfrak{b}n^{-1}$. Let $\mathcal{T}'^{\mathcal{O}} = \mathcal{H}^{2d_0 - 2\dim \mathcal{O}}\sigma'_!\bar{\mathbb{Q}}_l$, where $\sigma' : V \to \mathbf{A}$ is $x \mapsto \sigma(x)$. Let $\mathcal{T}''^{\mathcal{O}} = \mathcal{H}^{2d_0+2\dim H}\sigma''_!\bar{\mathbb{Q}}_l$, where $H = B \cap nBn^{-1}$ and $\sigma'' : G \times V \to \mathbf{A}$ is $(g, x) \mapsto \sigma(x)$. Consider the composition $G \times V \xrightarrow{pr_2} V \xrightarrow{\sigma'} \mathbf{A}$ (equal to σ'') and the composition $G \times V \xrightarrow{p} H\backslash(G \times V) = Z_{\mathcal{O}} \xrightarrow{\sigma^0} \mathbf{A}$ (equal to σ''), we obtain $\mathcal{T}''^{\mathcal{O}} = \mathcal{H}^{2d_0+2\dim H}(\sigma'_!pr_{2!}\bar{\mathbb{Q}}_l) = \mathcal{H}^{2d_0+2\dim H}(\sigma'_!\bar{\mathbb{Q}}_l[-2\dim G]) = \mathcal{T}'^{\mathcal{O}}, \mathcal{T}''^{\mathcal{O}} = \mathcal{H}^{2d_0+2\dim H}(\sigma^0_!p_!\bar{\mathbb{Q}}_l) = \mathcal{H}^{2d_0+2\dim H}(\sigma^0_!\bar{\mathbb{Q}}_l[-2\dim H]) = \mathcal{T}^{\mathcal{O}}$. It follows that $\mathcal{T}^{\mathcal{O}} = \mathcal{T}'^{\mathcal{O}}$. Now V is fibred over $\mathfrak{t}$ with fibers isomorphic to $\mathfrak{n} \cap n\mathfrak{n}n^{-1}$. The map $\sigma' : V \to \mathbf{A}$ factors through $\bar{\sigma} : \mathfrak{t} \to \mathbf{A}$. Since $\mathfrak{n} \cap n\mathfrak{n}n^{-1}$ is an affine space of dimension $d_0 - \dim \mathcal{O}$, we see that $\mathcal{T}^{\mathcal{O}} = \mathcal{T}'^{\mathcal{O}} \cong \mathcal{H}^0\bar{\sigma}_!\bar{\mathbb{Q}}_l$. Since $\bar{\sigma}$ is a finite covering (Lemma 1.6.8), we have $\mathcal{T}^{\mathcal{O}} \cong \bar{\sigma}_!\bar{\mathbb{Q}}_l$. □

Lemma 1.6.8. *The map $\bar{\sigma} : \mathfrak{t} \to \boldsymbol{A}$ is a finite covering.*

Proof. We show that for $x_1, x_2 \in \mathfrak{t}$, if $\sigma(x_1) = \sigma(x_2)$, then there exists $w \in W$ such that $x_2 = \mathrm{Ad}(w)x_1$. Since $\sigma(x_1) = \sigma(x_2)$, there exists $g \in G$ such that $x_1 = \mathrm{Ad}(g)(x_2)$. It follows that $Z_G^0(x_1) = gZ_G^0(x_2)g^{-1}$. We have $T \subset Z_G^0(x_1)$ and $gTg^{-1} \subset Z_G^0(x_1)$. Hence there exists $h \in Z_G^0(x_1)$ such that $hTh^{-1} = gTg^{-1}$. Let $n = g^{-1}h$, we have $x_2 = \mathrm{Ad}(g^{-1})x_1 = \mathrm{Ad}(nh^{-1})x_1 = \mathrm{Ad}(n)x_1$. □

Denote $\mathcal{T}_\varsigma$ and $\mathcal{T}_\varsigma^{\mathcal{O}}$ the stalk of $\mathcal{T}$ and $\mathcal{T}^{\mathcal{O}}$ at ς respectively.

Lemma 1.6.9. *For $w \in W$, let $\mathcal{O}_w$ be the G-orbit on $\mathcal{B}\times\mathcal{B}$ which contains $(B, n_wBn_w^{-1})$. There is a canonical isomorphism $\mathcal{T}_\varsigma \cong \bigoplus_{w\in W} \mathcal{T}_\varsigma^{\mathcal{O}_w}$.*

Proof. We have $\tilde{\sigma}^{-1}(\varsigma) = Z' = \{(x, B_1, B_2) \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} | x \in \mathfrak{n}_1 \cap \mathfrak{n}_2\}$. We have a partition $Z' = \sqcup_{w\in W} Z'_{\mathcal{O}_w}$, where $Z'_{\mathcal{O}_w} = \{(x, B_1, B_2) \in Z' | (B_1, B_2) \in \mathcal{O}_w\}$. Since $\dim Z' = d_0$, we have an isomorphism $H_c^{2d_0}(Z', \bar{\mathbb{Q}}_l) = \bigoplus_{w\in W} H_c^{2d_0}(Z'_{\mathcal{O}_w}, \bar{\mathbb{Q}}_l)$, which is $\mathcal{T}_\varsigma \cong \bigoplus_{w\in W} \mathcal{T}_\varsigma^{\mathcal{O}_w}$. □

Recall that we have $\bar{\mathbb{Q}}_l[W] = \mathrm{End}(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}}) = \mathrm{End}(\varphi_!\bar{\mathbb{Q}}_{lX})$. In particular, $\varphi_!\bar{\mathbb{Q}}_{lX}$ is naturally a W-module and $\varphi_!\bar{\mathbb{Q}}_{lX} \otimes \varphi_!\bar{\mathbb{Q}}_{lX}$ is naturally a W-module (with W acting on the first factor). This induces a W-module structure on $\mathcal{H}^{2d_0}\sigma_!(\varphi_!\bar{\mathbb{Q}}_{lX}\otimes\varphi_!\bar{\mathbb{Q}}_{lX})=\mathcal{T}$. Hence we obtain a W-module structure on the stalk $\mathcal{T}_\varsigma$.

Lemma 1.6.10. *Let $w\in W$. Multiplication by w in the W-module structure of $\mathcal{T}_\varsigma=\bigoplus_{w'\in W}\mathcal{T}_\varsigma^{\mathcal{O}_{w'}}$ defines for any $w'\in W$ an isomorphism $\mathcal{T}_\varsigma^{\mathcal{O}_{w'}}\xrightarrow{\sim}\mathcal{T}_\varsigma^{\mathcal{O}_{ww'}}$.*

Proof. We have an isomorphism $f: Z'_{\mathcal{O}_{w'}}\xrightarrow{\sim}Z'_{\mathcal{O}_{ww'}},(x,gBg^{-1},gn_{w'}Bn_{w'}^{-1}g^{-1})\mapsto(x,gn_w^{-1}Bn_wg^{-1},gn_{w'}Bn_{w'}^{-1}g^{-1})$. This induces an isomorphism $H_c^{2d_0}(Z'_{\mathcal{O}_{w'}},\bar{\mathbb{Q}}_l)\xrightarrow{\sim}H_c^{2d_0}(Z'_{\mathcal{O}_{ww'}},\bar{\mathbb{Q}}_l)$ which is just multiplication by w. □

1.6.5 Denote $\hat{W}$ the set of simple modules (up to isomorphism) for the Weyl group W (a description of $\hat{W}$ is given for example in [L2]). Given a semisimple object M of some abelian category such that M is a W-module, we write $M_\rho=\mathrm{Hom}_{\bar{\mathbb{Q}}_l[W]}(\rho,M)$ for $\rho\in\hat{W}$. We have $M=\oplus_{\rho\in\hat{W}}(\rho\otimes M_\rho)$ with W acting on the ρ-factor and M_ρ in our abelian category. In particular, we have $\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}}=\bigoplus_{\rho\in\hat{W}}(\rho\otimes(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}})_\rho)$ and $\varphi_!\bar{\mathbb{Q}}_{lX}=\bigoplus_{\rho\in\hat{W}}(\rho\otimes(\varphi_!\bar{\mathbb{Q}}_{lX})_\rho)$, where $(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}})_\rho$ is an irreducible local system on Y and $(\varphi_!\bar{\mathbb{Q}}_{lX})_\rho=IC(\mathfrak{g},(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}})_\rho)$. Moreover, for $a\in\mathbf{A}$, we have $\mathcal{T}_a=\bigoplus_{\rho\in\hat{W}}(\rho\otimes(\mathcal{T}_a)_\rho)$. Set $\bar{Y}^\varsigma=\{x\in\mathfrak{g}|\sigma(x)=\varsigma\},X^\varsigma=\varphi^{-1}(\bar{Y}^\varsigma)\subset X$. We have $\bar{Y}^\varsigma=\{x\in\mathfrak{g}|x\text{ nilpotent}\}:=\mathcal{N}$. Let $\varphi^\varsigma:X^\varsigma\to\mathcal{N}$ be the restriction of $\varphi:X\to\mathfrak{g}$.

Lemma 1.6.11. (i) *X^ς and $\mathcal{N}$ are irreducible varieties of dimension $d_0=\dim G-r$.*

(ii) *We have $(\varphi_!\bar{\mathbb{Q}}_{lX})|_{\mathcal{N}}=\varphi^\varsigma_!\bar{\mathbb{Q}}_{lX^\varsigma}$. Moreover, $\varphi^\varsigma_!\bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ is a semisimple perverse sheaf on $\mathcal{N}$.*

(iii) *We have $(\varphi_!\bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}\neq 0$ for any $\rho\in\hat{W}$.*

Proof. (i) $\mathcal{N}$, the nilpotent variety, is well-known to be irreducible of dimension d_0. We have $X^\varsigma=\{(x,gB)\in\mathfrak{g}\times\mathcal{B}|\mathrm{Ad}(g^{-1})(x)\in\mathfrak{n}\}$. By projection to the second coordinate, we see that $\dim X^\varsigma=\dim\mathfrak{n}+\dim\mathcal{B}=\dim G-r$. This proves (i).

The first assertion of (ii) follows from base change theorem. Since φ^ς is proper, by similar argument as in the proof of Proposition 1.6.5, to show that $\varphi^\varsigma_!\bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ is a perverse sheaf, it suffices to show $\forall\ i\geq 0,\dim\mathrm{supp}\mathcal{H}^i(\varphi^\varsigma_!\bar{\mathbb{Q}}_{lX^\varsigma})\leq\dim\mathcal{N}-i$. It is enough to show $\forall\ i\geq 0$, $\dim\{x\in\mathcal{N}|\dim(\varphi^\varsigma)^{-1}(x)\geq i/2\}\leq\dim\mathcal{N}-i$. If this is not true for some $i\geq 0$, it would follow that the variety $\{(x,B_1,B_2)\in\mathfrak{g}\times\mathcal{B}\times\mathcal{B}|x\in\mathfrak{n}_1\cap\mathfrak{n}_2\}$

has dimension greater than $\dim \mathcal{N} = \dim G - r$, which contradicts to Lemma 1.6.1. This proves that $\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ is a perverse sheaf. It is semisimple by the decomposition theorem ([BBD]). This proves (ii).

Now we prove (iii). By Lemma 1.6.7, we have $\mathcal{T}_\varsigma^{\mathcal{O}_1} = H_c^0(\mathfrak{t} \cap \sigma^{-1}(\varsigma), \bar{\mathbb{Q}}_l) \neq 0$. From Lemma 1.6.10, we see that the W-module structure defines an injective map $\bar{\mathbb{Q}}_l[W] \otimes \mathcal{T}_\varsigma^{\mathcal{O}_1} \to \mathcal{T}_\varsigma$. Since $\mathcal{T}_\varsigma^{\mathcal{O}_1} \neq 0$, we have $(\bar{\mathbb{Q}}_l[W] \otimes \mathcal{T}_\varsigma^{\mathcal{O}_1})_\rho \neq 0$ for any $\rho \in \hat{W}$, hence $(\mathcal{T}_\varsigma)_\rho \neq 0$. We have $\mathcal{T}_\varsigma = H_c^{2d_0}(\mathcal{N}, \varphi_! \bar{\mathbb{Q}}_{lX} \otimes \varphi_! \bar{\mathbb{Q}}_{lX})$, hence $\bigoplus_{\rho \in \hat{W}} \rho \otimes (\mathcal{T}_\varsigma)_\rho = \bigoplus_{\rho \in \hat{W}} \rho \otimes H_c^{2d_0}(\mathcal{N}, (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho \otimes \varphi_! \bar{\mathbb{Q}}_{lX})$. This implies that $(\mathcal{T}_\varsigma)_\rho = H_c^{2d_0}(\mathcal{N}, (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho \otimes \varphi_! \bar{\mathbb{Q}}_{lX})$. Thus it follows from $(\mathcal{T}_\varsigma)_\rho \neq 0$ that $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}} \neq 0$ for any $\rho \in \hat{W}$. □

Let $\mathfrak{A}_\mathfrak{g}$ be defined for G as in the introduction.

Proposition 1.6.12. (i) *The restriction map* $End_{\mathcal{D}(\mathfrak{g})}(\varphi_! \bar{\mathbb{Q}}_{lX}) \to End_{\mathcal{D}(\mathcal{N})}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma})$ *is an isomorphism.*

(ii) *For any* $\rho \in \hat{W}$, *there is a unique* $(\mathrm{c}, \mathcal{F}) \in \mathfrak{A}_\mathfrak{g}$ *such that* $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}[d_0]$ *is* $IC(\bar{\mathrm{c}}, \mathcal{F})[\dim \mathrm{c}]$ *regarded as a simple perverse sheaf on* $\mathcal{N}$ *(zero outside* $\bar{\mathrm{c}}$*). Moreover,* $\rho \mapsto (\mathrm{c}, \mathcal{F})$ *is an injective map* $\gamma : \hat{W} \to \mathfrak{A}_\mathfrak{g}$.

Proof. (i). Recall that we have $\varphi_! \bar{\mathbb{Q}}_{lX} = \bigoplus_{\rho \in \hat{W}} \rho \otimes (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho$ where $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho[\dim \mathfrak{g}]$ are simple perverse sheaves on $\mathfrak{g}$. Thus we have $\varphi_! \bar{\mathbb{Q}}_{lX}|_{\mathcal{N}} = \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma} = \bigoplus_{\rho \in \hat{W}} \rho \otimes (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}$ (we use Lemma 1.6.11 (ii)). The restriction map $\mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\varphi_! \bar{\mathbb{Q}}_{lX}) \to \mathrm{End}_{\mathcal{D}(\mathcal{N})}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma})$ is factorized as $\bigoplus_{\rho \in \hat{W}} \mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\rho \otimes (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho) \xrightarrow{b} \bigoplus_{\rho \in \hat{W}} \mathrm{End}_{\mathcal{D}(\mathcal{N})}(\rho \otimes (\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}) \xrightarrow{c} \mathrm{End}_{\mathcal{D}(\mathcal{N})}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma})$, where $b = \oplus_\rho b_\rho$, $b_\rho : \mathrm{End}(\rho) \otimes \mathrm{End}_{\mathcal{D}(\mathfrak{g})}((\varphi_! \bar{\mathbb{Q}}_{lX})_\rho) \to \mathrm{End}(\rho) \otimes \mathrm{End}_{\mathcal{D}(\mathcal{N})}((\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}})$. By Lemma 1.6.11 (iii), $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}} \neq 0$, thus $\mathrm{End}_{\mathcal{D}(\mathfrak{g})}((\varphi_! \bar{\mathbb{Q}}_{lX})_\rho) = \bar{\mathbb{Q}}_l \subset \mathrm{End}_{\mathcal{D}(\mathcal{N})}((\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}})$. It follows that b_ρ and thus b is injective. Since c is also injective, the restriction map in injective. Hence it remains to show that $\dim \mathrm{End}_{\mathcal{D}(\mathcal{N})}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}) = \dim \mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\varphi_! \bar{\mathbb{Q}}_{lX})$.

For A, A' two simple perverse sheaves on a variety X, we have $H_c^0(X, A \otimes A') = 0$ if and only if A is not isomorphic to $\mathfrak{D}(A')$ and $\dim H_c^0(X, A \otimes \mathfrak{D}(A)) = 1$ (see [L4, 7.4]). We apply this to the semisimple perverse sheaf $\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]$ on $\mathcal{N}$ and get $\dim \mathrm{End}_{\mathcal{D}(\mathcal{N})}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}) = \dim H_c^0(\mathcal{N}, \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0] \otimes \mathfrak{D}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]))$. We have $\dim H_c^0(\mathcal{N}, \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0] \otimes \mathfrak{D}(\varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0])) = \dim H_c^0(\mathcal{N}, \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0] \otimes \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}[d_0]) =$

$\dim H_c^{2d_0}(\mathcal{N}, \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma} \otimes \varphi_!^\varsigma \bar{\mathbb{Q}}_{lX^\varsigma}) = \dim H_c^{2d_0}(\mathcal{N}, \varphi_! \bar{\mathbb{Q}}_{lX} \otimes \varphi_! \bar{\mathbb{Q}}_{lX}) = \dim \mathcal{T}_\varsigma$ $= \sum_{w \in W} \dim \mathcal{T}_\varsigma^{\mathcal{O}_w}$ (The third equality follows from Lemma 1.6.11 (ii) and the last one follows from Lemma 1.6.9). We have $\mathcal{T}_\varsigma^{\mathcal{O}_w} = H_c^0(\bar{\sigma}^{-1}(\varsigma), \bar{\mathbb{Q}}_l)$ (see Lemma 1.6.7), hence $\dim \mathcal{T}_\varsigma^{\mathcal{O}_w} = 1$ and $\sum_{w \in W} \dim \mathcal{T}_\varsigma^{\mathcal{O}_w} = |W| = \dim \mathrm{End}_{\mathcal{D}(\mathfrak{g})}(\varphi_! \bar{\mathbb{Q}}_{lX})$. Thus (i) is proved.

From the proof of (i) we see that both b and c are isomorphisms. It follows that the perverse sheaf $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}[d_0]$ on $\mathcal{N}$ is simple and that for $\rho, \rho' \in \hat{W}$, we have $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}[d_0] \cong (\varphi_! \bar{\mathbb{Q}}_{lX})_{\rho'}|_{\mathcal{N}}[d_0]$ if and only if $\rho = \rho'$. Since the simple perverse sheaf $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}[d_0]$ is G-equivariant and $\mathcal{N}$ consists of finitely many nilpotent G-conjugacy classes, $(\varphi_! \bar{\mathbb{Q}}_{lX})_\rho|_{\mathcal{N}}[d_0]$ must be as in (ii). □

1.6.6 In this subsection let $G = SO_N(\mathbf{k})$ (resp. $Sp_{2n}(\mathbf{k})$) and $\mathfrak{g} = \mathfrak{o}_N(\mathbf{k})$ (resp. $\mathfrak{sp}_{2n}(\mathbf{k})$) the Lie algebra of G. Let G_{ad} be an adjoint group over $\mathbf{k}$ of the same type as G and $\mathfrak{g}_{ad}$ the Lie algebra of G_{ad}. Let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a split Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G, $\mathfrak{g}$. Let $G_{ad}(\mathbf{F}_q)$, $\mathfrak{g}_{ad}(\mathbf{F}_q)$ be defined like $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$. Let $\mathfrak{A}_\mathfrak{g}$ and $\mathfrak{A}_{\mathfrak{g}_{ad}}$ be defined as in the introduction for G and G_{ad} respectively. Denote the number of elements in $\mathfrak{A}_{\mathfrak{g}_{ad}}$ (resp. $\mathfrak{A}_\mathfrak{g}$) by $|\mathfrak{A}_{\mathfrak{g}_{ad}}|$ (resp. $|\mathfrak{A}_\mathfrak{g}|$). We show that $|\mathfrak{A}_{\mathfrak{g}_{ad}}| = |\mathfrak{A}_\mathfrak{g}|$.

We first show that $|\mathfrak{A}_\mathfrak{g}|$ is equal to the number of nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)$ (for q large). To see this we can assume $\mathbf{k} = \bar{\mathbf{F}}_2$. Pick representatives $x_1, \ldots, x_M$ for the nilpotent G-orbits in $\mathfrak{g}$. If q is large enough, the Frobenius map $\mathfrak{F}_q$ keeps x_i fixed and acts trivially on $Z_G(x_i)/Z_G^0(x_i)$. Then the number of $G(\mathbf{F}_q)$-orbits in the G-orbit of x_i is equal to the number of irreducible representations of $Z_G(x_i)/Z_G^0(x_i)$ hence to the number of G-equivariant irreducible local systems on the G-orbit of x_i. Similarly, $|\mathfrak{A}_{\mathfrak{g}_{ad}}|$ is equal to the number of nilpotent $G_{ad}(\mathbf{F}_q)$-orbits in $\mathfrak{g}_{ad}(\mathbf{F}_q)$.

On the other hand, the number of nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)$ is equal to the number of nilpotent $G_{ad}(\mathbf{F}_q)$-orbits in $\mathfrak{g}_{ad}(\mathbf{F}_q)$. In fact, we have a morphism $G \to G_{ad}$ which is an isomorphism of abstract groups and an obvious bijective morphism $\mathcal{N} \to \mathcal{N}_{ad}$ between the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$ and the nilpotent variety $\mathcal{N}_{ad}$ of $\mathfrak{g}_{ad}$. Thus the nilpotent orbits in $\mathfrak{g}$ and $\mathfrak{g}_{ad}$ are in bijection and the corresponding component groups of centralizers are isomorphic. It follows that $|\mathfrak{A}_\mathfrak{g}| = |\mathfrak{A}_{\mathfrak{g}_{ad}}|$.

Corollary 1.6.13. $|\mathfrak{A}_{\mathfrak{g}}| = |\mathfrak{A}_{\mathfrak{g}_{ad}}| = |\hat{W}|$.

Proof. Assume $G = SO_N(\mathbf{k})$. From Proposition 1.5.1, Corollary 1.5.2 and the above argument we see that $|\mathfrak{A}_{\mathfrak{g}}| = |\mathfrak{A}_{\mathfrak{g}_{ad}}| \leq |\hat{W}|$. On the other hand, by Proposition 1.6.12 (ii), we have $|\mathfrak{A}_{\mathfrak{g}_{ad}}| \geq |\hat{W}|$.

Assume $G = Sp_{2n}(\mathbf{k})$. It is known in [Spa1] that the number of nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)$ is equal to $|\hat{W}|$. The assertion follows from the above argument. □

Theorem 1.6.14. *The map γ in Proposition 1.6.12* (ii) *is a bijection.*

Corollary 1.6.15. *Proposition 1.4.1, Corollary 1.4.2, Proposition 1.5.1, Corollary 1.5.2 hold with all "at most" removed.*

Proof. For q large enough, this follows from Corollary 1.6.13. Now let q be an arbitrary power of 2. The Frobenius map $\mathfrak{F}_q$ acts trivially on W. Since we have the Springer correspondence map γ in Proposition 1.6.12 (ii), for each pair $(\mathrm{c}, \mathcal{F}) \in \mathfrak{A}_{\mathfrak{g}_{ad}}$, c is stable under the Frobenius map $\mathfrak{F}_q$ and we have $\mathfrak{F}_q^{-1}(\mathcal{F}) \cong \mathcal{F}$. Pick a rational point x in c. The G_{ad}-equivariant local systems on c are in 1-1 correspondence with the isomorphism classes of the irreducible representations of $Z_{G_{ad}}(x)/Z^0_{G_{ad}}(x)$. Since $Z_{G_{ad}}(x)/Z^0_{G_{ad}}(x)$ is abelian (see Proposition 1.7.1) and the Frobenius map $\mathfrak{F}_q$ acts trivially on the irreducible representations of $Z_{G_{ad}}(x)/Z^0_{G_{ad}}(x)$, $\mathfrak{F}_q$ acts trivially on $Z_{G_{ad}}(x)/Z^0_{G_{ad}}(x)$. Thus it follows that the number of nilpotent $G_{ad}(\mathbf{F}_q)$-orbits in $\mathfrak{g}_{ad}(\mathbf{F}_q)$ is independent of q hence it is equal to $|\mathfrak{A}_{\mathfrak{g}_{ad}}| = |\hat{W}|$. □

A corollary of Theorem 1.6.14 is that in this case there are no cuspidal local systems similarly defined as in [L3]. This result does not extend to exceptional Lie algebras. (In type F_4, characteristic 2, the results of [Spa3] suggest that a cuspidal local system exists on a nilpotent class.)

1.7 Component groups of centralizers

In this section we describe the component groups of centralizers $Z_G(x)/Z^0_G(x)$, where $G = SO_N(V)$ and $x \in \mathfrak{o}_N(V)$ is nilpotent.

Proposition 1.7.1. (i) *Assume V is defective. Let $x \in \mathfrak{o}(V)$ be a nilpotent element corresponding to the form module $(\lambda_1)^{m_1}_{\chi(\lambda_1)}(\lambda_2)^{m_2}_{\chi(\lambda_2)}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$. We have $Z_{O(V)}(x)/Z^0_{O(V)}(x) = (\mathbb{Z}_2)^{n_1}$, where n_1 is as in Proposition 1.4.1* (i).

(ii) *Assume V is non-defective. Let $x \in \mathfrak{o}(V)$ be a nilpotent element corresponding to the form module $(\lambda_1)^{m_1}_{\chi(\lambda_1)}(\lambda_2)^{m_2}_{\chi(\lambda_2)}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$. Then $Z_{SO(V)}(x)/Z^0_{SO(V)}(x) = (\mathbb{Z}_2)^{n_2-1}$, where n_2 is as in Proposition 1.4.1* (ii).

Proof. (i) We write $Z = Z_{O(V)}(x)$ and $Z^0 = Z^0_{O(V)}(x)$ for simplicity. We can assume that q is large enough. (i) is proved in two steps.

Step 1: we show that Z/Z^0 is an abelian group of order 2^{n_1}. The group Z/Z^0 has 2^{n_1} conjugacy classes, since the G-orbit of x splits into 2^{n_1} $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)$ (Corollary 1.6.15). We show $|Z/Z^0| = 2^{n_1}$ by showing that

$$(*) \qquad |Z(\mathbf{F}_q)| = 2^{n_1}q^{\dim(Z)} + \text{ lower terms in } q.$$

We prove $(*)$ by induction on n_1. Suppose $n_1 = 0$, then Z/Z^0 has only one conjugacy class. It follows that $Z/Z^0 = \{1\}$ and $(*)$ holds for $n_1 = 0$. Suppose $n_1 \geq 1$. Let t be the minimal integer such that $\chi(\lambda_t) \neq \lambda_t/2$ and $\chi(\lambda_t) + \chi(\lambda_{t+1}) \leq \lambda_t$. Let $V_1 = (\lambda_1)^{m_1}_{\chi(\lambda_1)}(\lambda_2)^{m_2}_{\chi(\lambda_2)}\cdots(\lambda_t)^{m_t}_{\chi(\lambda_t)}$ and $V_2 = (\lambda_{t+1})^{m_{t+1}}_{\chi(\lambda_{t+1})}\cdots(\lambda_s)^{m_s}_{\chi(\lambda_s)}$. Then V_1 is non-defective. Write $Z_i = Z_{O(V_i)}(V_i)$ and $Z_i^0 = Z^0_{O(V_i)}(V_i)$, $i = 1, 2$. We have $|Z_{SO(V_1)}(V_1)/Z^0_{SO(V_1)}(V_1)| = 1 \Rightarrow |Z_1/Z_1^0| = 2$. It follows that $|Z_1(\mathbf{F}_q)| = 2q^{\dim Z_1} +$ lower terms in q. We show that $|Z(\mathbf{F}_q)| = |Z_1(\mathbf{F}_q)| \cdot |Z_2(\mathbf{F}_q)| \cdot q^{\dim \mathrm{Hom}_A(V_1,V_2)}$. Then the assertion $(*)$ follows from induction hypothesis since we have $\dim Z_1 + \dim Z_2 + \dim \mathrm{Hom}_A(V_1, V_2) = \dim Z$.

Consider V_1 as an element in the Grassmannian variety $Gr(V, r)$ of dimension $r = \sum_{j=1}^t m_j\lambda_j$. Then $C(V) = \{g \in GL(V) | gx = xg\}$ acts on $Gr(V, r)$. We have $C(V)(V_1 \oplus V_2) = C(V)V_1 \oplus C(V)V_2 \cong V_1 \oplus V_2$. By our choice of V_1 and V_2, it follows that $C(V)V_1 \cong V_1$ and $C(V)V_2 \cong V_2$. Thus the orbit of V_1 under $C(V)$ coincides with the orbit of V_1 under the action of Z. It is easy to verify that this orbit consists of $q^{\dim \mathrm{Hom}_A(V_1,V_2)}$ elements (using $C(V)$ action). Since the stabilizer of V_1 in Z is the product of Z_1 and Z_2, we get $|Z(\mathbf{F}_q)| = |Z_1(\mathbf{F}_q)| \cdot |Z_2(\mathbf{F}_q)| \cdot q^{\dim \mathrm{Hom}_A(V_1,V_2)}$.

Step 2: we show that there is a subgroup $(\mathbb{Z}_2)^{n_1} \subset Z/Z^0$. Thus Z/Z^0 has to be $(\mathbb{Z}_2)^{n_1}$. Let $1 \leq i_1, \ldots, i_{n_1} \leq s-1$ be such that $\chi(\lambda_{i_j}) > \lambda_{i_j}/2$ and $\chi(\lambda_{i_j}) + \chi(\lambda_{i_j+1}) \leq$

λ_{i_j}, $j=1,\ldots,n_1$. Let $V_j=(\lambda_{i_{j-1}+1})^{m_{i_{j-1}+1}}_{\chi(\lambda_{i_{j-1}+1})}\cdots(\lambda_{i_j})^{m_{i_j}}_{\chi(\lambda_{i_j})}$, $j=1,\ldots,n_1+1$, where $i_0=0$, $i_{n_1+1}=s$. Then $V=V_1\oplus V_2\oplus\cdots\oplus V_{n_1+1}$, where V_i, $i=1,\ldots,n_1$ are non-defective and V_{n_1+1} is defective and non-degenerate. We have $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)=\mathbb{Z}_2$, $i=1,\ldots,n_1$, and $Z_{O(V_{n_1+1})}(V_{n_1+1})/Z^0_{O(V_{n_1+1})}(V_{n_1+1})=\{1\}$. Take $g_i\in Z_{O(V_i)}(V_i)$ such that $g_iZ^0_{O(V_i)}(V_i)$ generates $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)$, $i=1,\ldots,n_1$. We know each V_i, $i=1,\ldots,n_1$, has two isomorphism classes V_i^0,V_i^δ over $\mathbf{F}_q$. Let m_i^0 and m_i^δ be two elements in $\mathfrak{o}(V_i)(\mathbf{F}_q)$ corresponding to V_i^0 and V_i^δ respectively. We can assume $x=m_1^0\oplus m_2^0\oplus\cdots\oplus m_{n_1}^0\oplus m_{n_1+1}$. Let $\tilde{g}_i=Id\oplus\cdots\oplus g_i\oplus\cdots\oplus Id$, $i=1,\ldots,n_1$. Then we have $\tilde{g}_i\in Z$ and $\tilde{g}_i\notin Z^0$, since $V_1^0\oplus\cdots\oplus V_i^0\oplus\cdots\oplus V_{n_1}^0\oplus V_{n_1+1}\not\cong V_1^0\oplus\cdots\oplus V_i^\delta\oplus\cdots\oplus V_{n_1}^0\oplus V_{n_1+1}$ (Corollary 1.6.15). We also have that the images of $\tilde{g}_{i_1}\cdots\tilde{g}_{i_p}$'s, $1\leq i_1<\cdots<i_p\leq n_1$, $p=1,\ldots,n_1$, in Z/Z^0 are not equal to each other. Moreover $\tilde{g}_i^2\in Z^0$. Thus the $\tilde{g}_iZ^0$'s generate a subgroup $(\mathbb{Z}_2)^{n_1}$ in Z/Z^0.

(ii) Let us write $Z=Z_{SO(V)}(x)$ and $Z^0=Z^0_{SO(V)}(x)$ for simplicity. Assume $n_2\geq 1$. We know that the group Z/Z^0 has 2^{n_2-1} conjugacy classes since the G-orbit of x splits into 2^{n_2-1} $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)$ (we can assume q large enough). The same argument as in (i) shows that $|Z(\mathbf{F}_q)|=2^{n_2-1}q^{\dim(Z)}$+lower terms. Then it follows that Z/Z^0 is an abelian group of order 2^{n_2-1}. It is enough to show that there is a subgroup $(\mathbb{Z}_2)^{n_2-1}\subset Z/Z^0$. Let $1\leq i_1,\ldots,i_{n_2}\leq s$ be such that $\chi(\lambda_{i_j})>\lambda_{i_j}/2$ and $\chi(\lambda_{i_j})+\chi(\lambda_{i_j+1})\leq\lambda_{i_j}$, $j=1,\ldots,n_2$.

Case 1: $\chi(\lambda_s)=\lambda_s/2$. Then $i_{n_2}<s$. Let $V_j=(\lambda_{i_{j-1}+1})^{m_{i_{j-1}+1}}_{\chi(\lambda_{i_{j-1}+1})}\cdots(\lambda_{i_j})^{m_{i_j}}_{\chi(\lambda_{i_j})}$, $j=1,\ldots,n_2+1$, where $i_0=0$, $i_{n_2+1}=s$. Then $V=V_1\oplus\cdots\oplus V_{n_2+1}$. We have $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)=\mathbb{Z}_2$, $i=1,\ldots,n_2$ and $Z_{O(V_{n_2+1})}(V_{n_2+1})/Z^0_{O(V_{n_2+1})}(V_{n_2+1})=\{1\}$. Take $g_i\in Z_{O(V_i)}(V_i)$ such that $g_iZ^0_{O(V_i)}(V_i)$ generates $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)$, $i=1,\ldots,n_2$. Let $\tilde{g}_i=g_1\oplus Id\oplus\cdots\oplus g_i\oplus\cdots\oplus Id$, $i=2,\ldots,n_2$. We have $\tilde{g}_i\in Z$ and $\tilde{g}_i\notin Z^0$. We also have the images of $\tilde{g}_{i_1}\cdots\tilde{g}_{i_p}$'s, $2\leq i_1<\cdots<i_p\leq n_2$, $p=1,\ldots,n_2-1$, in Z/Z^0 are not equal to each other. Moreover $\tilde{g}_i^2\in Z^0$. Hence the $\tilde{g}_iZ^0$'s generate a subgroup $(\mathbb{Z}_2)^{n_2-1}$ in Z/Z^0.

Case2: $\chi(\lambda_s)>\lambda_s/2$. Then $i_{n_2}=s$. Let $V_j=(\lambda_{i_{j-1}+1})^{m_{i_{j-1}+1}}_{\chi(\lambda_{i_{j-1}+1})}\cdots(\lambda_{i_j})^{m_{i_j}}_{\chi(\lambda_{i_j})}$, $j=1,\ldots,n_2$, where $i_0=0$. Then $V=V_1\oplus\cdots\oplus V_{n_2}$. We have $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)=\mathbb{Z}_2$, $i=1,\ldots,n_2$. Take $g_i\in Z_{O(V_i)}(V_i)$ such that $g_iZ^0_{O(V_i)}(V_i)$ generates $Z_{O(V_i)}(V_i)/Z^0_{O(V_i)}(V_i)$,

$i = 1, \ldots, n_2$. Let $\tilde{g}_i = g_1 \oplus Id \oplus \cdots \oplus g_i \oplus \cdots \oplus Id$, $i = 2, \cdots, n_2$. The $\tilde{g}_i Z^0$'s generate a subgroup $(\mathbb{Z}_2)^{n_2 - 1}$ in Z/Z^0. $\square$

1.8 Example

We list the unipotent classes in $SO(9)$ and nilpotent classes in $\mathfrak{so}(9)$. We use the notation in [H], see also section 1.1.

Unipotent classes in characteristic 2: $\underline{8_5 1_1}$, $\underline{6_4 2_2 1_1}$, $\underline{6_4 1_1^3}$, $\underline{4_3^2 1_1}$, $4_2^2 1_1$, $4_3 2_2^2 1_1$, $\underline{4_3 2_1^2 1_1}$, $4_3 2_2 1_1^3$, $\underline{4_3 1_1^5}$, $3_2^2 2_2 1_1$, $\underline{3_2^2 1_1^3}$, $2_2^4 1_1$, $2_1^4 1_1$, $2_2^3 1_1^3$, $2_2^2 1_1^5$, $2_1^2 1_1^5$, $2_2 1_1^7$, 1_1^9.

Nilpotent classes in characteristic 2: $5_5 4_4$, $4_4^2 1_1$, $\underline{4_3^2 1_1}$, $4_2^2 1_1$, $4_4 3_3 1_1^2$, $3_3^2 1_1^3$, $\underline{3_2^2 1_1^3}$, $3_3^2 2_2 1_1$, $3_2^2 2_2 1_1$, $3_3 2_2^3$, $3_3 2_2 1_1^4$, $2_2^4 1_1$, $2_1^4 1_1$, $2_2^3 1_1^3$, $2_2^2 1_1^5$, $2_1^2 1_1^5$, $2_2 1_1^7$, 1_1^9.

Unipotent/nilpotent classes in characteristic not 2: 9_5, $\underline{7_4^1 1_1^2}$, $\underline{\underline{5_3^1 3_2^1 1_1^1}}$, $5_3^1 2_1^2$, $\underline{5_3^1 1_1^4}$, $4_2^2 1_1^1$, 3_2^3, $\underline{3_2^2 1_1^3}$, $\underline{3_2^1 2_1^2 1_1^2}$, $\underline{3_2^1 1_1^6}$, $2_1^4 1_1^1$, $2_1^2 1_1^5$, 1_1^9.

In each case the component group of the centralizer is trivial except that the component groups for the underlined ones are $\mathbb{Z}/2\mathbb{Z}$ and for the double underlined one is $(\mathbb{Z}/2\mathbb{Z})^2$. In this case the number of unipotent classes and nilpotent classes over an algebraically closed field of characteristic 2 happen to be the same, but this is not true in higher ranks. Note that the component groups are already quite different.

Chapter 2

Dual of Classical Lie Algebras

Throughout this chapter, $\mathbf{k}$ denotes a field of characteristic 2.

2.1 Introduction

Let G be a connected algebraic group of type B, C or D defined over $\mathbf{k}$ and $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. We have a natural coadjoint action of G on $\mathfrak{g}^*$, $g.\xi(x) = \xi(\mathrm{Ad}(g)^{-1}x)$ for $g \in G, \xi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. Fix a Borel subgroup B of G. Let $\mathfrak{b}$ be the Lie algebra of B and let $\mathfrak{n}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{b}) = 0\}$. An element ξ in $\mathfrak{g}^*$ is called nilpotent if there exists $g \in G$ such that $g.\xi \in \mathfrak{n}^*$ (see [KW]). We classify the nilpotent orbits in $\mathfrak{g}^*$ under the coadjoint action of G in the cases where $\mathbf{k}$ is algebraically closed and where $\mathbf{k}$ is a finite field $\mathbf{F}_q$. In particular, we obtain the number of nilpotent orbits over $\mathbf{F}_q$ and the structure of component groups of the centralizers of nilpotent elements.

We have constructed a Springer correspondence for $\mathfrak{g}$ in chapter 1 using a similar construction as in [L3, L5]. Let $\mathfrak{A}_{\mathfrak{g}^*}$ be the set of all pairs $(\mathrm{c}, \mathcal{F})$ where c is a nilpotent G-orbit in $\mathfrak{g}^*$ and $\mathcal{F}$ is an irreducible G-equivariant local system on c (up to isomorphism). We use a similar construction to construct a Springer correspondence for $\mathfrak{g}^*$, which is a bijective map from the set of isomorphism classes of irreducible representations of the Weyl group of G to the set $\mathfrak{A}_{\mathfrak{g}^*}$.

2.2 Symplectic groups

In this section we study the nilpotent orbits in $\mathfrak{g}^*$ where G is a symplectic group.

2.2.1 Let V be a vector space of dimension $2n$ over $\mathbf{k}$ equipped with a non-degenerate symplectic form $\beta : V \times V \to \mathbf{k}$. The symplectic group is defined as $G = Sp(2n) = \{g \in GL(V) \mid \beta(gv, gw) = \beta(v, w), \forall\, v, w \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{sp}(2n) = \{x \in \mathrm{End}_{\mathbf{k}}(V) \mid \beta(xv, w) + \beta(v, xw) = 0, \forall\, v, w \in V\}$.

Let $\xi \in \mathfrak{g}^*$. There exists $X \in \mathrm{End}_{\mathbf{k}}(V)$ such that $\xi(x) = \mathrm{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a quadratic form $\alpha_\xi : V \to \mathbf{k}$ by

$$\alpha_\xi(v) = \beta(v, Xv).$$

Lemma 2.2.1. *The quadratic form α_ξ is well-defined.*

Proof. Recall that the space $\mathrm{Quad}(V)$ of quadratic forms on V coincides with the second symmetric power $S^2(V^*)$ of V^*. Consider the following linear mapping

$$\Phi : \mathrm{End}_{\mathbf{k}}(V) \to S^2(V^*) = \mathrm{Quad}(V), \quad X \mapsto \alpha_X$$

where $\alpha_X(v) = \beta(v, Xv)$. It is easy to see that Φ is $G = Sp(V)$-equivariant. One can show that $\ker \Phi$ coincides with the orthogonal complement $\mathfrak{g}^\perp$ of $\mathfrak{g} = \mathfrak{sp}(V)$ in $\mathrm{End}_{\mathbf{k}}(V)$ under the nondegenerate trace form. It follows that α_ξ does not depend on the choice of X. □

Remark. *The present coordinate free proofs of Lemmas 2.2.1, 2.2.5, 2.3.1, 2.3.5, 2.3.6, 2.3.12 and 2.4.2 are suggested by the referee of [X2]. These proofs replace my earlier proofs for which coordinates are used.*

Let β_ξ be the symmetric bilinear form associated to α_ξ, namely, $\beta_\xi(v, w) = \alpha_\xi(v + w) + \alpha_\xi(v) + \alpha_\xi(w)$, $v, w \in V$. Define a linear map $T_\xi : V \to V$ by

$$\beta(T_\xi v, w) = \beta_\xi(v, w).$$

Assume $\xi \in \mathfrak{g}^*$. We denote by $(V_\xi, \beta, \alpha_\xi)$ the vector space V equipped with the symplectic form β and the quadratic form α_ξ.

Definition 2.2.2. Assume $\xi, \zeta \in \mathfrak{g}^*$. We say that $(V_\xi, \beta, \alpha_\xi)$ is equivalent to $(V_\zeta, \beta, \alpha_\zeta)$ if there exists a vector space isomorphism $g : V_\xi \to V_\zeta$ such that $\beta(gv, gw) = \beta(v, w)$ and $\alpha_\zeta(gv) = \alpha_\xi(v)$ for all $v, w \in V_\xi$.

Lemma 2.2.3. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if there exists $g \in G$ such that $\alpha_\xi(g^{-1}v) = \alpha_\zeta(v)$, $\forall\, v \in V$.*

Proof. The two elements ξ, ζ lie in the same G-orbit if and only if there exists $g \in G$ such that $g.\xi(x) = \xi(g^{-1}xg) = \zeta(x)$, $\forall\, x \in \mathfrak{g}$. Assume $\xi(x) = \mathrm{tr}(X_\xi x)$ and $\zeta(x) = \mathrm{tr}(X_\zeta x)$. Similar argument as in the proof of Lemma 2.2.1 shows that $g.\xi(x) = \mathrm{tr}(gX_\xi g^{-1}x) = \zeta(x)$ if and only if $\beta(gX_\xi g^{-1}v, v) + \beta(X_\zeta v, v) = 0$ if and only if $\alpha_\xi(g^{-1}v) = \alpha_\zeta(v)$, $\forall\, v \in V$. □

Corollary 2.2.4. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if $(V_\xi, \beta, \alpha_\xi)$ is equivalent to $(V_\zeta, \beta, \alpha_\zeta)$.*

2.2.2 From now on we assume that $\xi \in \mathfrak{g}^*$ is nilpotent.

Lemma 2.2.5. *Let $\xi \in \mathfrak{g}^*$ be nilpotent. Then T_ξ is a nilpotent element in $End(V_\xi)$.*

Proof. Note that the nilpotent elements in $\mathfrak{g}^*$ (resp. $\mathfrak{g}$) are precisely the "unstable" vectors ξ (resp. x), namely, those ξ (resp. x) for which the closure of the G-orbit $G.\xi$ (resp. $\mathrm{Ad}(G)x$) contains 0. By Hilbert's criterion for instability, there exists a co-character $\phi : \mathbf{G}_m \to G$ such that $\lim_{a\to 0} \phi(a).\xi = 0$. To show that T_ξ is nilpotent, it is enough to show that $\lim_{a\to 0} \mathrm{Ad}(\phi(a))T_\xi = 0$.

For any G-representation M and $i \in \mathbb{Z}$, we write $M(\phi; i)$ for the i-weight space of the torus $\{\phi(a)\}_{a\in\mathbf{G}_m}$ and $M(\phi; > i) = \oplus_{j>i} M(\phi; j)$, and similarly for $M(\phi; \geq i)$, $M(\phi; \leq i)$ etc.

Since $\xi \in \mathfrak{g}^*(\phi, > 0)$, we may choose $X \in \mathrm{End}_{\mathbf{k}}(V)(\phi; > 0)$ such that $\xi(x) = \mathrm{tr}(Xx)$ for all $x \in \mathfrak{g}$. Notice that $\beta((\mathrm{Ad}(\phi(a))T_\xi)v, w) = \beta(X\phi(a)^{-1}v, \phi(a)^{-1}w) + \beta(\phi(a)^{-1}v, X\phi(a)^{-1}w) = \beta((\mathrm{Ad}(\phi(a))X)v, w) + \beta(v, (\mathrm{Ad}(\phi(a))X)w)$. Since $X \in \mathrm{End}_{\mathbf{k}}(V)(\phi; > 0)$, $\mathrm{Ad}(\phi(a))X \to 0$ as $a \to 0$ and thus $\beta(\mathrm{Ad}(\phi(a))T_\xi v, w) \to 0$ as $a \to 0$ for any $v, w \in V$. It follows that $\mathrm{Ad}(\phi(a))T_\xi \to 0$ as $a \to 0$, since the bilinear form β is nondegenerate. Thus T_ξ is nilpotent. □

Recall that $A = \mathbf{k}[[t]]$ denotes the ring of formal power series in the indeterminate t and the A-module E is the vector space spanned by the linear functionals $t^{-k} : A \to \mathbf{k}$, $\sum a_i t^i \mapsto a_k$, $k \geq 0$ (see 1.2.2). We consider V_ξ as an A-module by $(\sum a_k t^k)v =$

$\sum a_k T_\xi^k v$. Let E_0 and E_1 be the subspace $\sum \mathbf{k}t^{-2k}$ and $\sum \mathbf{k}t^{-2k-1}$ of E respectively. Denote $\pi_i : E \to E_i$, $i = 0, 1$ the natural projections. Define $\varphi : V \times V \to E, \psi : V \to E_1$, $\varphi_\xi : V \times V \to E, \psi_\xi : V \to E_0$ by

$$\varphi(v, w) = \textstyle\sum_{k \geq 0} \beta(t^k v, w) t^{-k}, \ \ \psi(v) = \sum_{k \geq 0} \beta(t^{k+1} v, t^k v) t^{-2k-1}$$

$$\varphi_\xi(v, w) = \textstyle\sum_{k \geq 0} \beta_\xi(t^k v, w) t^{-k}, \ \ \psi_\xi(v) = \sum_{k \geq 0} \alpha_\xi(t^k v) t^{-2k}.$$

Note that we have $\beta(T_\xi v, v) = \beta_\xi(v, v) = 0$ and $\beta_\xi(T_\xi v, v) = \beta(T_\xi v, T_\xi v) = 0$. By [H, Proposition 2.7], we can identify $(V_\xi, \alpha = 0, \beta)$ with (V_ξ, φ, ψ), $(V_\xi, \alpha_\xi, \beta_\xi)$ with $(V_\xi, \varphi_\xi, \psi_\xi)$, hence $(V_\xi, \beta, \alpha_\xi)$ with $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$. The mappings φ, ψ and φ_ξ, ψ_ξ satisfy the following properties ([H]): for all $v, w \in V_\xi$,

(i) the maps $\varphi(\cdot, w)$ and $\varphi_\xi(\cdot, w)$ are A-linear,

(ii) $\varphi(v, w) = \varphi(w, v)$, $\varphi_\xi(v, w) = \varphi_\xi(w, v)$, $\varphi(v, v) = \psi(v)$, $\varphi_\xi(v, v) = 0$,

(iii) $\psi(v + w) = \psi(v) + \psi(w)$, $\psi_\xi(v + w) = \psi_\xi(v) + \psi_\xi(w) + \pi_0(\varphi_\xi(v, w))$,

(vi) $\psi(av) = a^2 \psi(v)$,$\psi_\xi(av) = a^2 \psi_\xi(v)$ for all $a \in A$.

Following [H], we call $(V_\xi, \beta, \alpha_\xi)$ a form module and $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$ an abstract form module. Corollary 2.2.4 says that classifying the nilpotent G-orbits in $\mathfrak{g}^*$ is equivalent to classifying the equivalence classes of the form modules $(V_\xi, \beta, \alpha_\xi)$. In the following we classify the form modules $(V_\xi, \beta, \alpha_\xi)$ via the identification with $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$. We write $V_\xi = (V_\xi, \beta, \alpha_\xi)$.

Since T_ξ is nilpotent (Lemma 2.2.5), there exists a unique sequence of integers $p_1 \geq \cdots \geq p_s \geq 1$ and a family of vectors $v_1, \ldots, v_s$ such that $T_\xi^{p_i} v_i = 0$ and the vectors $T_\xi^{q_i} v_i$, $0 \leq q_i \leq p_i - 1$ form a basis of V. We define $p(V_\xi) = p(T_\xi) = (p_1, \ldots, p_s)$. Define an index function $\chi_{V_\xi} : \mathbb{Z} \to \mathbb{N}$ for $(V_\xi, \beta, \alpha_\xi)$ by

$$\chi_{V_\xi}(m) = \min\{i \geq 0 | T_\xi^m v = 0 \Rightarrow \alpha_\xi(T_\xi^i v) = 0\}.$$

Define $\mu(V_\xi)$ to be the minimal integer $m \geq 0$ such that $T_\xi^m V_\xi = 0$. For $v \in V_\xi$, we define $\mu(v) = \mu(Av)$. We define $\mu(E)$ for E and $\mu(u)$ for $u \in E$ similarly.

Lemma 2.2.6. *We have $\psi(v) = 0$ and $\varphi_\xi(v, w) = t\varphi(v, w)$ for all $v, w \in V_\xi$.*

Proof. The first assertion follows since $\beta(T_\xi v, v) = 0$, $\forall\ v \in V_\xi$. The second assertion follows since $\beta_\xi(T_\xi^k v, w) = \beta(T_\xi^{k+1} v, w)$. □

We study the orthogonal decomposition of V_ξ with respect to φ, which is also an orthogonal decomposition of V_ξ with respect to φ_ξ since $\varphi(v,w) = 0$ implies $\varphi_\xi(v,w) = 0$ (Lemma 2.2.6). Recall that every form module V has some orthogonal decomposition $V = \sum_{i=1}^r V_i$ in indecomposable submodules $V_1, V_2, \ldots, V_r$ (see 1.2.4).

We first classify the indecomposable modules (with respect to φ) that appear in the orthogonal decompositions of form modules $(V_\xi, \beta, \alpha_\xi)$. Let $(V_\xi, \varphi, \psi, \varphi_\xi, \psi_\xi)$ be an indecomposable module. Since $\psi(v) = 0$ for all $v \in V_\xi$ (Lemma 2.2.6), by the classification of modules (V_ξ, φ, ψ), there exist v_1, v_2 such that $V_\xi = Av_1 \oplus Av_2$ with $\mu(v_1) = \mu(v_2) = m$ and $\varphi(v_1, v_2) = t^{1-m}$ (see [H, 3.5], notice that β is non-degenerate on V_ξ). Denote $\psi_\xi(v_1) = \Psi_1$, $\psi_\xi(v_2) = \Psi_2$ and $\varphi_\xi(v_1, v_2) = t^{2-m} = \Phi_\xi$.

2.2.3 In this subsection assume $\mathbf{k}$ is algebraically closed.

Proposition 2.2.7. *The indecomposable modules are* $^*W_l(m) = Av_1 \oplus Av_2$, $[\frac{m}{2}] \leq l \leq m$, *with* $\mu(v_1) = \mu(v_2) = m$, $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = 0$ *and* $\varphi(v_1, v_2) = t^{1-m}$. *We have* $\chi_{^*W_l(m)} = [m;l]$, *where* $[m;l] : \mathbb{N} \to \mathbb{Z}$ *is defined by* $[m;l](k) = \max\{0, \min\{k - m + l, l\}\}$.

Proof. Assume $\mu(\Psi_1) \geq \mu(\Psi_2)$. Let $v_2' = v_2 + av_1$. The equation $\psi_\xi(v_2') = \Psi_2 + a^2\Psi_1 + \pi_0(a\Phi_\xi) = 0$ has a solution for a, hence we can assume $\Psi_2 = 0$. Assume $\Psi_1 = \sum_{i=0}^l a_i t^{-2i}$, $a_i \in \mathbf{k}, a_l \neq 0$. Let $v_1' = av_1$, $a \in A$. We can take a invertible in A such that $\psi_\xi(v_1') = t^{-2l}$. Let $v_2'' = a^{-1}v_2'$. One verifies that $\psi_\xi(v_1') = t^{-2l}$, $\psi_\xi(v_2'') = 0$ and $\varphi(v_1', v_2'') = t^{1-m}$. Furthermore, we can assume $[m/2] - 1 \leq l \leq m-1$. In fact, we have $l \leq m-1$ since $t^m v = 0, \forall\, v \in V$; if $l < [\frac{m}{2}] - 1$, let $v_1' = v_1 + t^{m-2l-2}v_2 + t^{m-2[\frac{m}{2}]}v_2$, then $\psi_\xi(v_1') = t^{-2([\frac{m}{2}]-1)}$ and $\varphi(v_1', v_2) = t^{1-m}$. One can verify that the modules $^*W_l(m)$, $[m/2] \leq l \leq m$ exist and are not equivalent to each other. ☐

Lemma 2.2.8. *Assume* $m_1 \geq m_2$.

(i) *If* $l_1 < l_2$, *we have* $^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \cong {}^*W_{l_2}(m_1) \oplus {}^*W_{l_2}(m_2)$.

(ii) *If* $m_1 - l_1 < m_2 - l_2$, *we have* $^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \cong {}^*W_{l_1}(m_1) \oplus {}^*W_{m_2-m_1+l_1}(m_2)$.

Proof. Assume $^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) = Av_1 \oplus Aw_1 \oplus Av_2 \oplus Aw_2$ with $\psi_\xi(v_i) = t^{2-2l_i}, \psi_\xi(w_i) = 0$ and $\varphi(v_i, w_j) = \delta_{i,j}t^{1-m_i}, \varphi(v_i, v_j) = \varphi(w_i, w_j) = 0$, $i, j = 1, 2$. Let $\tilde{v}_1 = v_1 + (1 + t^{l_2 - l_1})v_2$, $\tilde{w}_1 = w_1$, $\tilde{v}_2 = v_2$, $\tilde{w}_2 = w_2 + (t^{m_1-m_2} + t^{m_1-l_1-m_2+l_2})w_1$. Then we have $\psi_\xi(\tilde{v}_i) = t^{2-2l_2}$, $\psi_\xi(\tilde{w}_i) = 0$ and $\varphi(\tilde{v}_i, \tilde{w}_j) = \delta_{i,j} t^{1-m_i}, \varphi(\tilde{v}_i, \tilde{v}_j) = \varphi(\tilde{w}_i, \tilde{w}_j) = 0$, $i, j = 1, 2$. This proves (i). One can prove (ii) similarly. □

Remark. *Notice that we do not have a "Krull-Schmidt" type theorem here, namely, the indecomposable summands of a form module V are not uniquely determined by V. (See also Lemma 2.2.12* (ii)*.)*

By Proposition 2.2.7 and Lemma 2.2.8, for every module V, there exists a unique sequence of modules ${}^*W_{l_i}(m_i)$ such that V is equivalent to ${}^*W_{l_1}(m_1) \oplus {}^*W_{l_2}(m_2) \oplus \cdots \oplus {}^*W_{l_s}(m_s)$, $[\frac{m_i}{2}] \leq l_i \leq m_i$, $m_1 \geq m_2 \geq \cdots \geq m_s$, $l_1 \geq l_2 \geq \cdots \geq l_s$ and $m_1 - l_1 \geq m_2 - l_2 \geq \cdots \geq m_s - l_s$. We call this the *normal form* of the module V. Two form modules are equivalent if and only if their normal forms are the same. It follows that $p(V_\xi) = m_1^2 \cdots m_s^2$, $\chi_{V_\xi}(k) = \sup_i \chi_{{}^*W_{l_i}(m_i)}(k)$ for all $k \in \mathbb{N}$ and $\chi_V(m_i) = \chi_{{}^*W_{l_i}(m_i)} = l_i$. Thus the equivalence class of V is characterized by the symbol

$$(m_1)^2_{\chi(m_1)} \cdots (m_s)^2_{\chi(m_s)}.$$

A symbol of the above form is the symbol of a form module if and only if $[\frac{m_i}{2}] \leq \chi(m_i) \leq m_i$, $\chi(m_i) \geq \chi(m_{i+1})$ and $m_i - \chi(m_i) \geq m_{i+1} - \chi(m_{i+1})$, $i = 1, \ldots, s$.

Proposition 2.2.9. *Two nilpotent elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if T_ξ, T_ζ are conjugate by $GL(V)$ and $\chi(V_\xi) = \chi(V_\zeta)$.*

We associate to the orbit $(m_1)^2_{\chi(m_1)} \cdots (m_s)^2_{\chi(m_s)}$ a pair of partitions $(\chi(m_1), \ldots, \chi(m_s))(m_1 - \chi(m_1), \ldots, m_s - \chi(m_s))$. In this way we construct a bijection from the set of nilpotent orbits in $\mathfrak{g}^*$ to the set $\{(\mu, \nu) | |\mu| + |\nu| = n, \nu_i \leq \mu_i + 1\}$, which has cardinality $p_2(n) - p_2(n-2)$. Recall that $p_2(n)$ denotes the number of pairs of partitions (μ, ν) such that $|\mu| + |\nu| = n$.

2.2.4 In this subsection, let $\mathbf{k} = \mathbf{F}_q$. Let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G, $\mathfrak{g}$. We study the nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)^*$. Fix $\delta \in \mathbf{F}_q \backslash \{x^2 + x | x \in \mathbf{F}_q\}$. We have the following statements whose proofs are entirely similar to those in section 1.3. For completeness, we also include the proofs here.

Proposition 2.2.10. *The indecomposable modules over $\mathbf{F}_q$ are*

(i) $^*W_l^0(m) = Av_1 \oplus Av_2$, $(m-1)/2 \le l \le m$ *with* $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = 0$ *and* $\varphi(v_1, v_2) = t^{1-m}$;

(ii) $^*W_l^\delta(m) = Av_1 \oplus Av_2$, $(m-1)/2 < l < m$ *with* $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = \delta t^{-2(m-1-l)}$ *and* $\varphi(v_1, v_2) = t^{1-m}$.

Proof. Let $V_\xi = Av_1 \oplus Av_2$ be an indecomposable module as in the last paragraph of subsection 2.2.2. We have $\Phi_\xi = t^{2-m}$. We can assume that $\mu(\Psi_1) \ge \mu(\Psi_2)$. We have the following cases:

Case 1: $\Psi_1 = \Psi_2 = 0$. Let $\tilde{v}_1 = v_1 + t^{m-2[\frac{m}{2}]}v_2, \tilde{v}_2 = v_2$, then we have $\psi_\xi(\tilde{v}_1) = t^{2-2[\frac{m}{2}]}$, $\psi_\xi(\tilde{v}_2) = 0$ and $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 2: $\Psi_1 \ne 0$, $\Psi_2 = 0$. There exist $a, b \in A$ invertible, such that $\psi_\xi(av_1) = t^{-2k}, \psi_\xi(bv_2) = 0$ and $\varphi(av_1, bv_2) = t^{1-m}$. Hence we can assume $\Psi_1 = t^{-2k}$ where $k \le m-1$. If $k < [\frac{m}{2}] - 1$, let $\tilde{v}_1 = v_1 + t^{m-2[\frac{m}{2}]}v_2 + t^{m-2k-2}v_2$, $\tilde{v}_2 = v_2$; otherwise, let $\tilde{v}_1 = v_1$, $\tilde{v}_2 = v_2$. Then we get $\psi_\xi(\tilde{v}_1) = t^{-2k}$, $[\frac{m}{2}] - 1 \le k \le m-1$, $\psi_\xi(\tilde{v}_2) = 0$, $\varphi(\tilde{v}_1, \tilde{v}_2) = t^{1-m}$.

Case 3: $\Psi_1 \ne 0$, $\Psi_2 \ne 0$. There exist $a, b \in A$ invertible, such that $\psi_\xi(av_1) = t^{-2l_1}$ and $\varphi(av_1, bv_2) = t^{1-m}$. Hence we can assume $\Psi_1 = t^{-2l_1}$ and $\Psi_2 = \sum_{i=0}^{l_2} a_i t^{-2i}$ where $l_2 \le l_1 \le m-1$. Let $\tilde{v}_2 = v_2 + \sum_{i=0}^{m-1} x_i t^i v_1$. Assume $l_1 < \frac{m-2}{2}$, then $\psi_\xi(\tilde{v}_2) = 0$ has a solution for x_i's and we get to Case 2. Assume $l_1 \ge \frac{m-2}{2}$. If $a_{m-l_1-2} \in \{x^2 + x | x \in \mathbf{F}_q\}$, then $\psi_\xi(\tilde{v}_2) = 0$ has a solution for x_i's and we get to Case 2; if $a_{m-l_1-2} \notin \{x^2 + x | x \in \mathbf{F}_q\}$, then $\psi_\xi(\tilde{v}_2) = \delta t^{-2(m-l_1-2)}$ has a solution for x_i's.

Summarizing Cases 1-3, we have normalized $V_\xi = Av_1 \oplus Av_2$ with $\mu(v_1) = \mu(v_2) = m$ as follows:

(i) $(m-1)/2 \le \chi(m) = l \le m$, $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = 0$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $^*W_l^0(m)$.

(ii) $(m-1)/2 < \chi(m) = l < m$, $\psi_\xi(v_1) = t^{2-2l}$, $\psi_\xi(v_2) = \delta t^{-2(m-l-1)}$, $\varphi(v_1, v_2) = t^{1-m}$, denoted by $^*W_l^\delta(m)$.

We show that $^*W_l^0(m)$ and $^*W_l^\delta(m)$, where $\frac{m-1}{2} < l < m$, are not equivalent. Take v_i, w_i, $i = 1, 2$, such that $^*W_l^0(m) = Av_1 \oplus Aw_1$, $^*W_l^\delta(m) = Av_2 \oplus Aw_2$, $\mu(v_i) = \mu(w_i) = m$, $\psi_\xi(v_i) = t^{2-2l}, \psi_\xi(w_1) = 0, \psi_\xi(w_2) = \delta t^{2l-2m+2}$ and $\varphi(v_i, w_i) = t^{1-m}$, $i = 1, 2$. The modules $^*W_l^0(m)$ and $^*W_l^\delta(m)$ are equivalent if and only if there exists a linear isomorphism $g : {}^*W_l^0(m) \to {}^*W_l^\delta(m)$ such that $\psi_\xi(gv) = \psi_\xi(v)$ and $\varphi(gv, gw) = \varphi(v, w)$ for all $v, w \in {}^*W_l^0(m)$. Assume $gv_1 = \sum_{i=0}^{m-1}(a_i t^i v_2 + b_i t^i w_2), gw_1 = \sum_{i=0}^{m-1}(c_i t^i v_2 + d_i t^i w_2)$. Then a straightforward calculation shows that if $l = \frac{m}{2}$, among the equations $\psi_\xi(gv_1) = \psi_\xi(v_1), \psi_\xi(gw_1) = \psi_\xi(w_1), \varphi(gv_1, gw_1) = \varphi(v_1, w_1)$, the following equations appear $c_0^2 + \delta d_0^2 + c_0 d_0 = 0,\ a_0 d_0 + b_0 c_0 = 1$. It follows that $c_0, d_0 \neq 0$ and thus the first equation becomes an "Artin-Schreier" equation $(\frac{c_0}{d_0})^2 + \frac{c_0}{d_0} = \delta$ which has no solutions over $\mathbf{F}_q$. Similarly if $\frac{m}{2} < l < m$, an "Artin-Schreier" equation $c_{2l-m}^2 + c_{2l-m} = \delta$ appears. It follows that ${}^*W_l^0(m)$ and ${}^*W_l^\delta(m)$, where $\frac{m-1}{2} < l < m$, are not equivalent. □

Remark 2.2.11. It follows that the equivalence class of the form module ${}^*W_l(m)$ over $\bar{\mathbf{F}}_q$ remains as one equivalence class over $\mathbf{F}_q$ when $l = \frac{m-1}{2}$ or $l = m$ and decomposes into two equivalence classes ${}^*W_l^0(m)$ and ${}^*W_l^\delta(m)$ over $\mathbf{F}_q$ otherwise.

Lemma 2.2.12. *Assume $l_1 \geq l_2$ and $m_1 - l_1 \geq m_2 - l_2$.*

(i) *If $l_1 + l_2 < m_1$, we have that ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^0(m_2)$, ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$, ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ are not equivalent to each other.*

(ii) *If $l_1 + l_2 \geq m_1$, we have ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^0(m_2) \cong {}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2) \cong {}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$. The two pairs are not equivalent to each other.*

Proof. We show that ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ are equivalent if and only if $l_1 + l_2 \geq m_1$. The other statements are proved entirely similarly. Assume ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ correspond to ξ and ξ' respectively. Take v_1, w_1 and v_2, w_2 such that ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2) = Av_1 \oplus Aw_1 \oplus Av_2 \oplus Aw_2$ and $\psi_\xi(v_1) = t^{2-2l_1}, \psi_\xi(w_1) = 0, \varphi(v_1, w_1) = t^{1-m_1}, \psi_\xi(v_2) = t^{2-2l_2}, \psi_\xi(w_2) = \delta t^{-2(m_2-l_2-1)}, \varphi(v_2, w_2) = t^{1-m_2}, \varphi(v_1, v_2) = \varphi(v_1, w_2) = \varphi(w_1, v_2) = \varphi(w_1, w_2) = 0$. Similarly, take v_1', w_1' and v_2', w_2' such that ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2) = Av_1' \oplus Aw_1' \oplus Av_2' \oplus Aw_2'$ and $\psi_{\xi'}(v_1') = t^{2-2l_1}, \psi_{\xi'}(w_1') = t^{-2(m_1-l_1-1)}, \varphi(v_1', w_1') = t^{1-m_1}, \psi_{\xi'}(v_2') = t^{2-2l_2}, \psi_{\xi'}(w_2') = 0, \varphi(v_2', w_2') = t^{1-m_2}, \varphi(v_1', v_2') = \varphi(v_1', w_2') = \varphi(w_1', v_2') = \varphi(w_1', w_2') = 0$.

The form modules ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ are equivalent if and only if there exists an A-module isomorphism $g : V \to V$ such that $\psi_{\xi'}(gv) = \psi_\xi(v), \varphi(gv, gw) = \varphi(v, w)$ for any $v, w \in V$. Assume $gv_j = \sum_{i=0}^{m_1-1}(a_{j,i}t^iv_1' + b_{j,i}t^iw_1') + \sum_{i=0}^{m_2-1}(c_{j,i}t^iv_2' + d_{j,i}t^iw_2')$, $gw_j = \sum_{i=0}^{m_1-1}(e_{j,i}t^iv_1' + f_{j,i}t^iw_1') + \sum_{i=0}^{m_2-1}(g_{j,i}t^iv_2' + h_{j,i}t^iw_2'), j = 1, 2$. Then ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ are equivalent if and only if the equations $\psi_{\xi'}(gv_i) = \psi_\xi(v_i), \psi_{\xi'}(gw_i) = \psi_\xi(w_i), \varphi(gv_i, gv_j) = \varphi(v_i, v_j), \varphi(gv_i, gw_j) = \varphi(v_i, w_j)$, $\varphi(gw_i, gw_j) = \varphi(w_i, w_j), i, j = 1, 2$, have solutions.

If $l_1 + l_2 < m_1$, some equations are $e_{1,2l_1-m_1}^2 + e_{1,2l_1-m_1} = \delta$ (if $l_1 \neq \frac{m_1}{2}$) or $e_{1,0}^2 + e_{1,0}f_{1,0} + \delta f_{1,0}^2 = 0, a_{1,0}f_{1,0} + b_{1,0}e_{1,0} = 1$ (if $l_1 = \frac{m_1}{2}$). As in the proof of Proposition 2.2.10, we get "Artin-Schreier" equations which have no solutions over $\mathbf{F}_q$. Hence ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2)$ and ${}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$ are not equivalent.

If $l_1 + l_2 \geq m_1$, let $gv_1 = v_1', gw_1 = w_1' + \sqrt{\delta}t^{l_1+l_2-m_1}v_2', gv_2 = v_2', gw_2 = w_2' + \sqrt{\delta}t^{l_1+l_2-m_2}v_1'$, then this is a solution for the equations. It follows that ${}^*W_{l_1}^0(m_1) \oplus {}^*W_{l_2}^\delta(m_2) \cong {}^*W_{l_1}^\delta(m_1) \oplus {}^*W_{l_2}^0(m_2)$. □

Proposition 2.2.13. *The equivalence class of the module*

$$ {}^*W_{l_1}(m_1) \oplus \cdots \oplus {}^*W_{l_s}(m_s), m_i \geq m_{i+1}, l_i = \chi(m_i), i = 1, \ldots, s, $$

over $\bar{\boldsymbol{F}}_q$ *decomposes into at most* 2^k *equivalence classes over* $\boldsymbol{F}_q$, *where*

$$ k = \#\{1 \leq i \leq s | l_i + l_{i+1} < m_i \text{ and } l_i > \tfrac{m_i-1}{2}\}. $$

Proof. By Proposition 2.2.10 and Remark 2.2.11, it is enough to show that form modules of the form ${}^*W_{l_1}^{\epsilon_1'}(m_1) \oplus \cdots \oplus {}^*W_{l_s}^{\epsilon_s'}(m_s)$, where $\epsilon_i' = 0$ or δ, have at most 2^k equivalence classes. Suppose $i_1, i_2, \ldots, i_k$ are such that $1 \leq i_j \leq s, l_{i_j} + l_{i_j+1} < m_{i_j}$, $l_{i_j} > \frac{m_{i_j}-1}{2}, j = 1, \ldots, k$. Using Lemma 2.2.12 one can easily show that a module of the above form is isomorphic to one of the following modules: $V_1^{\epsilon_1} \oplus \cdots \oplus V_k^{\epsilon_k}$, where $V_t^{\epsilon_t} = {}^*W_{l_{i_{t-1}+1}}^0(m_{i_{t-1}+1}) \oplus \cdots \oplus {}^*W_{l_{i_t-1}}^0(m_{i_t-1}) \oplus {}^*W_{m_{i_t}}^{\epsilon_t}(m_{i_t})$, $t = 1, \ldots, k-1$, $i_0 = 0$, and $V_k = {}^*W_{l_{i_{k-1}+1}}^0(m_{i_{k-1}+1}) \oplus \cdots \oplus {}^*W_{l_{i_k}}^{\epsilon_k}(m_{i_k}) \oplus {}^*W_{l_{i_k+1}}^0(m_{i_k+1}) \oplus \cdots \oplus {}^*W_{l_s}^0(m_s)$, $\epsilon_t = 0$ or δ, $t = 1, \ldots, k$. Thus the proposition is proved. □

Corollary 2.2.14. *The nilpotent orbit* $(m_1)_{l_1}^2 \cdots (m_s)_{l_s}^2$ *in* $\mathfrak{g}(\bar{\boldsymbol{F}}_q)^*$ *splits into at most* 2^k $G(\boldsymbol{F}_q)$*-orbits in* $\mathfrak{g}(\boldsymbol{F}_q)^*$.

Proposition 2.2.15. *The number of nilpotent* $G(\boldsymbol{F}_q)$*-orbits in* $\mathfrak{g}(\boldsymbol{F}_q)^*$ *is at most* $p_2(n)$.

Proof. Recall that we have mapped the nilpotent orbits in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ bijectively to the set $\{(\mu,\nu) | |\mu| + |\nu| = n, \nu_i \leq \mu_i + 1\} := \Delta$. By Corollary 2.2.14, a nilpotent orbit in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ corresponding to $(\mu,\nu) \in \Delta, \mu = (\mu_1, \mu_2, \ldots, \mu_s), \nu = (\nu_1, \nu_2, \ldots, \nu_s)$ splits into at most 2^k orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{1 \leq i \leq s | \mu_{i+1} + 1 \leq \nu_i < \mu_i + 1\}$. We associate to the orbit 2^k pairs of partitions as follows. Suppose $r_1, r_2, ..., r_k$ are such that $\mu_{r_i+1} + 1 \leq \nu_{r_i} < \mu_{r_i} + 1, i = 1, ..., k$ and let $\mu^{1,i} = (\mu_{r_{i-1}+1}, \ldots, \mu_{r_i}), \nu^{1,i} = (\nu_{r_{i-1}+1}, \ldots, \nu_{r_i}), \mu^{2,i} = (\nu_{r_{i-1}+1} - 1, \ldots, \nu_{r_i} - 1), \nu^{2,i} = (\mu_{r_{i-1}+1} + 1, \ldots, \mu_{r_i} + 1), i = 1, \ldots, k, \mu^{k+1} = (\mu_{r_k+1}, \ldots, \mu_s), \nu^{k+1} = (\nu_{r_k+1}, \ldots, \nu_s)$. We associate to (μ,ν) the pairs of partitions $(\tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k}, \tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k})$, $\tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k} = (\mu^{\epsilon_1,1}, \mu^{\epsilon_2,2}, \ldots, \mu^{\epsilon_k,k}, \mu^{k+1}), \tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k} = (\nu^{\epsilon_1,1}, \nu^{\epsilon_2,2}, \ldots, \nu^{\epsilon_k,k}, \nu^{k+1})$, where $\epsilon_i \in \{1,2\}, i = 1, \ldots, k$. Notice that the pairs of partitions $(\tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k}, \tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k})$ are distinct and among them only $(\mu,\nu) = (\tilde{\mu}^{1,\ldots,1}, \tilde{\nu}^{1,\ldots,1})$ is in Δ. One can verify that the set of all pairs of partitions constructed as above for all $(\mu,\nu) \in \Delta$ is in bijection with the set $\{(\mu,\nu) | |\mu| + |\nu| = n\}$, which has cardinality $p_2(n)$. It follows that the number of nilpotent orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is less that $p_2(n)$. □

2.3 Odd orthogonal groups

In this section we study the nilpotent orbits in $\mathfrak{g}^*$ where G is an odd orthogonal group.

2.3.1 Let V be a vector space of dimension $2n+1$ over $\mathbf{k}$ equipped with a non-degenerate quadratic form $\alpha : V \to \mathbf{k}$. Let $\beta : V \times V \to \mathbf{k}$ be the bilinear form associated to α. Recall that the odd orthogonal group is defined as $G = O(2n+1) = O(V,\alpha) = \{g \in GL(V) \mid \alpha(gv) = \alpha(v), \forall\ v \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{o}(2n+1) = \mathfrak{o}(V,\alpha) = \{x \in \mathfrak{gl}(V) \mid \beta(xv,v) = 0, \forall\ v \in V \text{ and } \mathrm{tr}(x) = 0\}$. Let ξ be an element of $\mathfrak{g}^*$. There exists $X \in \mathfrak{gl}(V)$ such that $\xi(x) = \mathrm{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a bilinear form

$$\beta_\xi : V \times V \to \mathbf{k}, (v,w) \mapsto \beta(Xv,w) + \beta(v,Xw).$$

Lemma 2.3.1. *The bilinear form β_ξ is well-defined.*

Proof. Recall that the space $\mathrm{Alt}(V)$ of alternate bilinear forms on V coincides with the second exterior power $\wedge^2(V^*)$ of V^*. Consider the following linear mapping

$$\Phi : \mathrm{End}_{\mathbf{k}}(V) \to \wedge^2(V^*) = \mathrm{Alt}(V), \quad X \mapsto \beta_X$$

where $\beta_X(v,w) = \beta(Xv,w) + \beta(v,Xw)$ for $v,w \in V$. It is easy to see that Φ is $G = O(V)$-equivariant. One can show that $\ker\Phi$ coincides with the orthogonal complement $\mathfrak{g}^\perp$ of $\mathfrak{g} = \mathfrak{o}(V)$ in $\mathrm{End}_{\mathbf{k}}(V)$ under the nondegenerate trace form. It follows that β_ξ does not depend on the choice of X. □

Assume $\xi \in \mathfrak{g}^*$. We denote $(V_\xi, \alpha, \beta_\xi)$ the vector space V equipped with the quadratic form α and the bilinear form β_ξ.

Definition 2.3.2. Assume $\xi, \zeta \in \mathfrak{g}^*$. We say that $(V_\xi, \alpha, \beta_\xi)$ and $(V_\zeta, \alpha, \beta_\zeta)$ are equivalent if there exists a vector space isomorphism $g : V_\xi \to V_\zeta$ such that $\alpha(gv) = \alpha(v)$ and $\beta_\zeta(gv, gw) = \beta_\xi(v,w)$ for any $v, w \in V_\xi$.

Lemma 2.3.3. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if there exists $g \in G$ such that $\beta_\zeta(gv, gw) = \beta_\xi(v,w)$ for any $v, w \in V$.*

Proof. Assume $\xi(x) = \mathrm{tr}(Xx), \zeta(x) = \mathrm{tr}(X'x)$, $\forall x \in \mathfrak{g}$. Using similar argument as in the proof of Lemma 2.3.1, one can see that ξ, ζ lie in the same G-orbit if and only if there exists $g \in G$ such that $\beta((gXg^{-1} + X')v, w) + \beta(v, (gXg^{-1} + X')w) = 0$, $\forall\, v, w \in V$. □

Corollary 2.3.4. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if $(V_\xi, \alpha, \beta_\xi)$ is equivalent to $(V_\zeta, \alpha, \beta_\zeta)$.*

2.3.2 From now on we assume that $\xi \in \mathfrak{g}^*$ is nilpotent. Let $(V_\xi, \alpha, \beta_\xi)$ be defined as in subsection 2.3.1. Let λ be a formal parameter. There exists a smallest integer m such that there exists a set of vectors $v_0, \ldots, v_m$ for which $\beta_\xi(\sum_{i=0}^m v_i\lambda^i, v) + \lambda\beta(\sum_{i=0}^m v_i\lambda^i, v) = 0$ for any $v \in V$ (see Lemma 2.3.5 below). Lemmas 2.3.5-2.3.10 in the following extend some results in [LS]. (Most parts of the proofs are included in [LS]. We add some conditions about the quadratic form α.)

Lemma 2.3.5. *The vectors $v_0, \ldots, v_m$ (up to multiple) and $m \geq 0$ are uniquely determined by β_ξ and β. Moreover, $\beta(v_i, v_j) = \beta_\xi(v_i, v_j) = 0$, $i, j = 0, \ldots, m$, $\alpha(v_i) = 0$, $i = 0, \ldots, m-1$ and we can assume $\alpha(v_m) = 1$.*

Proof. Since ξ is nilpotent, we can find a cocharacter $\phi : \mathbf{G}_m \to G$ for which $\xi \in \mathfrak{g}^*(\phi, > 0)$. Moreover, we can find $X \in \mathrm{End}_{\mathbf{k}}(V)(\phi, > 0)$ such that $\xi(x) = \mathrm{tr}(Xx)$ for all $x \in \mathfrak{g}$.

Let w_0 be a non-zero vector such that $\beta(w_0, -) = 0$. Then w_0 is unique up to a multiple. We have that $w_0 \in V(\phi, 0)$. If $\beta_\xi(w_0, v) = 0$ for all $v \in V$, then $m = 0$ and we are done.

Now assume $\beta_\xi(w_0, -)$ does not vanish on V. Fix $v \in V$. It is easy to show that if $\beta_\xi(v, w_0) = 0$, then there is $v' \in V$ for which $\beta(v', -) = \beta_\xi(v, -)$. Moreover, if $\tilde{w}_{i-1} \in V(\phi, \geq i-1)$, then one can show that for all $\tilde{w}_i$ such that $\beta(\tilde{w}_i, -) = \beta_\xi(\tilde{w}_{i-1}, -)$, we have $\tilde{w}_i \in V(\phi, \geq i) + \mathbf{k}w_0$.

We define inductively a set of vectors w_i, $i = 0, \ldots, m$, such that $\beta(w_0, -) = 0$, $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$, $\beta_\xi(w_m, -) = 0$ and m is minimal. We have defined w_0. Assume $w_{i-1} \in V(\phi, \geq i-1)$ is found. Then $\beta_\xi(w_{i-1}, w_0) = \beta(w_{i/2}, w_{i/2}) = 0$ if i is even and $\beta_\xi(w_{i-1}, w_0) = \beta_\xi(w_{(i-1)/2}, w_{(i-1)/2}) = 0$ if i is odd. We define w_i to be the unique vector such that $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$ and $w_i \in V(\phi, \geq i)$. One readily sees that we find a unique (up to multiple) set of vectors w_i, $i = 0, \ldots, m$, such that $\beta(w_0, -) = 0$, $\beta(w_i, -) = \beta_\xi(w_{i-1}, -)$, $\beta_\xi(w_m, -) = 0$ and m is minimal.

Since all $w_i \in V(\phi, \geq 0)$, we see that $\beta(w_i, w_j) = 0$. Since for $i > 0$, $w_i \in V(\phi, > 0)$, we see that $\alpha(w_i) = 0$ for $i > 0$. Since $X \in \mathrm{End}_{\mathbf{k}}(\phi, > 0)$, it follows that $\beta_\xi(w_i, w_j) = 0$. We take $v_i = w_{m-i}$. Moreover, we can assume $\alpha(v_m) = \alpha(w_0) = 1$. ☐

Lemma 2.3.6. *Assume $m \geq 1$. There exist $u_0, u_1, \ldots, u_{m-1}$ such that $\beta(v_i, u_j) = \beta_\xi(v_{i+1}, u_j) = \delta_{i,j}$, $\beta(u_i, u_j) = \beta_\xi(u_i, u_j) = 0$, $i, j = 0, \ldots, m-1$, $\alpha(u_i) = 0, i = 0, \ldots, m-1$, and furthermore, $\beta(u_i, v) = \beta_\xi(u_{i-1}, v)$, $i = 1, \ldots, m-1$, for all $v \in V$.*

Proof. Choose u_0 such that $\beta(u_0, v_i) = 0, i = 1, \ldots, m-1$, $\beta(u_0, v_0) = 1$ and $\alpha(u_0) = 0$ (such u_0 exists). We find inductively a set of vectors u_i, $1 \leq i \leq m-1$ such that $\beta(u_i, -) = \beta_\xi(u_{i-1}, -)$ and $\alpha(u_i) = 0$. Assume u_{i-1}, $1 \leq i \leq m-1$ is found. Since $\beta_\xi(u_{i-1}, v_m) = \beta(u_0, v_{m-i}) = 0$ (note that $m - i \geq 1$), there exist a unique u_i such that $\beta(u_i, -) = \beta_\xi(u_{i-1}, -)$ and $\alpha(u_i) = 0$. (The existence is as in the proof of Lemma 2.3.5 and the uniqueness is guaranteed by the condition $\alpha(u_i) = 0$.)

Now it follows that if $i < j$, $\beta(v_i, u_j) = \beta_\xi(v_{i-1}, u_{j-2}) = \beta_\xi(v_0, u_{j-i-1}) = 0$; if $i > j$, $\beta(v_i, u_j) = \beta(v_{i+1}, v_{j+1}) = \beta(v_m, u_{j-i+m}) = 0$; if $i = j$, $\beta(v_i, u_i) = \beta(v_{i-1}, u_{i-1}) = \beta(v_0, u_0) = 1$. Moreover, $\beta(u_i, u_{i+2k}) = \beta(u_{i+k}, u_{i+k}) = 0$, $\beta(u_i, u_{i+2k+1}) = \beta_\xi(u_{i+k}, u_{i+k}) = 0$. It follows that $\beta(u_i, u_j) = 0$. Similarly $\beta_\xi(u_i, u_j) = 0$. The u_i's satisfy the conditions desired. □

Lemma 2.3.7. *The vectors $v_0, v_1, \ldots, v_m, u_0, u_1, \ldots, u_{m-1}$ are linearly independent.*

Proof. Assume $\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i = 0$. Then $\beta(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, u_j) = a_j = 0$, $\beta(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, v_j) = b_j = 0$, $j = 0, \ldots, m-1$ and $\beta_\xi(\sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i, u_{m-1}) = a_m = 0$. □

Let V_{2m+1} be the vector subspace of V spanned by $v_0, v_1, \ldots, v_m, u_0, u_1, \ldots, u_{m-1}$. If $m = 0$, let W be a complementary subspace of V_{2m+1} in V. If $m \geq 1$, let $W = \{w \in V_\xi | \beta(w, v) = \beta_\xi(w, v) = 0, \ \forall\ v \in V_{2m+1}\}$.

Lemma 2.3.8. *We have $V_\xi = V_{2m+1} \perp_{\beta, \beta_\xi} W$.*

Proof. Assume $m = 0$. Lemma follows since by the definition of v_0 we have $\beta(v_0, v) = \beta_\xi(v_0, v) = 0$ for any $v \in V$. Assume $m \geq 1$. A vector w is in W if and only if $\beta(w, v_i) = \beta_\xi(w, v_i) = 0$, $i = 0, \ldots, m$ and $\beta(w, u_i) = \beta_\xi(w, u_i) = 0$, $i = 0, \ldots, m-1$. By our choice of v_i and u_i's, we have $\beta(v_m, w) = \beta_\xi(v_0, w) = 0$, $\beta(w, v_i) = \beta_\xi(w, v_{i+1})$ and $\beta(w, u_i) = \beta_\xi(w, u_{i-1})$. Hence $w \in W$ if and only if $\beta(w, u_i) = \beta(w, v_i) = 0$, $i = 0, \ldots, m-1$ and $\beta_\xi(w, u_{m-1}) = 0$. Thus $\dim W \geq \dim V_\xi - (2m+1)$. Now we show $V_{2m+1} \cap W = \{0\}$. Let $w = \sum_{i=0}^m a_i v_i + \sum_{i=0}^{m-1} b_i u_i \in V_{2m+1} \cap W$. We have $\beta(w, u_j) = a_j = 0$, $\beta(w, v_j) = b_j = 0$, $j = 0, \ldots, m-1$, and $\beta_\xi(w, u_{m-1}) = a_m = 0$. Hence together with the dimension condition we get the conclusion. □

Let $V_\xi = V_{2m+1} \oplus W$ be as in Lemma 2.3.8. Then we get a $2(n-m)$ dimensional vector space W, equipped with a quadratic form $\alpha|_W$ and a bilinear form $\beta_\xi|_{W \times W}$. It is easily seen that the quadratic form $\alpha|_W$ is non-defective on W, namely, $\beta|_{W \times W}$ is non-degenerate. Define a linear map $T_\xi : W \to W$ by

$$\beta(T_\xi w, w') = \beta_\xi(w, w'), w, w' \in W.$$

Remark 2.3.9. Assume $m \geq 1$. Note that the set of vectors $\{u_i\}_{i=0}^{m-1}$ in Lemma 2.3.6 and the corresponding subspace W depend on the choice of u_0 and are uniquely determined by u_0. Suppose we take $\tilde{u}_0 = u_0 + w_0$, where $w_0 \in W$ and $\alpha(w_0) = 0$, then $\tilde{u}_i = u_i + T_\xi^i w_0$ defines another set of vectors as in Lemma 2.3.6. Let $\tilde{V}_{2m+1}$, $\tilde{W}$, $\tilde{T}_\xi$ be defined as V_{2m+1}, W, T_ξ replacing u_i by $\tilde{u}_i$. Then $V = \tilde{V}_{2m+1} \oplus \tilde{W}$ and $\tilde{W} = \{\sum_{i=0}^m \beta(w, T_\xi^i w_0)v_i + w | w \in W\}$. On $\tilde{W}$, we have $\tilde{T}_\xi(\sum_{i=0}^m \beta(w, T_\xi^i w_0)v_i + w) = \sum_{i=0}^m \beta(w, T_\xi^{i+1} w_0)v_i + T_\xi w$.

Lemma 2.3.10. *Assume $V_\xi = V_{2m_\xi+1,\xi} \oplus W_\xi$ is equivalent to $V_\zeta = V_{2m_\zeta+1,\zeta} \oplus W_\zeta$, then $m_\xi = m_\zeta$ and $(W_\xi, \beta, \beta_\xi)$ is equivalent to $(W_\zeta, \beta, \beta_\zeta)$.*

Proof. Assume $V_{2m_\xi+1,\xi} = \text{span}\{v_i^1, u_i^1\}$ and $V_{2m_\zeta+1,\zeta} = \text{span}\{v_i^2, u_i^2\}$, where v_i^1, v_i^2 are as in Lemma 2.3.5 and u_i^1, u_i^2 are as in Lemma 2.3.6. By assumption, there exists $g : V_{2m_\xi+1,\xi} \oplus W_\xi \to V_{2m_\zeta+1,\zeta} \oplus W_\zeta$ such that $\beta(gv, gw) = \beta(v, w)$ and $\beta_\zeta(gv, gw) = \beta_\xi(v, w)$. Since for all $v \in V$, $\beta_\zeta(\sum_{i=0}^{m_\zeta} v_i^2 \lambda^i, v) + \lambda\beta(\sum_{i=0}^{m_\zeta} v_i^2 \lambda^i, v) = 0$, we get $\beta_\xi(\sum_{i=0}^{m_\zeta} g^{-1}v_i^2 \lambda^i, v) + \lambda\beta(\sum_{i=0}^{m_\zeta} g^{-1}v_i^2 \lambda^i, v) = 0$. Hence by Lemma 2.3.5, $m_\xi = m_\zeta$ and $g^{-1}v_i^2 \in V_{2m_\xi+1,\xi}$.

For $w \in W_\xi$, suppose $gw = \sum a_i v_i^2 + \sum b_i u_i^2 + w'$ where $w' \in W_\zeta$. Since $g^{-1}v_i^2 \in V_{2m_\xi+1,\xi}$, we have $\beta_\xi(g^{-1}v_i^2, w) = 0$, $i = 0, \ldots, m_\xi$. It follows that $\beta_\zeta(v_i^2, gw) = b_i = 0$, $i = 0, \ldots, m_\xi - 1$. We get $gw = \sum a_i v_i^2 + w'$. Define $\varphi : W_\xi \to W_\zeta$, $w \mapsto gw$ projects to W_ζ. Let $w_1, w_2 \in W_\xi$. Assume $gw_1 = \sum a_i^1 v_i^2 + w_1'$, $gw_2 = \sum a_i^2 v_i^2 + w_2'$. We have $\beta(gw_1, gw_2) = \beta(w_1', w_2') = \beta(w_1, w_2), \beta_\zeta(gw_1, gw_2) = \beta_\zeta(w_1', w_2') = \beta_\xi(w_1, w_2)$, namely, $\beta(\varphi(w_1), \varphi(w_2)) = \beta(w_1, w_2)$, $\beta_\zeta(\varphi(w_1), \varphi(w_2)) = \beta_\xi(w_1, w_2)$. Now we show that φ is a bijection. Let $w \in W_\xi$ be such that $\varphi(w) = 0$. Then for any $v \in W_\xi$, $\beta(v, w) = \beta(\varphi(v), \varphi(w)) = 0$. Since $\beta|_{W_\xi \times W_\xi}$ is nondegenerate, $w = 0$. Thus φ is injective. On the other hand, we have $\dim W_\xi = \dim W_\zeta$. Hence φ is bijective. □

Corollary 2.3.11. *Assume $V_\xi = V_{2m_\xi+1,\xi} \oplus W_\xi$ is equivalent to $V_\zeta = V_{2m_\zeta+1,\zeta} \oplus W_\zeta$, then $m_\xi = m_\zeta$ and T_ξ, T_ζ are conjugate.*

Lemma 2.3.12. *Assume ξ is nilpotent. Then T_ξ is nilpotent.*

Proof. We replace $G = Sp(V)$ by $G = O(V)$ and β by $\beta|_{W\times W}$ in the proof of Lemma 2.2.5. Moreover, when apply $\mathrm{Ad}(\phi(a))$ to T_ξ, we regard $\phi(a)$ as a linear map restricting to the subspace W of V so that $\phi(a) \in O(W)$. Also notice that $T_\xi \in \mathfrak{o}(W) = \{x \in \mathfrak{gl}(W) | \beta(xw, w) = 0, \ \forall \ w \in W\}$, since $\beta(T_\xi w, w) = \beta_\xi(w, w) = 0$ for all $w \in W$. Then the same argument as in the proof of Lemma 2.2.5 applies since $\beta|_{W\times W}$ is nondegenerate. □

By Lemma 2.3.8, every form module $(V_\xi, \alpha, \beta_\xi)$ can be reduced to the form $V_\xi = V_{2m+1} \oplus W_\xi$, where V_{2m+1} has a basis $\{v_i, i = 0, \dots, m, u_i, i = 0, \dots, m-1\}$ as in Lemmas 2.3.5 and 2.3.6. We have that $(V_\xi, \alpha, \beta_\xi)$ is determined by V_{2m+1} and $(W_\xi, \alpha|_{W_\xi}, \beta_\xi|_{W_\xi\times W_\xi})$. Now we consider $(W_\xi, \alpha|_{W_\xi}, \beta_\xi|_{W_\xi\times W_\xi}) := (W, \alpha|_W, \beta_\xi|_{W\times W})$ and let $T_\xi : W \to W$ be defined as above. It follows that $\beta_\xi|_{W\times W}$ is determined by T_ξ and $\beta|_{W\times W}$.

2.3.3 In this subsection assume $\mathbf{k}$ is algebraically closed. Let $V_\xi = V_{2m+1} \oplus W$ and T_ξ be as in the last paragraph of 2.3.2. Since $T_\xi \in \mathfrak{o}(W)$ is nilpotent (Lemma 2.3.12), we can view W as a $k[[T_\xi]]-$module (see subsection 1.2.2). By the classification of nilpotent orbits in $\mathfrak{o}(W)$ (see [H, 3.5 and 3.9]), W is equivalent to $W_{l_1}(m_1) \oplus \cdots \oplus W_{l_s}(m_s)$ (notation as in Proposition 1.2.2) for some $m_1 \geq \cdots \geq m_s$, $l_1 \geq \cdots \geq l_s$ and $m_1 - l_1 \geq \cdots \geq m_s - l_s$, where $[(m_i+1)/2] \leq l_i \leq m_i$.

Lemma 2.3.13. *Assume $m < k - l$. We have $V_{2m+1} \oplus W_l(k) \cong V_{2m+1} \oplus W_{k-m}(k)$.*

Proof. Assume $V_{2m+1} = \mathrm{span}\{v_0, \dots, v_m, u_0, \dots, u_{m-1}\}$, where v_i, u_i are chosen as in Lemma 2.3.5 and Lemma 2.3.6. Assume $V_{2m+1} \oplus W_l(k)$ and $V_{2m+1} \oplus W_{k-m}(k)$ correspond to ξ_1 and ξ_2 respectively. Let $T_1 = T_{\xi_1} : W_l(k) \to W_l(k)$ and $T_2 = T_{\xi_2} : W_{k-m}(k) \to W_{k-m}(k)$. There exist ρ_1, ρ_2 such that $W_l(k) = \mathrm{span}\{\rho_1, \dots, T_1^{k-1}\rho_1, \rho_2, \dots, T_1^{k-1}\rho_2\}$, $T_1^k\rho_1 = T_1^k\rho_2 = 0$, $\alpha(T_1^i\rho_1) = \delta_{i,l-1}$, $\alpha(T_1^i\rho_2) = 0$, $\beta(T_1^i\rho_1, T_1^j\rho_1) = \beta(T_1^i\rho_2, T_1^j\rho_2) = 0$ and $\beta(T_1^i\rho_1, T_1^j\rho_2) = \delta_{i+j,k-1}$. There exist τ_1, τ_2 such that $W_{k-m}(k) = \mathrm{span}\{\tau_1, \dots, T_2^{k-1}\tau_1, \tau_2, \dots, T_2^{k-1}\tau_2\}$, $T_2^k\tau_1 = T_2^k\tau_2 = 0$, $\alpha(T_2^i\tau_1) = \delta_{i,k-m-1}$, $\alpha(T_2^i\tau_2) = 0$, $\beta(T_2^i\tau_1, T_2^j\tau_1) = \beta(T_2^i\tau_2, T_2^j\tau_2) = 0$ and $\beta(T_2^i\tau_1, T_2^j\tau_2) = \delta_{i+j,k-1}$. Define $g : V_{2m+1} \oplus W_l(k) \to V_{2m+1} \oplus W_{k-m}(k)$ by $gv_i = v_i$, $gu_i = u_i + (T_2^{k-(m+l)+i} + T_2^i)\tau_2, gT_1^j\rho_2 =$

$T_2^j\tau_2$, $gT_1^j\rho_1 = T_2^j\tau_1 + v_{k-1-j} + v_{m+l-1-j}$, where $v_i = 0$, if $i < 0$ or $i > m$. Then g is the isomorphism we want. $\square$

Lemma 2.3.14. *Assume $m \geq k - l_i, i = 1, 2$. We have $V_{2m+1} \oplus W_{l_1}(k) \cong V_{2m+1} \oplus W_{l_2}(k)$ if and only if $l_1 = l_2$.*

Proof. Assume $k - m \leq l_1 < l_2$. We show that $V_{2m+1} \oplus W_{l_1}(k) \not\cong V_{2m+1} \oplus W_{l_2}(k)$. Let $(V_1, \alpha, \beta_1) = V_{2m+1} \oplus W_{l_1}(k)$ and $(V_2, \alpha, \beta_2) = V_{2m+1} \oplus W_{l_2}(k)$. Let $T_1 = T_{\xi_1} : W_{l_1}(k) \to W_{l_1}(k)$ and $T_2 = T_{\xi_2} : W_{l_2}(k) \to W_{l_2}(k)$. Assume there exists $g : V_{2m+1} \oplus W_{l_1}(k) \to V_{2m+1} \oplus W_{l_2}(k)$ a linear isomorphism satisfying $\beta_2(gv, gw) = \beta_1(v, w)$ and $\alpha(gv) = \alpha(v)$. Define $\varphi : W_{l_1}(k) \to W_{l_2}(k)$ by $w_1 \mapsto$ (gw_1 projects to $W_{l_2}(k)$). Then we have $\beta(\varphi(w_1), \varphi(w_1')) = \beta(w_1, w_1')$, $\beta_2(\varphi(w_1), \varphi(w_1')) = \beta_1(w_1, w_1')$ and $T_2(\varphi(w)) = \varphi(T_1(w))$ (see the proof of Lemma 2.3.10).

Let v_i, $i = 0, \ldots, m$, and u_i, $i = 0, \ldots, m-1$, be a basis of V_{2m+1} as in Lemmas 2.3.5 and 2.3.6. Choose a basis $T_i^j\rho_i, T_i^j\tau_i$, $j = 0, \ldots, k-1$, $i = 1, 2$ of $W_{l_i}(k)$ such that $T_i^k\rho_i = T_i^k\tau_i = 0$, $\beta(T_i^{j_1}\rho_i, T_j^{j_2}\tau_j) = \delta_{j_1+j_2,k-1}\delta_{i,j}$, $\beta(T_i^{j_1}\rho_i, T_j^{j_2}\rho_j) = \beta(T_i^{j_1}\tau_i, T_j^{j_2}\tau_j) = 0$, $\alpha(T_i^j\rho_i) = \delta_{j,l_i-1}$ and $\alpha(T_i^j\tau_i) = 0$. We have $gv_i = av_i, i = 0, \ldots, m$, $gu_i = u_i/a + \sum_{l=0}^{m} a_{il}v_l + \sum_{l=0}^{k-1} x_{il}T_2^l\rho_2 + \sum_{l=0}^{k-1} y_{il}T_2^l\tau_2$. Now we can assume $gT_1^j\rho_1 = \sum_{i=0}^{k-1-j} a_iT_2^{i+j}\rho_2 + \sum_{i=0}^{k-1-j} b_iT_2^{i+j}\tau_2 + \sum_{i=0}^{m} c_{ij}v_i + \sum_{i=0}^{m-1} d_{ij}u_i$, $j = 0, \ldots, k-1, gT_1^j\tau_1 = \sum_{i=0}^{k-1-j} e_iT_2^{i+j}\rho_2 + \sum_{i=0}^{k-1-j} f_iT_2^{i+j}\tau_2 + \sum_{i=0}^{m} g_{ij}v_i + \sum_{i=0}^{m-1} h_{ij}u_i$, $j = 0, \ldots, k-1$. A straightforward calculation shows that we have Since $l_2 > k - m$, it follows that $c_{m,l_2-1} = g_{m,l_2-1} = 0$ (if $l_2 = k$) or $c_{m,l_2-1} = c_{0,l_2+m-1} = 0$, $g_{m,l_2-1} = g_{0,l_2+m-1} = 0$ (if $l_2 < k$). Thus $a_0 = e_0 = 0$. On the other hand, one can show that $a_0 = c_{m,l_2-1}$ and $e_0 = g_{m,l_2-1}$ (note $l_2 > l_1 \geq [k+1]/2$). But from $\beta(g\rho_1, gT_1^{k-1}\tau_1) = \beta(\rho_1, T_1^{k-1}\tau_1) = 1$ we have $a_0f_0 + e_0b_0 = 1$. This is a contradiction. $\square$

It follows that for any $V = (V_\xi, \alpha, \beta_\xi)$, there exist a unique $m \geq 0$ and a unique sequence of modules $W_{l_i}(k_i)$, $i = 1, \ldots, s$ such that

$$V \cong V_{2m+1} \oplus W_{l_1}(k_1) \oplus \cdots \oplus W_{l_s}(k_s),$$

$[(k_i+1)/2] \leq l_i \leq k_i$, $k_1 \geq k_2 \geq \cdots \geq k_s$, $l_1 \geq l_2 \geq \cdots \geq l_s$ and $m \geq k_1 - l_1 \geq$

$k_2 - l_2 \geq \cdots \geq k_s - l_s$. We call this the *normal form* of the module V. Two form modules are equivalent if and only if their normal forms are the same.

Hence to each nilpotent orbits we associate a pair of partitions $(m, k_1-l_1, \ldots, k_s-l_s)(l_1, \ldots, l_s)$, where $l_1 \geq l_2 \geq \cdots \geq l_s \geq 0$ and $m \geq k_1 - l_1 \geq k_2 - l_2 \geq \cdots \geq k_s - l_s \geq 0$. This defines a bijection from the set of nilpotent orbits to the set $\{(\nu, \mu) | \nu = (\nu_0, \nu_1, \ldots, \nu_s), \mu = (\mu_1, \mu_2, \ldots, \mu_s), |\mu| + |\nu| = n, \nu_i \leq \mu_i, i = 1, \ldots, s\}$, which has cardinality $p_2(n) - p_2(n-2)$.

2.3.4 In this subsection, we classify the form modules $(V_\xi, \alpha, \beta_\xi)$ over $\mathbf{F}_q$. Let $V_\xi = V_{2m+1} \oplus W_\xi$ and T_ξ be as in the last paragraph of 2.3.2. By the classification of $(W_\xi, \alpha|_{W_\xi}, T_\xi)$ over $\mathbf{F}_q$, we have $W_\xi \cong \oplus W_{l_i}^{\epsilon_i}(k_i)$ where $\epsilon_i = 0$ or δ, $m_1 \geq \cdots \geq m_s$, $l_1 \geq \cdots \geq l_s$, $m_1 - l_1 \geq \cdots \geq m_s - l_s$ and $[(m_i+1)/2] \leq l_i \leq m_i$ (notation as in Proposition 1.3.1).

Lemma 2.3.15. *Assume $m \geq k - l$ and $l > m$. We have $V_{2m+1} \oplus W_l^0(k) \cong V_{2m+1} \oplus W_l^\delta(k)$.*

Proof. Let $W_l^0(k) = (W_1, \alpha, T_1)$ and $W_l^\delta(k) = (W_2, \alpha, T_2)$. Take ρ_1, ρ_2 such that $W_l^0(k) = \operatorname{span}\{\rho_1, \ldots, T_1^{k-1}\rho_1, \rho_2, \ldots, T_1^{k-1}\rho_2\}$, $T_1^k\rho_1 = T_1^k\rho_2 = 0$, $\alpha(T_1^i\rho_1) = \delta_{i,l-1}$, $\alpha(T_1^i\rho_2) = 0$, $\beta(T_1^i\rho_1, T_1^j\rho_1) = \beta(T_1^i\rho_2, T_1^j\rho_2) = 0$ and $\beta(T_1^i\rho_1, T_1^j\rho_2) = \delta_{i+j,k-1}$. Take τ_1, τ_2 such that $W_l^\delta(k) = \operatorname{span}\{\tau_1, \ldots, T_2^{k-1}\tau_1, \tau_2, \ldots, T_2^{k-1}\tau_2\}$, $T_2^k\tau_1 = T_2^k\tau_2 = 0$, $\alpha(T_2^i\tau_1) = \delta_{i,l-1}$, $\alpha(T_2^i\tau_2) = \delta_{i,k-l}\delta$, $\beta(T_2^i\tau_1, T_2^j\tau_1) = \beta(T_2^i\tau_2, T_2^j\tau_2) = 0$ and $\beta(T_2^i\tau_1, T_2^j\tau_2) = \delta_{i+j,k-1}$. Let v_i, u_i be a basis of V_{2m+1} as in Lemmas 2.3.5 and 2.3.6. Define $g : V_{2m+1} \oplus W_l^0(k) \to V_{2m+1} \oplus W_l^\delta(k)$ by $gv_i = v_i$, $gu_i = u_i + \sqrt{\delta}T_2^{l-m-1+i}\tau_1, gT_1^i\rho_1 = T_2^i\tau_1$, $gT_1^i\rho_2 = T_2^i\tau_2 + \sqrt{\delta}v_{k-l+m-i}$, where $v_i = 0$ if $i < 0$ or $i > m$. ☐

Lemma 2.3.16. *Assume $m \geq k - l$ and $l \leq m$. We have $V_{2m+1} \oplus W_l^0(k) \ncong V_{2m+1} \oplus W_l^\delta(k)$.*

Proof. Let v_i, $i = 0, \ldots, m$ and u_i, $i = 0, \ldots, m-1$ be a basis of V_{2m+1} as in Lemmas 2.3.5 and 2.3.6. Let $W_l^0(k) = (W_0, \alpha, T_0)$ and $W_l^\delta(k) = (W_\delta, \alpha, T_\delta)$. Choose a basis $T_\epsilon^j\rho_\epsilon, T_\epsilon^j\tau_\epsilon$, $j = 0, \ldots, k-1$, $\epsilon = 0, \delta$ of $W_l^\epsilon(k)$ such that $T_\epsilon^k\rho_\epsilon = T_\epsilon^k\tau_\epsilon = 0$, $\beta(T_{\epsilon_1}^{j_1}\rho_{\epsilon_1}, T_{\epsilon_2}^{j_2}\tau_{\epsilon_2}) = \delta_{j_1+j_2,k-1}\delta_{\epsilon_1,\epsilon_2}$, $\beta(T_{\epsilon_1}^{j_1}\rho_{\epsilon_1}, T_{\epsilon_2}^{j_2}\rho_{\epsilon_2}) = \beta(T_{\epsilon_1}^{j_1}\tau_{\epsilon_1}, T_{\epsilon_2}^{j_2}\tau_{\epsilon_2}) = 0$, $\alpha(T_\epsilon^j\rho_\epsilon) = \delta_{j,l-1}$ and $\alpha(T_\epsilon^j\tau_\epsilon) = \epsilon\delta_{j,k-l}\delta_{\epsilon,\delta}$. Assume there exists $g : V_{2m+1} \oplus W_l^0(k) \to V_{2m+1} \oplus$

$W_{l_2}^{\delta}(k)$ a linear isomorphism satisfying $\beta(gv,gw) = \beta(v,w)$, $\beta_\delta(gv,gw) = \beta_0(v,w)$ and $\alpha(gv) = \alpha(v)$. We have $gv_i = av_i$, $gu_i = u_i/a + \sum_{l=0}^{m} a_{il}v_l + \sum_{l=0}^{k-1} x_{il}T_\delta^l\rho_\delta + \sum_{l=0}^{k-1} y_{il}T_\delta^l\tau_\delta$. Now we can assume $gT_0^j\rho_0 = \sum_{i=0}^{k-1-j} a_iT_\delta^{i+j}\rho_\delta + \sum_{i=0}^{k-1-j} b_iT_\delta^{i+j}\tau_\delta + \sum_{i=0}^{m} c_{ij}v_i + \sum_{i=0}^{m-1} d_{ij}u_i$, $j = 0,\dots,k-1, gT_0^j\tau_0 = \sum_{i=0}^{k-1-j} e_iT_\delta^{i+j}\rho_\delta + \sum_{i=0}^{k-1-j} f_iT_\delta^{i+j}\tau_\delta + \sum_{i=0}^{m} g_{ij}v_i + \sum_{i=0}^{m-1} h_{ij}u_i$, $j = 0,\dots,k-1$. By similar argument as in the proof of Lemma 2.3.14, we get that $c_{m,k-1} = 0, g_{m,k-1} = 0, c_{ij} = c_{i+1,j-1}, g_{ij} = g_{i+1,j-1}, i = 0,\dots,m-1, j = 0,\dots,k-1$. Since we have $m \geq l$, $c_{m,i} = c_{0,i+m} = 0$ and $g_{m,i} = g_{0,i+m} = 0$ when $i \geq k-l$. We get some of the equations are $a_0^2 + a_0b_0 + \delta b_0^2 = 1, e_0^2 + e_0f_0 + \delta f_0^2 = 0$ and $a_0f_0 + b_0e_0 = 1$ (when $l = (k+1)/2$) or $a_{l-1-i}^2 + \sum_{j=0}^{k-1-2i} a_jb_{k-1-2i-j} + \delta b_{k-l-i}^2 = \delta_{i,l-1}$, $e_{l-1-i}^2 + \sum_{j=0}^{k-1-2i} e_jf_{k-1-2i-j} + \delta f_{k-l-i}^2 = 0$, $k-l \leq i \leq l-1$ and $a_0f_0 + b_0e_0 = 1$ (when $l > (k+1)/2$). We get $a_0 = f_0 = 1, e_i = 0, i = 0,\dots,2l-k-2$ and $e_{2l-k-1}^2 + e_{2l-k-1} + \delta = 0$. This is a contradiction. □

Let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G, $\mathfrak{g}$.

Proposition 2.3.17. *The nilpotent orbit in $\mathfrak{g}^*$ corresponding to the pair of partitions $(\nu_0,\nu_1,\dots,\nu_s)(\mu_1,\mu_2,\dots,\mu_s)$ splits into at most 2^k $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$.*

Proof. Let $V = V_{2m+1} \oplus W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_s}(\lambda_s)$ be the normal form of a module corresponding to (ν,μ) over $\bar{\mathbf{F}}_q$. We show that the equivalence class of V over $\bar{\mathbf{F}}_q$ decomposes into at most 2^k equivalence classes over $\mathbf{F}_q$. It is enough to show that form modules of the form $V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(\lambda_1) \oplus \cdots \oplus W_{l_s}^{\epsilon_s}(\lambda_s)$, $\epsilon_i = 0$ or δ, have at most 2^k equivalence classes over $\mathbf{F}_q$. Suppose $i_1,\dots,i_k$ are such that $\beta_{i_j} < \mu_{i_j} \leq \beta_{i_j-1}$, $j = 1,\dots,k$. Using Lemma 2.3.15, Lemma 2.3.16 and Lemma 1.4.3 (iii), one can easily verify that a form module of the above form is isomorphic to one of the following modules: $V_1^{\epsilon_1} \oplus \cdots \oplus V_k^{\epsilon_k} \oplus V_{k+1}$, where $V_1^{\epsilon_1} = V_{2m+1} \oplus W_{l_1}^0(\lambda_1) \oplus \cdots \oplus W_{l_{i_1-1}}^0(\lambda_{i_1-1}) \oplus W_{l_{i_1}}^{\epsilon_1}(\lambda_{i_1})$, $V_t^{\epsilon_t} = W_{l_{i_{t-1}+1}}^0(\lambda_{i_{t-1}+1}) \oplus \cdots \oplus W_{l_{i_t-1}}^0(\lambda_{i_t-1}) \oplus W_{l_{i_t}}^{\epsilon_t}(\lambda_{i_t})$, $t = 2,\dots,k$, $\epsilon_i = 0$ or δ, $i = 1,\dots,k$, and $V_{k+1} = W_{l_{i_k+1}}^0(\lambda_{i_k+1}) \oplus \cdots \oplus W_{l_s}^0(\lambda_s)$. Thus the proposition follows. □

Proposition 2.3.18. *The number of nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is at most $p_2(n)$.*

Proof. We have mapped the nilpotent orbits in $\mathfrak{g}(\bar{\mathbf{F}}_q)^*$ bijectively to the set $\{(\nu,\mu)|\nu = (\nu_0,\nu_1,\ldots,\nu_s), \mu = (\mu_1,\mu_2,\ldots,\mu_s), |\mu|+|\nu| = n, \nu_i \leq \mu_i, i = 1,\ldots,s\} := \Delta$. Let $(\nu,\mu) \in \Delta$, $\nu = (\nu_0,\nu_1,\ldots,\nu_s), \mu = (\mu_1,\mu_2,\cdots,\mu_s)$. By Proposition 2.3.17, the nilpotent orbit corresponding to (ν,μ) splits into at most 2^k orbits in $\mathfrak{g}(\mathbf{F}_q)^*$, where $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$. We associate 2^k pairs of partitions to this orbit as follows. Suppose $r_1, r_2, \ldots, r_k$ are such that $\nu_{r_i} < \mu_{r_i} \leq \nu_{r_i-1}, i = 1,\ldots,k$. Let $\nu^0 = (\nu_0,\ldots,\nu_{r_1-1}), \mu^0 = (\mu_1,\ldots,\mu_{r_1-1}), \nu^{1,i} = (\nu_{r_i},\ldots,\nu_{r_{i+1}-1}), \mu^{1,i} = (\mu_{r_i},\ldots,\mu_{r_{i+1}-1})$, $\nu^{2,i} = (\mu_{r_i},\ldots,\mu_{r_{i+1}-1}), \mu^{2,i} = (\nu_{r_i},\ldots,\nu_{r_{i+1}-1}), i = 1,\ldots,k-1, \nu^{1,k} = (\nu_{r_k},\ldots,\nu_s)$, $\mu^{1,k} = (\mu_{r_k},\ldots,\mu_s), \nu^{2,k} = (\mu_{r_k},\ldots,\mu_s), \mu^{2,k} = (\nu_{r_k},\ldots,\nu_s)$. We associate to (ν,μ) the pairs of partitions $(\tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k}, \tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k})$, $\tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k} = (\nu^0, \nu^{\epsilon_1,1}, \nu^{\epsilon_2,2}, \ldots, \nu^{\epsilon_k,k})$, $\tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k} = (\mu^0, \mu^{\epsilon_1,1}, \mu^{\epsilon_2,2}, \ldots, \mu^{\epsilon_k,k})$, where $\epsilon_i \in \{1,2\}$, $i = 1,\ldots,k$. Notice that the pairs of partitions $(\tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k}, \tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k})$ are distinct and among them only $(\nu,\mu) = (\tilde{\nu}^{1,\ldots,1}, \tilde{\mu}^{1,\ldots,1})$ is in Δ. One can verify that $\{(\tilde{\nu}^{\epsilon_1,\ldots,\epsilon_k}, \tilde{\mu}^{\epsilon_1,\ldots,\epsilon_k}) | (\nu,\mu) \in \Delta\} = \{(\nu,\mu) | |\nu|+|\mu| = n\}$. □

2.4 Even orthogonal groups

Let V be a vector space of dimension $2n$ over $\mathbf{k}$ equipped with a non-defective quadratic form $\alpha : V \to \mathbf{k}$. Let $\beta : V \times V \to \mathbf{k}$ be the non-degenerate bilinear form associated to α. Recall that the even orthogonal group is defined as $G = O(2n) = O(V,\alpha) = \{g \in GL(V) \mid \alpha(gv) = \alpha(v), \forall\ v \in V\}$ and its Lie algebra is $\mathfrak{g} = \mathfrak{o}(2n) = \mathfrak{o}(V,\alpha) = \{x \in \mathfrak{gl}(V) \mid \beta(xv,v) = 0, \forall\ v \in V\}$. Let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a split Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G, $\mathfrak{g}$.

Proposition 2.4.1. *The numbers of nilpotent $G(\boldsymbol{F}_q)$-orbits in $\mathfrak{g}(\boldsymbol{F}_q)$ and in $\mathfrak{g}(\boldsymbol{F}_q)^*$ are the same.*

The proposition can be proved in two ways.

First proof. There exists a G-invariant non-degenerate bilinear form on $\mathfrak{g} = \mathfrak{o}(2n)$ (G. Lusztig). Hence we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via this bilinear form and the proposition follows. Consider the vector space $\bigwedge^2 V$ on which G acts in a natural way: $g(a \wedge b) = ga \wedge gb$. On $\bigwedge^2 V$ there is a G-invariant non-degenerate bilinear form

$$\langle a \wedge b, c \wedge d\rangle = \det \begin{bmatrix} \beta(a,c) & \beta(a,d) \\ \beta(b,c) & \beta(b,d) \end{bmatrix}.$$

Define a map $\phi : \bigwedge^2 V \to \mathfrak{o}(2n)$ by $a \wedge b \mapsto \phi_{a\wedge b}$ and extending by linearity where $\phi_{a\wedge b}(v) = \beta(a,v)b + \beta(b,v)a$. This map is G-equivariant since we have $\phi_{ga\wedge gb} = g\phi_{a\wedge b}g^{-1}$. One can easily verify that ϕ is a bijection. Define $\langle \phi_{a\wedge b}, \phi_{c\wedge d}\rangle_{\mathfrak{o}(2n)} = \langle a \wedge b, c \wedge d\rangle$ and extend it to $\mathfrak{o}(2n)$ by linearity. This defines a G-invariant non-degenerate bilinear form on $\mathfrak{o}(2n)$.

Second proof. Let ξ be an element of $\mathfrak{g}^*$. There exists $X \in \mathfrak{gl}(V)$ such that $\xi(x) = \mathrm{tr}(Xx)$ for any $x \in \mathfrak{g}$. We define a linear map $T_\xi : V \to V$ by

$$\beta(T_\xi v, v') = \beta(Xv, v') + \beta(v, Xv'), \text{ for all } v, v' \in V.$$

Lemma 2.4.2. *T_ξ is well-defined.*

Proof. The same proof as in Lemma 2.3.1 shows that $\beta(T_\xi v, v')$ is well-defined and thus T_ξ is well-defined. □

Lemma 2.4.3. *Two elements $\xi, \zeta \in \mathfrak{g}^*$ lie in the same G-orbit if and only if there exists $g \in G$ such that $gT_\xi g^{-1} = T_\zeta$.*

Proof. Assume $\xi(x) = \mathrm{tr}(X_\xi x), \zeta(x) = \mathrm{tr}(X_\zeta x)$, $\forall x \in \mathfrak{g}$. Then ξ, ζ lie in the same G-orbit if and only if there exists $g \in G$ such that $\mathrm{tr}(gX_\xi g^{-1}x) = \mathrm{tr}(X_\zeta x)$, $\forall\, x \in \mathfrak{g}$. This is equivalent to $\beta((gX_\xi g^{-1} + X_\zeta)v, w) + \beta(v, (gX_\xi g^{-1} + X_\zeta)w) = 0$, $\forall\, v, w \in V$, which is true if and only if $gT_\xi g^{-1} = T_\zeta$. □

Note that $\beta(T_\xi v, v) = 0$ for any $v \in V$. Thus $T_\xi \in \mathfrak{g}$. We have in fact defined a bijection $\theta : \mathfrak{g}^* \to \mathfrak{g}$, $\xi \mapsto T_\xi$. This induces a bijection $\theta|_{\mathcal{N}'} : \mathcal{N}' \to \mathcal{N}$, where $\mathcal{N}'$(resp. $\mathcal{N}$) is the set of all nilpotent elements (unstable vectors) in $\mathfrak{g}^*$(resp. $\mathfrak{g}$). Moreover, $\theta|_{\mathcal{N}'}$ is G-equivariant by Lemma 2.4.3. The proposition follows.

2.5 Springer correspondence

In this section, we assume $\mathbf{k}$ is algebraically closed. Let G, $\mathfrak{g}$, $\mathfrak{g}^*$, B, $\mathfrak{b}$ and $\mathfrak{n}^*$ be as in the introduction. In subsections 2.5.1 and 2.5.2, we assume that G is simply connected. Let $B = TU$ be a Levi decomposition of the Borel subgroup B and U^- a maximal unipotent subgroup opposite to B. Let $\mathfrak{t}$, $\mathfrak{n}$ and $\mathfrak{n}^-$ be the Lie algebra of T, U and U^- respectively. Let $\mathcal{B}$ be the variety of Borel subgroups of G. Define $\mathfrak{t}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n} \oplus \mathfrak{n}^-) = 0\}$. An element ξ in $\mathfrak{g}^*$ is called semisimple if there exists $g \in G$ such that $g.\xi \in \mathfrak{t}^*$ (see [KW]). Let $r = \dim T$. The proofs in this section are entirely similar to those in section 1.6 (see also [L3, L5]). We omit much detail and refer the readers to [X2] for complete proofs.

2.5.1 Recall that a semisimple element ξ in $\mathfrak{g}^*$ is called regular if the connected centralizer $Z_G^0(\xi)$ in G is a maximal torus of G ([KW]). By [KW, Lemma 3.2], there exist regular semisimple elements in $\mathfrak{g}^*$ and they form an open dense subset in $\mathfrak{g}^*$. Note that this is not always true when G is not simply connected.

Let $\mathfrak{t}_0', Y'$ be the set of regular semisimple elements in $\mathfrak{t}^*, \mathfrak{g}^*$ respectively. Let $\widetilde{Y}' = \{(\xi, gT) \in Y' \times G/T | g^{-1}.\xi \in \mathfrak{t}_0'\}$. Define $\pi' : \widetilde{Y}' \to Y'$ by $\pi'(\xi, gT) = \xi$. The Weyl group $W = NT/T$ acts (freely) on $\widetilde{Y}'$ by $n : (\xi, gT) \mapsto (\xi, gn^{-1}T)$.

Similar to the map π in section 1.6, π' is a finite covering. Thus $\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'}$ is a well-defined local system on Y' and the intersection cohomology complex $IC(\mathfrak{g}^*, \pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'})$ is well-defined.

Let $X' = \{(\xi, gB) \in \mathfrak{g}^* \times G/B | g^{-1}.\xi \in \mathfrak{b}^*\}$, where $\mathfrak{b}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}) = 0\}$. Define $\varphi' : X' \to \mathfrak{g}^*$ by $\varphi'(\xi, gB) = \xi$. The map φ' is G-equivariant with G-action on X' given by $g_0 : (\xi, gB) \mapsto (g_0.\xi, g_0 gB)$.

Proposition 2.5.1. *$\varphi'_!\bar{\mathbb{Q}}_{lX'}$ is canonically isomorphic to $IC(\mathfrak{g}^*, \pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'})$. Moreover, $End(\varphi'_!\bar{\mathbb{Q}}_{lX'}) = End(\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'}) = \bar{\mathbb{Q}}_l[W]$.*

2.5.2 Recall that we denote $\hat{W}$ the set of simple modules (up to isomorphism) for the Weyl group W. Similarly as in 1.6.5, we have $\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'} = \bigoplus_{\rho \in \hat{W}} (\rho \otimes (\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'})_\rho)$ and $\varphi'_!\bar{\mathbb{Q}}_{lX'} = \bigoplus_{\rho \in \hat{W}} (\rho \otimes (\varphi'_!\bar{\mathbb{Q}}_{lX'})_\rho)$, where $(\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'})_\rho$ is an irreducible local system on Y and $(\varphi'_!\bar{\mathbb{Q}}_{lX'})_\rho = IC(\mathfrak{g}^*, (\pi'_!\bar{\mathbb{Q}}_{l\widetilde{Y}'})_\rho)$.

Let $\mathcal{N}'$ be the set of nilpotent elements in $\mathfrak{g}^*$. Set $X'^{\varsigma} = \varphi'^{-1}(\mathcal{N}') \subset X'$. Let $\varphi'^{\varsigma} : X'^{\varsigma} \to \mathcal{N}'$ be the restriction of $\varphi' : X' \to \mathfrak{g}^*$.

Lemma 2.5.2. *There exists a nilpotent element ξ in $\mathfrak{g}^*$ such that the set $\{B_1 \in \mathcal{B} | \xi \in \mathfrak{n}_1'\}$ is finite.*

Proof. Let R be the root system of G relative to T. We have a weight space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \oplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | Ad(t)x = \alpha(t)x, \forall t \in T\}$ is one dimensional for $\alpha \in R$ (see for example [Sp2]). Let $\alpha_i, i = 1, \ldots, r$ be a set of simple roots in R such that $\mathfrak{b} = \mathfrak{t} \oplus \oplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and $x_\alpha, \alpha \in R, h_{\alpha_i}$ a Chevalley basis in $\mathfrak{g}$. Let x_α^* and $h_{\alpha_i}^*$ be the dual basis in $\mathfrak{g}^*$. Set $\xi = \sum_{i=1}^r x_{-\alpha_i}^*$. Then $\xi \in \mathfrak{n}^*$.

We show that $\{B_1 \in \mathcal{B} | \xi \in \mathfrak{n}_1'\} = \{B\}$. Assume $g.\xi \in \mathfrak{n}^*$. We have $\xi(g^{-1}\mathfrak{b}g) = 0$. By Bruhat decomposition, we can write $g^{-1} = vn_wb$, where $v \in U \cap wUw^{-1}$ and $n_w \in NT$ is a representative for $w \in W$. Assume $w \neq 1$. There exists $1 \leq i \leq r$ such that $w^{-1}\alpha_i < 0$. Let $\alpha = -w^{-1}\alpha_i > 0$. We have $\xi(\mathrm{Ad}(vn_w)x_\alpha) = \xi(c\mathrm{Ad}(v)x_{-\alpha_i}) = \xi(cx_{-\alpha_i}) = c$, where c is a nonzero constant. This contradicts $\xi(g^{-1}\mathfrak{b}g) = 0$. Thus $w = 1$ and $g^{-1}.\mathfrak{n}^* = \mathfrak{n}^*$. □

Lemma 2.5.3. (i) *X'^{ς} and $\mathcal{N}'$ are irreducible varieties of dimension $d_0 = \dim G - r$.*

(ii) *We have $(\varphi_!'\bar{\mathbb{Q}}_{lX'})|_{\mathcal{N}'} = \varphi_!'^{\varsigma}\bar{\mathbb{Q}}_{lX'^{\varsigma}}$. Moreover, $\varphi_!'^{\varsigma}\bar{\mathbb{Q}}_{lX'^{\varsigma}}[d_0]$ is a semisimple perverse sheaf on $\mathcal{N}'$.*

(iii) *We have $(\varphi_!'\bar{\mathbb{Q}}_{lX'})_\rho|_{\mathcal{N}'} \neq 0$ for any $\rho \in \hat{W}$.*

Proof. We only prove (i). We have $X'^{\varsigma} = \{(\xi, gB) \in \mathfrak{g}^* \times \mathcal{B} | g^{-1}.\xi \in \mathfrak{n}\}$. By projection to the second coordinate, we see that $\dim X'^{\varsigma} = \dim \mathfrak{n}^* + \dim \mathcal{B} = \dim G - r$. The map φ'^{ς} is surjective and the fiber at some point ξ is finite (see Lemma 2.5.2). It follows that $\dim \mathcal{N}' = \dim G - r$. This proves (i). □

Let $\mathfrak{A}_{\mathfrak{g}^*}$ be the set defined for $\mathfrak{g}^*$ as in the introduction.

Proposition 2.5.4. (i) *The restriction map $End_{\mathcal{D}(\mathfrak{g}^*)}(\varphi_!'\bar{\mathbb{Q}}_{lX'}) \to End_{\mathcal{D}(\mathcal{N}')}(\varphi_!'^{\varsigma}\bar{\mathbb{Q}}_{lX'^{\varsigma}})$ is an isomorphism.*

(ii) *For any $\rho \in \hat{W}$, there is a unique $(\mathrm{c}, \mathcal{F}) \in \mathfrak{A}_{\mathfrak{g}^*}$ such that $(\varphi_!'\bar{\mathbb{Q}}_{lX'})_\rho|_{\mathcal{N}'}[d_0]$ is $IC(\bar{\mathrm{c}}, \mathcal{F})[\dim \mathrm{c}]$ regarded as a simple perverse sheaf on $\mathcal{N}'$ (zero outside $\bar{\mathrm{c}}$). Moreover, $\rho \mapsto (\mathrm{c}, \mathcal{F})$ is an injective map $\gamma : \hat{W} \to \mathfrak{A}_{\mathfrak{g}^*}$.*

2.5.3 In this subsection let $G = SO_N(\mathbf{k})$ (resp. $Sp_{2n}(\mathbf{k})$) and $\mathfrak{g} = \mathfrak{o}_N(\mathbf{k})$ (resp. $\mathfrak{sp}_{2n}(\mathbf{k})$) the Lie algebra of G. Let G_s be a simply connected group over $\mathbf{k}$ of the same type as G and $\mathfrak{g}_s$ the Lie algebra of G_s. For q a power of 2, let $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$ be the fixed points of a split Frobenius map $\mathfrak{F}_q$ relative to $\mathbf{F}_q$ on G, $\mathfrak{g}$. Let $G_s(\mathbf{F}_q)$, $\mathfrak{g}_s(\mathbf{F}_q)$ be defined like $G(\mathbf{F}_q)$, $\mathfrak{g}(\mathbf{F}_q)$. Let $\mathfrak{A}_{\mathfrak{g}^*}$ (resp. $\mathfrak{A}_{\mathfrak{g}_s^*}$) be defined for $\mathfrak{g}^*$ (resp. $\mathfrak{g}_s^*$) as in the introduction. For a finite set S, we denote its cardinality by $|S|$.

By similar argument as in subsection 1.6.6, $|\mathfrak{A}_{\mathfrak{g}^*}|$ (resp. $|\mathfrak{A}_{\mathfrak{g}_s^*}|$) is equal to the number of nilpotent $G(\mathbf{F}_q)$ (resp. $G_s(\mathbf{F}_q)$)-orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ (resp. $\mathfrak{g}_s(\mathbf{F}_q)^*$) (for q large). On the other hand, the number of nilpotent $G(\mathbf{F}_q)$-orbits in $\mathfrak{g}(\mathbf{F}_q)^*$ is equal to the number of nilpotent $G_s(\mathbf{F}_q)$-orbits in $\mathfrak{g}_s(\mathbf{F}_q)^*$. In fact, we have a morphism $G_s \to G$ which is an isomorphism of abstract groups and an obvious bijective morphism $\mathcal{N}' \to \mathcal{N}'_s$ where $\mathcal{N}'$ (resp. $\mathcal{N}'_s$) is the set of nilpotent elements in $\mathfrak{g}^*$ (resp. $\mathfrak{g}_s^*$). Thus the nilpotent orbits in $\mathfrak{g}^*$ and $\mathfrak{g}_s^*$ are in bijection and the corresponding component groups of centralizers are isomorphic. It follows that $|\mathfrak{A}_{\mathfrak{g}^*}| = |\mathfrak{A}_{\mathfrak{g}_s^*}|$.

Corollary 2.5.5. $|\mathfrak{A}_{\mathfrak{g}^*}| = |\mathfrak{A}_{\mathfrak{g}_s^*}| = |\hat{W}|$.

Proof. Assume G is $SO_{2n}(\mathbf{k})$. The assertion follows from the above argument, Proposition 2.4.1, and Corollary 1.6.15. Assume G is $Sp_{2n}(\mathbf{k})$ or $O_{2n+1}(\mathbf{k})$. It follows from Proposition 2.5.4 (ii) that $|\mathfrak{A}_{\mathfrak{g}^*}| = |\mathfrak{A}_{\mathfrak{g}_s^*}| \geq |\hat{W}|$. On the other hand, it is known that $|\hat{W}| = p_2(n)$ (see [L2]). Hence $|\mathfrak{A}_{\mathfrak{g}^*}| = |\mathfrak{A}_{\mathfrak{g}_s^*}| \leq |\hat{W}|$ by Proposition 2.2.15, Proposition 2.3.18 and the above argument. □

Theorem 2.5.6. *The map γ in Proposition 2.5.4* (ii) *is a bijection.*

Corollary 2.5.7. *Proposition 2.2.13, Corollary 2.2.14, Proposition 2.2.15, Proposition 2.3.17 and Proposition 2.3.18 hold with all "at most" removed.*

Proof. For q large enough, this follows from Corollary 2.5.5. Now let q be an arbitrary power of 2. Let $(\mathrm{c}, \mathcal{F})$ be a pair in $\mathfrak{A}_{\mathfrak{g}_s^*}$. Since the Springer correspondence map γ in Proposition 2.5.4 (ii) is bijective (Theorem 2.5.6), there exists $\rho \in \hat{W}$ corresponding to $(\mathrm{c}, \mathcal{F})$ under the map γ. It follows that the pair $(\mathfrak{F}_q^{-1}(\mathrm{c}), \mathfrak{F}_q^{-1}(\mathcal{F}))$ corresponds to $\mathfrak{F}_q^{-1}(\rho) \in \hat{W}$. Since the Frobenius map $\mathfrak{F}_q$ acts trivially on W and γ is injective, it follows that c is stable under $\mathfrak{F}_q$ and $\mathfrak{F}_q^{-1}(\mathcal{F}) \cong \mathcal{F}$. Pick a rational point ξ in c. The G_s-equivariant local systems on c are in 1-1 correspondence with the isomorphism classes of the irreducible representations of $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$. Since $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$ is abelian (see Proposition 2.6.2 and 2.6.7) and the Frobenius map $\mathfrak{F}_q$ acts trivially on the irreducible representations of $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$, $\mathfrak{F}_q$ acts trivially on $Z_{G_s}(\xi)/Z_{G_s}^0(\xi)$. Thus it follows that the number of nilpotent $G_s(\mathbf{F}_q)$-orbits in $\mathfrak{g}_s(\mathbf{F}_q)^*$ is independent of q hence it is equal to $|\mathfrak{A}_{\mathfrak{g}_s^*}| = |\hat{W}|$. □

Remark. *Let G_{ad} be an adjoint algebraic group of type B, C or D over $\boldsymbol{k}$ and $\mathfrak{g}_{ad}$ its Lie algebra. Let $\mathfrak{g}_{ad}^*$ be the dual space of $\mathfrak{g}_{ad}$. In chapter 1, we have constructed a Springer correspondence for $\mathfrak{g}_{ad}$. One can construct a Springer correspondence for $\mathfrak{g}_{ad}^*$ using the result for $\mathfrak{g}_{ad}$ and the Deligne-Fourier transform. We expect the two Springer correspondences coincide (up to sign representation of the Weyl group). We use the approach presented above since this construction is more suitable for computing the explicit Springer correspondence (see chapter 3).*

2.6 Centralizers and component groups

In this section we study some properties of the centralizer $Z_G(\xi)$ for a nilpotent element $\xi \in \mathfrak{g}^*$ and the component group $Z_G(\xi)/Z_G^0(\xi)$.

2.6.1 In this subsection assume $G = Sp(2N)$. Let $V = {}^*W_{\chi(m_1)}(m_1) \oplus {}^*W_{\chi(m_2)}(m_2) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$, $m_1 \geq \cdots \geq m_s$, be a form module corresponding to $\xi \in \mathfrak{g}^*$. Let T_ξ be defined as in subsection 2.2.1. We have $Z_G(\xi) = Z(V) = \{g \in GL(V) | \beta(gv, gw) = \beta(v, w), \alpha_\xi(gv) = \alpha_\xi(g),\ \forall\ v, w \in V\}$.

Proposition 2.6.1. $\dim Z(V) = \sum_{i=1}^{s}((4i-1)m_i - 2\chi(m_i))$.

Proof. We argue by induction on s. The case $s = 1$ can be easily verified. Let $C(V) = \{g \in GL(V) \mid gT_\xi = T_\xi g\}$. Let $V_1 = {}^*W_{\chi(m_1)}(m_1)$ and $V_2 = {}^*W_{\chi(m_2)}(m_2) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$. We consider V_1 as an element in the Grassmannian variety $Gr(V, 2m_1)$ and consider the action of $C(V)$ on $Gr(V, 2m_1)$. Then the orbit of V_1 has dimension $\dim \mathrm{Hom}_A(V_1, V_2) = 4\sum_{i=2}^{s} m_i$. Now we consider the action of $Z(V)$ on $Gr(V, 2m_1)$.

The orbit $Z(V)V_1$ is open dense in $C(V)V_1$ and thus has dimension $4\sum_{i=2}^{s} m_i$. We claim that

(*) the stabilizer of V_1 in $Z(V)$ is the product of $Z(V_1)$ and $Z(V_2)$.

Thus using induction hypothesis and (*) we get $\dim Z(V) = \dim Z(V_1) + \dim Z(V_2) + \dim Z(V)V_1 = 3m_1 - 2\chi(m_1) + \sum_{i=2}^{s}((4i-5)m_i - 2\chi(m_i)) + 4\sum_{i=2}^{s} m_i = \sum_{i=1}^{s}((4i-1)m_i - 2\chi(m_i))$.

Proof of (*): Assume $g : V_1 \oplus V_2 \to V_1 \oplus V_2$ lies in the stabilizer of V_1 in $Z(V)$. Let p_{ij}, $i,j = 1,2$ be the obvious projection composed with g. Then $p_{12} = 0$. We claim that p_{11} is non-singular. It is enough to show that p_{11} is injective. Assume $p_{11}(v_1) = 0$ for some $v_1 \in V_1$. Then we have $\beta(gv_1, gv_1') = \beta(p_{11}v_1, gv_1') = 0 = \beta(v_1, v_1')$ for any $v_1' \in V_1$. Since $\beta|_{V_1}$ is non-degenerate, we get $v_1 = 0$. Now for any $v_2 \in V_2, v_1 \in V_1$, we have $\beta(gv_1, gv_2) = \beta(p_{11}v_1, p_{21}v_2 + p_{22}v_2) = \beta(p_{11}v_1, p_{21}v_2) = \beta(v_1, v_2) = 0$. Since $\beta|_{V_1}$ is non-degenerate and p_{11} is bijective on V_1, we get $p_{21}(v_2) = 0$. Then (*) follows. □

Let $r = \#\{1 \leq i \leq s | \chi(m_i) + \chi(m_{i+1}) < m_i \text{ and } \chi(m_i) > \frac{m_i - 1}{2}\}$.

Proposition 2.6.2. *The component group $Z(V)/Z^0(V)$ is $(\mathbb{Z}/2\mathbb{Z})^r$.*

Proof. Assume q large enough. By the same argument as in Proposition 1.7.1, one shows that $Z(V)/Z^0(V)$ is an abelian group of order 2^r. We show that there is a subgroup $(\mathbb{Z}/2\mathbb{Z})^r \subset Z(V)/Z(V)^0$. Thus $Z(V)/Z(V)^0$ has to be $(\mathbb{Z}/2\mathbb{Z})^r$. Let $1 \leq i_1, \ldots, i_r \leq s$ be such that $\chi(m_{i_j}) > (m_{i_j} - 1)/2$ and $\chi(m_{i_j}) + \chi(m_{i_j+1}) < m_{i_j}$, $j = 1, \ldots, r$. Let $V_j = {}^*W_{\chi(m_{i_{j-1}+1})}(m_{i_{j-1}+1}) \oplus \cdots \oplus {}^*W_{\chi(m_{i_j})}(m_{i_j})$, $j = 1, \ldots, r-1$, where $i_0 = 0$, and $V_r = {}^*W_{\chi(m_{i_{r-1}+1})}(m_{i_{r-1}+1}) \oplus \cdots \oplus {}^*W_{\chi(m_s)}(m_s)$. Then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. We have $Z(V_i)/Z^0(V_i) = \mathbb{Z}/2\mathbb{Z}$, $i = 1, \ldots, r$. Take $g_i \in Z(V_i)$ such that $g_i Z^0(V_i)$ generates $Z(V_i)/Z^0(V_i)$, $i = 1, \ldots, r$. Let $\tilde{g}_i = Id \oplus \cdots \oplus g_i \oplus \cdots \oplus Id$, $i = 1, \ldots, r$. Then we have $\tilde{g}_i \in Z(V)$ and $\tilde{g}_i \notin Z^0(V)$. We also have that the images of $\tilde{g}_{i_1} \cdots \tilde{g}_{i_p}$'s, $1 \leq i_1 < \cdots < i_p \leq r$, $p = 1, \ldots, r$, in $Z(V)/Z^0(V)$ are not equal to each other. Moreover $\tilde{g}_i^2 \in Z^0(V)$. Thus the $\tilde{g}_i Z^0(V)$'s generate a subgroup $(\mathbb{Z}/2\mathbb{Z})^r$ in $Z(V)/Z^0(V)$. □

2.6.2 In this subsection assume $G = O(2N+1)$. Let (V, α, β_ξ) be a form module corresponding to $\xi \in \mathfrak{g}^*$. Let $C(V) = \{g \in GL(V) | \beta(gv, gw) = \beta(v, w), \beta_\xi(gv, gw) = \beta_\xi(v, w), \ \forall\, v, w \in V\}$. We have $Z_G(\xi) = Z(V) = \{g \in C(V) | \alpha(gv) = \alpha(v), \ \forall\, v \in V\}$.

Lemma 2.6.3. *$|Z(V_{2m+1})(\boldsymbol{F}_q)| = q^m$ and $|C(V_{2m+1})(\boldsymbol{F}_q)| = q^{2m+1}$.*

Proof. Let $V_{2m+1} = \text{span}\{v_0, \cdots, v_m, u_0, \cdots, u_{m-1}\}$, where v_i, u_i are chosen as in Lemma 2.3.5 and Lemma 2.3.6. Let $g \in C(V_{2m+1})$. Then $g : V_{2m+1} \to V_{2m+1}$ satisfies $\beta(gv, gw) = \beta(v, w)$ and $\beta_\xi(gv, gw) = \beta_\xi(v, w)$ for all $v, w \in V_{2m+1}$. One easily shows that $gv_i = av_i$, $i = 0, \ldots, m$ and $gu_i = u_i/a + \sum_{j=0}^m a_{ij} v_j$, $i = 0, \ldots, m-1$, where $a_{ij} = a_{ji}$, $a_{i,j+1} = a_{j,i+1}$, $0 \leq i, j \leq m-1$. Hence $|C(V_{2m+1})(\mathbf{F}_q)| = q^{2m+1}$.

Now assume $g \in Z(V_{2m+1})$. Then we have additional conditions $a = 1$ and $a_{im}^2 + a_{ii}/a = 0$, $i = 0, \ldots, m-1$. Hence $|Z(V_{2m+1})(\mathbf{F}_q)| = q^m$. □

Write $V = V_{2m+1} \oplus W$ as in Lemma 2.3.8.

Lemma 2.6.4. *$|C(V)(\boldsymbol{F}_q)| = |C(V_{2m+1})(\boldsymbol{F}_q)| \cdot |C(W)(\boldsymbol{F}_q)| \cdot q^{\dim W}$.*

Proof. Let $g \in C(V)$. Let $p_{11} : V_{2m+1} \to V_{2m+1}, p_{12} : V_{2m+1} \to W$, $p_{21} : W \to V_{2m+1}$ and $p_{22} : W \to W$ be the projections composed with g. Let v_i, u_i be a basis of V_{2m+1} as before. By the same argument as in Lemma 2.3.10, we have $gv_i = av_i$ for some a and $p_{22} \in C(W)$. Moreover, one easily shows that we have $gu_i = \sum_{j=0}^m a_{ij} v_j + u_i/a + T_\xi^i p_{12}(u_0)$, $i = 0, \ldots, m-1, gw = \sum_{i=0}^m a\beta(T_\xi^i p_{12}(u_0), p_{22}w)v_i + p_{22}(w)$, $\forall\, w \in W$. Now note that $p_{12}(u_0)$ can be any vector in W. It is easily verified that the lemma holds. □

Proposition 2.6.5. $\dim Z(V) = \nu_0 + \sum_{i=1}^s \nu_i(4i+1) + \sum_{i=1}^s \mu_i(4i-1)$.

Proof. Let $V = V_{2m+1} \oplus W_{l_1}(m_1) \oplus \cdots \oplus W_{l_s}(m_s) = (V, \alpha, \beta_\xi)$. Let $W = W_{l_1}(m_1) \oplus \cdots \oplus W_{l_s}(m_s)$. We have $\dim C(W) = \sum_{i=1}^s (4i-1)m_i$ and $\dim C(V_{2m+1}) = 2m+1$. By Lemma 2.6.4, $\dim C(V) = \dim C(W) + \dim V_{2m+1} + \dim W = \sum_{i=1}^s (4i-1)m_i + 2m+1 + 2\sum_{i=1}^s m_i$. Consider V_{2m+1} as an element in the Grassmannian variety $Gr(V, 2m+1)$. Let $C(V)V_{2m+1}$ be the orbit of V_{2m+1} under the action of $C(V)$. The stabilizer of V_{2m+1} in $C(V)$ is the product of $C(V_{2m+1})$ and $C(W)$. Hence $\dim C(V)V_{2m+1} = \dim C(V) - \dim C(V_{2m+1}) - \dim C(W) = 2\sum_{i=1}^s m_i$. We have $\dim Z(V)V_{2m+1} =$

$\dim C(V)V_{2m+1}$. Hence $\dim Z(V) = \dim Z(V_{2m+1}) + \dim Z(W) + \dim Z(V)V_{2m+1} = m + \sum_{i=1}^{s}((4i+1)m_i - 2l_i) = \nu_0 + \sum_{i=1}^{s} \nu_i(4i+1) + \sum_{i=1}^{s} \mu_i(4i-1)$. $\square$

Lemma 2.6.6. $|Z(V)(\boldsymbol{F}_q)| = 2^k q^{\dim Z(V)}+$ *lower terms, where* $k = \#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$.

Proof. If $\#\{i \geq 1 | \nu_i < \mu_i \leq \nu_{i-1}\}$=0, the assertion follows from the classification of nilpotent orbits. Assume $1 \leq t \leq s$ is the minimal integer such that $\nu_t < \mu_t \leq \nu_{t-1}$. Let $V_1 = V_{2m+1} \oplus W_1$ where $W_1 = W_{l_1}(m_1) \oplus \cdots \oplus W_{l_{t-1}}(m_{t-1})$ and $W_2 = W_{l_t}(m_t) \oplus \cdots \oplus W_{l_s}(m_s)$. We show that

$$|Z(V)(\mathbf{F}_q)| = |Z(V_1)(\mathbf{F}_q)| \cdot |Z(W_2)(\mathbf{F}_q)| \cdot q^{r_1}, \tag{2.1}$$

where $r_1 = \dim W_2 + \dim \mathrm{Hom}_A(W_1, W_2)$. We consider V_1 as an element in the Grassmannian variety $Gr(V, \dim V_1)$. We have

$$\begin{aligned} |C(V)V_1(\mathbf{F}_q)| &= \frac{|C(V)(\mathbf{F}_q)|}{|C(V_1)(\mathbf{F}_q)| \cdot |C(W_2)(\mathbf{F}_q)|} \qquad (2.2) \\ &= \frac{|C(V_{2m+1})(\mathbf{F}_q)| \cdot |C(W_1 \oplus W_2)(\mathbf{F}_q)| \cdot q^{\dim(W_1+W_2)}}{|C(V_{2m+1})(\mathbf{F}_q)| \cdot |C(W_1)(\mathbf{F}_q)| \cdot q^{\dim(W_1)} \cdot |C(W_2)(\mathbf{F}_q)|} = q^{r_1}. \end{aligned}$$

In fact, let p_{ij}, $i, j = 1, 2, 3$ be the projections of $g \in C(V)$. Assume g is in the stabilizer of V_1 in $C(V)$. Then we have $p_{13} = p_{23} = 0$. It follows from the same argument as in Lemma 2.6.4 that p_{22} is nonsingular and $gv_i = av_i, i = 0, \ldots, m$, $gu_i = \sum_{j=0}^{m} a_{ij}v_j + u_i/a + T_\xi^i p_{12}(u_0)$, $i = 0, \ldots, m-1, gw_1 = \sum_{i=0}^{m} a\beta(T_\xi^i p_{12}(u_0), p_{22}w_1)v_i + p_{22}(w_1)$, $\forall\ w_1 \in W_1, gw_2 = \sum_{i=0}^{m} a\beta(T_\xi^i p_{12}(u_0), p_{22}w_2 + p_{23}w_2)v_i + p_{22}(w_2) + p_{23}(w_2)$, $\forall\ w_2 \in W_2$. Now $\beta(gw_1, gw_2) = \beta(p_{22}(w_1), p_{22}(w_2) + p_{23}(w_2)) = \beta(p_{22}(w_1), p_{22}(w_2)) = 0$, for any $w_1 \in W_1$ and $w_2 \in W_2$. Since p_{22} is nonsingular and $\beta|_{W_1 \times W_1}$ is nondegenerate, we get $p_{22}(w_2) = 0$ for any $w_2 \in W_2$. Thus the stabilizer of V_1 in $C(V)$ is the product of $C(V_1)$ and $C(W_2)$ and (2.2) follows.

We have $C(V)(V_1 \oplus W_2) \cong C(V)(V_1) \oplus C(V)(W_2)$ implies $C(V)(V_1) \cong V_1$ and $C(V)(W_2) \cong W_2$. Thus $|C(V)(V_1)(\mathbf{F}_q)| = |Z(V)V_1(\mathbf{F}_q)| = q^{r_1}$. Since the stabilizer of V_1 in $Z(V)$ is the product of $Z(V_1)$ and $Z(W_2)$, (4.1) follows. Now the lemma follows by induction hypothesis since we have $\dim Z(V) = \dim Z(V_1) + \dim Z(W_2) + r_1$. $\square$

Proposition 2.6.7. *The component group* $Z(V)/Z^0(V)$ *is* $(\mathbb{Z}/2\mathbb{Z})^k$*, where* $k = \#\{i \geq 1|\nu_i < \mu_i \leq \nu_{i-1}\}$.

Proof. Lemma 2.6.6 and the classification of nilpotent orbits in $\mathfrak{g}(\mathbf{F}_q)^*$($q$ large) show that $Z(V)/Z^0(V)$ is an abelian group of order 2^k. It is enough to show that there exists a subgroup $(\mathbb{Z}/2\mathbb{Z})^k \subset Z(V)/Z^0(V)$. Assume $V = V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(m_1) \oplus \cdots \oplus W_{l_s}^{\epsilon_s}(m_s)$. Let $i_1 < i_2 < \cdots < i_k$ be the i's such that $\nu_i < \mu_i \leq \nu_{i-1}$. Let $V_0 = V_{2m+1} \oplus W_{l_1}^{\epsilon_1}(m_1) \oplus \cdots \oplus W_{l_{i_1-1}}^{\epsilon_{i_1-1}}(m_{i_1-1})$ and $W_j = W_{l_{i_j}}^{\epsilon_{i_j}}(m_{i_j}) \oplus \cdots \oplus W_{l_{i_{j+1}-1}}^{\epsilon_{i_{j+1}-1}}(m_{i_{j+1}-1})$, $j = 1, \ldots, k$, where $i_{k+1} = s + 1$. We have $Z(V_0)/Z^0(V_0) = \{1\}$ and $Z(W_j)/Z^0(W_j) = \mathbb{Z}/2\mathbb{Z}$, $j = 1, \ldots, k$. Take $g_j \in Z(W_j)$ such that $g_j Z^0(W_j)$ generates $Z(W_j)/Z^0(W_j)$. Let $\tilde{g}_j = Id \oplus \cdots \oplus g_j \oplus \cdots \oplus Id$, $j = 1, \ldots, k$. Then $\tilde{g}_j \in Z(V)$, $\tilde{g}_j \notin Z^0(V)$, $\tilde{g}_j^2 \in Z^0(V)$ and $\tilde{g}_{j_1}\tilde{g}_{j_2} \cdots \tilde{g}_{j_r} \notin Z^0(V)$ for any $1 \leq j_1 < j_2 < \cdots < j_r \leq s$, $r = 1, \ldots, k$. Thus $\tilde{g}_j Z^0(W_j)$, $j = 1, \ldots, k$ generate a subgroup $(\mathbb{Z}/2\mathbb{Z})^k$. □

Chapter 3

Combinatorics of the Springer Correspondence for Classical Lie Algebras and their Duals

Throughout this chapter, $\mathbf{k}$ denotes an algebraically closed field of characteristic 2.

3.1 Introduction

Assume G is a connected algebraic group of type B, C or D defined over an algebraically closed field of characteristic p, $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$. When p is large, Shoji [Sh1] describes an algorithm to compute the Springer correspondence for $\mathfrak{g}$ which does not provide a close formula. A combinatorial description of the generalized correspondence for G is given by Lusztig [L3] for $p \neq 2$ and by Lusztig, Spaltenstein [LS2] for $p = 2$. Spaltenstein [Spa1] describes a part of the Springer correspondence for $\mathfrak{g}$ (when $p = 2$) under the assumption that the theory of Springer representation is still valid in this case. We describe the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ constructed in chapter 1 and chapter 2 (when $p = 2$) using similar combinatorics that appears in [L3, LS2]. It is very nice that this combinatorics gives a unified description for (generalized) Springer correspondence of classical groups in all cases, namely, in G, $\mathfrak{g}$ and $\mathfrak{g}^*$ in all characteristic. Moreover, it gives rise to close formulas for computing the correspondence.

3.2 Recollections and outline

3.2.1 For a finite group H, we denote $H^\wedge$ the set of isomorphism classes of irreducible representations of H.

For $n \geq 1$, let $\mathbf{W}_n$ be a Weyl group of type B_n (or C_n). The set $\mathbf{W}_n^\wedge$ is parametrized by ordered pairs of partitions (μ, ν) with $\sum \mu_i + \sum \nu_i = n$. We use the convention that the trivial representation corresponds to (μ, ν) with $\mu = (n)$ and the sign representation corresponds to (μ, ν) with $\nu = (1^n)$. Moreover, $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$, $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq 0)$.

For $n \geq 2$, let $\mathbf{W}'_n \subset \mathbf{W}_n$ be a Weyl group of type D_n. Let $\mathbf{W}'_0 = \mathbf{W}'_1 = \{1\}$. Let $\mathbf{W}_n^{\wedge\prime}$ be the quotient of $(\mathbf{W}'_n)^\wedge$ by the natural action of $\mathbf{W}_n/\mathbf{W}'_n$. The parametrization of $\mathbf{W}_n$ by ordered pairs of partitions induces a parametrization of $\mathbf{W}_n^{\wedge\prime}$ by unordered pairs of partitions $\{\mu, \nu\}$. Moreover, $\{\mu, \nu\}$ corresponds to one (resp. two) element(s) of $(\mathbf{W}'_n)^\wedge$ if and only if $\mu \neq \nu$ (resp. $\mu = \nu$). We say that $\{\mu, \nu\}$ and the corresponding elements of $\mathbf{W}_n^{\wedge\prime}$ and $(\mathbf{W}'_n)^\wedge$ are non-degenerate (resp. degenerate).

3.2.2 Recall that $\mathfrak{A}_\mathfrak{g}$ (resp. $\mathfrak{A}_{\mathfrak{g}^*}$) denotes the set of all pairs $(\mathrm{c}, \mathcal{F})$ with c a nilpotent G-orbit in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) and $\mathcal{F}$ an irreducible G-equivariant local system on c (up to isomorphism). The set $\mathfrak{A}_\mathfrak{g}$ (resp. $\mathfrak{A}_{\mathfrak{g}^*}$) is the same as the set of all pairs (x, ϕ) (resp. (ξ, ϕ)) with $x \in \mathfrak{g}$ (resp. $\xi \in \mathfrak{g}^*$) nilpotent (up to G-action) and $\phi \in A_G(x)^\wedge$ (resp. $\phi \in A_G(\xi)^\wedge$), where $A_G(x) = Z_G(x)/Z_G^0(x)$, $A_G(\xi) = Z_G(\xi)/Z_G^0(\xi)$, $Z_G(x) = \{g \in G | \mathrm{Ad}(g)x = x\}$ and $Z_G(\xi) = \{g \in G | g.\xi = \xi\}$.

We denote $\mathcal{N}_\mathfrak{g}$ (resp. $\mathcal{N}_{\mathfrak{g}^*}$) the variety of nilpotent elements in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$).

3.2.3 Assuming G is adjoint (resp. simply connected), in chapter 1 (resp. chapter 2) we have constructed a Springer correspondence for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$), which is a bijective map from $\mathfrak{A}_\mathfrak{g}$ (resp. $\mathfrak{A}_{\mathfrak{g}^*}$) to $\mathbf{W}_G^\wedge$ (we denote $\mathbf{W}_G$ the Weyl group of G). This induces a Springer correspondence for any $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) with G of the same type. In fact, there are natural bijections between the sets of nilpotent orbits in two Lie algebras (resp. duals of the Lie algebras) of groups of the same type, and the corresponding component groups of centralizers are isomorphic. Hence the sets $\mathfrak{A}_{\mathfrak{g}}$ (resp. $\mathfrak{A}_{\mathfrak{g}^*}$) are naturally identified.

3.2.4 Assume $G = Sp(2n)$ or $G = O(2n+1)$. The Springer correspondence for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is a bijective map $\gamma_{\mathfrak{g}} : \mathfrak{A}_{\mathfrak{g}} \xrightarrow{\sim} \mathbf{W}_n^\wedge$ (resp. $\gamma_{\mathfrak{g}^*} : \mathfrak{A}_{\mathfrak{g}^*} \xrightarrow{\sim} \mathbf{W}_n^\wedge$).

Assume $G = SO(2n)$. Let $\tilde{G} = O(2n)$. The group $\tilde{G}/G$ acts on $\mathfrak{A}_{\mathfrak{g}}$ and on the set of all nilpotent G-orbits in $\mathfrak{g}$. An element in $\mathfrak{A}_{\mathfrak{g}}$ or a nilpotent orbit in $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if it is fixed (resp. not fixed) by this action. Then $(x,\phi) \in \mathfrak{A}_{\mathfrak{g}}$ is degenerate if and only if x is degenerate, in this case $A_G(x) = 1$ and thus $\phi = 1$. Let $\tilde{\mathfrak{A}}_{\mathfrak{g}}$ be the quotient of $\mathfrak{A}_{\mathfrak{g}}$ by $\tilde{G}/G$. The Springer correspondence for $\mathfrak{g}$ (or $\mathfrak{g}^*$) is a bijective map $\gamma_{\mathfrak{g}} : \mathfrak{A}_{\mathfrak{g}} \cong \mathfrak{A}_{\mathfrak{g}^*} \xrightarrow{\sim} (\mathbf{W}_n')^\wedge$, which induces a bijection $\tilde{\gamma}_{\mathfrak{g}} : \tilde{\mathfrak{A}}_{\mathfrak{g}} \xrightarrow{\sim} \mathbf{W}_n^{\wedge\prime}$.

3.2.5 For a Borel subgroup B of G, we write $B = TU$ a Levi decomposition of B and denote $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{n}$ the Lie algebra of B, T and U respectively. Recall that $\mathfrak{n}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{b}) = 0\}$ and $\mathfrak{b}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}) = 0\}$.

For a parabolic subgroup P of G, we denote U_P the unipotent radical of P, $\mathfrak{p}$ and $\mathfrak{n}_P$ the Lie algebra of P and U_P respectively. For a Levi subgroup L of P, we denote $\mathfrak{l}$ the Lie algebra of L. Define $\mathfrak{p}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}_P) = 0\}$, $\mathfrak{n}_P^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{l} \oplus \mathfrak{n}_P) = 0\}$ and $\mathfrak{l}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}_P \oplus \mathfrak{n}_P^-) = 0\}$ where $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}_P \oplus \mathfrak{n}_P^-$. We have $\mathfrak{p}^* = \mathfrak{l}^* \oplus \mathfrak{n}_P^*$.

3.2.6 Let P be a parabolic subgroup of G with a Levi decomposition $P = LU_P$ such that the semisimple rank of L is 1 less than that of G. Let $x \in \mathcal{N}_{\mathfrak{g}}$ and $x' \in \mathcal{N}_{\mathfrak{l}}$. Consider the variety

$$Y_{x,x'} = \{g \in G | \mathrm{Ad}(g^{-1})(x) \in x' + \mathfrak{n}_P\}.$$

Let $d_{x,x'} = (\dim Z_G(x) + \dim Z_L(x'))/2 + \dim \mathfrak{n}_P$. We have $\dim Y_{x,x'} \leq d_{x,x'}$ (see Proposition 3.3.1 (ii)). Let $S_{x,x'}$ be the set of all irreducible components of $Y_{x,x'}$ of dimension $d_{x,x'}$. The action of $Z_G(x) \times Z_L(x')U_P$ on $Y_{x,x'}$ by $(g_0, g_1).g = g_0 g g_1^{-1}$ induces an action of $A_G(x) \times A_L(x')$ on $S_{x,x'}$. Denote $\varepsilon_{x,x'}$ the corresponding $A_G(x) \times A_L(x')$-representation. We prove in section 3.3 the following restriction formula

$$\textbf{(R)} \qquad \langle \phi \otimes \phi', \varepsilon_{x,x'} \rangle = \langle \mathrm{Res}_{\mathbf{W}_L}^{\mathbf{W}_G} \rho_{x,\phi}^G, \rho_{x',\phi'}^L \rangle_{\mathbf{W}_L},$$

where $\phi \in A_G(x)^\wedge, \phi' \in A_L(x')^\wedge$, $\rho_{x,\phi}^G = \gamma_{\mathfrak{g}}((x,\phi)) \in \mathbf{W}_G^\wedge$, $\rho_{x',\phi'}^L = \gamma_{\mathfrak{l}}((x',\phi')) \in \mathbf{W}_L^\wedge$.

To prove the restriction formula **(R)** it suffices to assume that G is adjoint (see

3.2.3). The proof is essentially the same as that of the restriction formula in unipotent case [L3].

3.2.7 Let P, L be as in 3.2.6. Let $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ and $\xi' \in \mathcal{N}_{\mathfrak{l}^*}$. We define $Y_{\xi,\xi'}, S_{\xi,\xi'}, \varepsilon_{\xi,\xi'}$ as $Y_{x,x'}, S_{x,x'}, \varepsilon_{x,x'}$ replacing x, x',$\mathfrak{p}, \mathfrak{n}_p$, adjoint G-action on $\mathfrak{g}$ by ξ, ξ',$\mathfrak{p}^*, \mathfrak{n}_P^*$, coadjoint G-action on $\mathfrak{g}^*$ respectively. We have the following restriction formula

$$(\mathbf{R}') \qquad \langle \phi \otimes \phi', \varepsilon_{\xi,\xi'} \rangle = \langle \mathrm{Res}_{\mathbf{W}_L}^{\mathbf{W}_G} \rho_{\xi,\phi}^G, \rho_{\xi',\phi'}^L \rangle_{\mathbf{W}_L},$$

where $\phi \in A_G(\xi)^\wedge, \phi' \in A_L(\xi')^\wedge$, $\rho_{\xi,\phi}^G = \gamma_{\mathfrak{g}^*}((\xi,\phi)) \in \mathbf{W}_G^\wedge, \rho_{\xi',\phi'}^L = \gamma_{\mathfrak{l}^*}((\xi',\phi')) \in \mathbf{W}_L^\wedge$. The proof of $(\mathbf{R}')$ is entirely similar to that of $(\mathbf{R})$ and is omitted.

3.2.8 Assume $G = SO(V,Q)$ (see 1.2.1). Let $\tilde{G} = O(V,Q)$. Note that $\tilde{G} \neq G$ if and only if $\dim(V)$ is even. Let $\Sigma \subset V$ be a line such that $Q|_\Sigma = 0$. Let $\tilde{P}$ be the stabilizer of Σ in $\tilde{G}$ and P the identity component of $\tilde{P}$. Then P is a parabolic subgroup of G. Let L be a Levi subgroup of P and $\tilde{L} = N_{\tilde{P}}(L)$. Then $\tilde{P} = \tilde{L}U_P$ and $L = \tilde{L}^0$. Fix a Borel subgroup $B \subset P$ and let $\tilde{B} = N_{\tilde{G}}(B)$. Denote $\tilde{\mathcal{B}} = \{g\tilde{B}g^{-1} | g \in \tilde{G}\}$, $\tilde{\mathcal{P}} = \{g\tilde{P}g^{-1} | g \in \tilde{G}\}$.

Let $x \in \mathcal{N}_{\mathfrak{g}}$. Define $\tilde{\mathcal{B}}_x = \{g\tilde{B}g^{-1} \in \tilde{\mathcal{B}} | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{b}\}$ and $\tilde{\mathcal{P}}_x = \{g\tilde{P}g^{-1} \in \tilde{\mathcal{P}}_x | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{p}\}$. The natural morphism $\varrho_x : \tilde{\mathcal{B}}_x \to \tilde{\mathcal{P}}_x$, $g\tilde{B}g^{-1} \to g\tilde{P}g^{-1}$ is $Z_{\tilde{G}}(x)$ equivariant. We have a well defined map

$$f_x : \tilde{\mathcal{P}}_x \to \mathcal{CN}(\mathfrak{p}/\mathfrak{n}_P),\ g\tilde{P}g^{-1} \mapsto \text{ orbit of } Ad(g^{-1})x + \mathfrak{n}_P,$$

where $\mathcal{CN}(\mathfrak{p}/\mathfrak{n}_P)$ is the set of nilpotent $\tilde{P}/U_P$-orbits in $\mathfrak{p}/\mathfrak{n}_P$. Let $\mathrm{c}' \in f_x(\tilde{\mathcal{P}}_x)$ be a nilpotent orbit. Define $\mathbf{Y} = f_x^{-1}(\mathrm{c}')$ and $\mathbf{X} = \varrho_x^{-1}(\mathbf{Y})$.

We can assume $\tilde{P} \in \mathbf{Y}$. We identify $\tilde{L}$ with $\tilde{P}/U_P$, $\mathfrak{l}$ with $\mathfrak{p}/\mathfrak{n}_P$. Let x' be the image of x in $\mathfrak{l}$ and let $\tilde{A}'(x') = A_{\tilde{L}}(x') = Z_{\tilde{L}}(x')/Z_{\tilde{L}}^0(x'), H = Z_{\tilde{G}}(x) \cap \tilde{P} = Z_{\tilde{P}}(x), K = Z_{\tilde{G}}^0(x) \cap \tilde{P}$. The natural morphisms $H \to Z_{\tilde{G}}(x)$, $H \to Z_{\tilde{L}}(x')$ and $K \to Z_{\tilde{L}}(x')$ induce morphisms $H \to A_{\tilde{G}}(x)$, $H \to A_{\tilde{L}}(x')$ and $K \to A_{\tilde{L}}(x')$. Let $\tilde{A}_P$ be the image of H in $A_{\tilde{G}}(x)$ and $\tilde{A}'_P$ the image of K in $A_{\tilde{L}}(x')$. Then we have a natural morphism $\tilde{A}_P \to \tilde{A}'(x')/\tilde{A}'_P$.

If $G = \tilde{G}$, then we omit the tildes from the notations, for example, $A_P = \tilde{A}_P$ and etc.

3.2.9 We preserve the notations in 3.2.8. Let $\tilde{Y}_{x,x'}$ and $\tilde{S}_{x,x'}$ be defined as in 3.2.6 replacing G by $\tilde{G}$ and L by $\tilde{L}$. Note that $\tilde{S}_{x,x'} \neq \emptyset$ if and only if $\dim \mathbf{X} = \dim \tilde{\mathcal{B}}_x$, where $\mathbf{X}$ is defined as in 3.2.8 with c' the orbit of x'. If $\tilde{S}_{x,x'} \neq \emptyset$, then $\tilde{Y}_{x,x'}$ is a single orbit under the action of $Z_{\tilde{G}}(x) \times Z_{\tilde{L}}(x')U_P$ (see Proposition 3.4.2). It follows that $\tilde{S}_{x,x'}$ is a single $A_{\tilde{G}}(x) \times A_{\tilde{L}}(x')$-orbit. Hence $\tilde{S}_{x,x'} = A_{\tilde{G}}(x) \times A_{\tilde{L}}(x')/\tilde{H}_{x,x'}$ for some subgroup $\tilde{H}_{x,x'} \subset A_{\tilde{G}}(x) \times A_{\tilde{L}}(x')$. The subgroup $\tilde{H}_{x,x'}$ is described as follows.

If A, B are groups, a subgroup C of $A \times B$ is characterized by the triple (A_0, B_0, h) where $A_0 = \mathrm{pr}_1(C)$, $B_0 = B \cap C$ and $h : A_0 \to N_B(B_0)/B_0$ is defined by $a \mapsto bB_0$ if $(a, b) \in C$. Then $\tilde{H}_{x,x'}$ is characterized by the triple $(\tilde{A}_P, \tilde{A}'_P, h)$, where h is the natural morphism $\tilde{A}_P \to \tilde{A}'(x')/\tilde{A}'_P$ described in 3.2.8.

Assume $G = SO(2n)$. The subset $S_{x,x'}$ of $\tilde{S}_{x,x'}$ is the image in $\tilde{S}_{x,x'}$ of the subgroup of $A_{\tilde{G}}(x) \times A_{\tilde{L}}(x')$ consisting of the elements that can be written as a product of even number of generators. This is also the image of $A_G(x) \times A_L(x')$.

3.2.10 Assume $G = Sp(V)$ or $O(V)$. The definitions in 3.2.8 apply to $\mathfrak{g}^*$ (if $G = Sp(V)$, Σ is an arbitrary line). Let $\varrho_\xi, f_\xi, A_P, A'_P$ etc. be defined in this way. Then $Y_{\xi,\xi'}$, $S_{\xi,\xi'}$ are described in the same way as $Y_{x,x'}$, $S_{x,x'}$ in 3.2.9.

3.2.11 The correspondence for symplectic Lie algebras is determined by Spaltenstein [Spa1] since in this case the centralizer of a nilpotent element is connected and $\mathfrak{A}_{\mathfrak{g}} = \{(c, \bar{\mathbb{Q}}_l)\}$. We rewrite his results in section 3.8 using different combinatorics. The Springer correspondence for orthogonal Lie algebras is described in section 3.9. The proof will essentially be as in [L3], which is based on the restriction formula (**R**) and the following observation of Shoji: if $n \geq 3$, an irreducible character of $\mathbf{W}_n$ (resp. a nondegenerate irreducible character of $\mathbf{W}'_n$) is completely determined by its restriction to $\mathbf{W}_{n-1}$ (resp. $\mathbf{W}'_{n-1}$). We need to study the representations $\varepsilon_{x,x'}$, which require a description of the groups $\tilde{A}_P$ and $\tilde{A}'_P$. Extending some methods in [Spa2], we describe these groups for orthogonal Lie algebras, duals of symplectic Lie algebras and duals of odd orthogonal Lie algebras in section 3.4, 3.5 and 3.6 respectively.

3.2.12 The Springer correspondence for the duals of symplectic Lie algebras and orthogonal Lie algebras is described in section 3.10. The proofs are very similar to the Lie algebra case. We omit much detail.

3.3 Restriction formula

Assume G is adjoint. Fix a Borel subgroup B of G and a maximal torus $T \subset B$. Recall that $\mathcal{B}$ denotes the variety of Borel subgroups of G. A proof of the restriction formula in unipotent case is given in [L3]. The proof for nilpotent case is essentially the same. For completeness, we include the proof here.

3.3.1 We prove first a dimension formula following [L3]. Let $\mathcal{P}$ be a G-conjugacy class of parabolic subgroups of G. For $P \in \mathcal{P}$, let $\bar{P} = P/U_P$, $\bar{\mathfrak{p}} = \mathfrak{p}/\mathfrak{n}_P$ and $\pi_{\mathfrak{p}} : \mathfrak{p} \to \bar{\mathfrak{p}}$ the natural projection. Let c be a nilpotent G-orbit in $\mathfrak{g}$. Assume for each $P \in \mathcal{P}$, given a nilpotent $\bar{P}$-orbit $\mathrm{c}_{\bar{\mathfrak{p}}} \subset \bar{\mathfrak{p}}$ with the following property: for any $P_1, P_2 \in \mathcal{P}$ and any $g \in G$ such that $P_2 = gP_1g^{-1}$, we have $\pi_{\mathfrak{p}_2}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}_2}) = \mathrm{Ad}(g)(\pi_{\mathfrak{p}_1}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}_1}))$. Let

$$Z' = \{(x, P_1, P_2) \in \mathfrak{g} \times \mathcal{P} \times \mathcal{P} | x \in \pi_{\mathfrak{p}_1}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}_1}) \cap \pi_{\mathfrak{p}_2}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}_2})\}.$$

We have a partition $Z' = \cup_{\mathcal{O}} Z'_{\mathcal{O}}$, where $\mathcal{O}$ runs through the G-orbits on $\mathcal{P} \times \mathcal{P}$ and $Z'_{\mathcal{O}} = \{(x, P_1, P_2) \in Z' | (P_1, P_2) \in \mathcal{O}\}$.

We denote ν_G the number of positive roots in G and set $\bar{\nu} = \nu_{\bar{P}}$ $(P \in \mathcal{P})$. Let $c = \dim \mathrm{c}$ and $\bar{c} = \dim \mathrm{c}_{\bar{\mathfrak{p}}}$ for $P \in \mathcal{P}$.

Proposition 3.3.1. (i) *Given* $P \in \mathcal{P}$ *and* $\bar{x} \in \mathrm{c}_{\bar{\mathfrak{p}}}$, *we have* $\dim(\mathrm{c} \cap \pi_{\mathfrak{p}}^{-1}(\bar{x})) \leq \frac{1}{2}(c - \bar{c})$.

(ii) *Given* $x \in \mathrm{c}$, *we have* $\dim\{P \in \mathcal{P} | x \in \pi_{\mathfrak{p}}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}})\} \leq (\nu_G - \frac{c}{2}) - (\bar{\nu} - \frac{\bar{c}}{2})$.

(iii) *If* $d_0 = 2\nu_G - 2\bar{\nu} + \bar{c}$, *then* $\dim Z'_{\mathcal{O}} \leq d_0$ *for all* $\mathcal{O}$. *Hence* $\dim Z' \leq d_0$.

Proof. We prove the proposition by induction on the dimension of the group. Assume $\mathcal{P} = \{G\}$, the proposition is clear. Thus we can assume that $\mathcal{P}$ is a class of proper parabolic subgroups of G and that the proposition holds when G is replaced by a group of strictly smaller dimension.

Consider the map $Z'_{\mathcal{O}} \to \mathcal{O}$, $(x, P_1, P_2) \mapsto (P_1, P_2)$. We see that proving (iii) for $Z'_{\mathcal{O}}$ is equivalent to proving that for a fixed $(P', P'') \in \mathcal{O}$, we have

$$\dim \pi_{\mathfrak{p}'}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}'}) \cap \pi_{\mathfrak{p}''}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}''}) \leq 2\nu_G - 2\bar{\nu} + \bar{c} - \dim \mathcal{O}. \tag{3.1}$$

Choose Levi subgroups L' of P' and L'' of P'' such that L' and L'' contain a common maximal torus. An element in $\mathfrak{p}' \cap \mathfrak{p}''$ can be written both in the form $x' + n'$ $(x' \in$

$\mathfrak{l}', n' \in \mathfrak{n}_{P'}$) and in the form $x''+n''$ ($x'' \in \mathfrak{l}'', n'' \in \mathfrak{n}_{P''}$). It is easy to see that there are unique elements $z \in \mathfrak{l}' \cap \mathfrak{l}'', u'' \in \mathfrak{l}' \cap \mathfrak{n}_{P''}, u' \in \mathfrak{l}'' \cap \mathfrak{n}_{P'}$, such that $x' = z+u'', x'' = z+u'$. Hence (3.1) is equivalent to

$$\dim\{(n', n'', u'', u', z) \in \mathfrak{n}_{P'} \times \mathfrak{n}_{P''} \times (\mathfrak{l}' \cap \mathfrak{n}_{P''}) \times (\mathfrak{l}'' \cap \mathfrak{n}_{P'}) \times (\mathfrak{l}' \cap \mathfrak{l}'') |\ u'' + n' = u' + n'', \\ z + u'' \in \mathrm{c}_{\bar{\mathfrak{p}}'}, z + u' \in \mathrm{c}_{\bar{\mathfrak{p}}''}\} \leq 2\nu_G - 2\bar{\nu} + \bar{c} - \dim \mathcal{O}. \tag{3.2}$$

(We identify $\mathfrak{l}' = \bar{\mathfrak{p}}', \mathfrak{l}'' = \bar{\mathfrak{p}}''$, and then view $\mathrm{c}_{\bar{\mathfrak{p}}'} \subset \mathfrak{l}', \mathrm{c}_{\bar{\mathfrak{p}}''} \subset \mathfrak{l}''$.) When $(u'', u') \in (\mathfrak{l}' \cap \mathfrak{n}_{P''}) \times (\mathfrak{l}'' \cap \mathfrak{n}_{P'})$ is fixed, the variety $\{(n', n'') \in \mathfrak{n}_{P'} \times \mathfrak{n}_{P''} | u'' + n' = u' + n''\}$ is isomorphic to $\mathfrak{n}_{P'} \cap \mathfrak{n}_{P''}$. Since $\dim(\mathfrak{n}_{P'} \cap \mathfrak{n}_{P''}) = 2\nu_G - 2\bar{\nu} - \dim \mathcal{O}$, we see that (3.2) is equivalent to

$$\dim\{(u'', u', z) \in (\mathfrak{l}' \cap \mathfrak{n}_{P''}) \times (\mathfrak{l}'' \cap \mathfrak{n}_{P'}) \times (\mathfrak{l}' \cap \mathfrak{l}'') |\ z + u'' \in \mathrm{c}_{\bar{\mathfrak{p}}'}, z + u' \in \mathrm{c}_{\bar{\mathfrak{p}}''}\} \leq \bar{c}. \tag{3.3}$$

By the finiteness of the number of nilpotent orbits, the projection pr_3 of the variety in (3.3) on the z-coordinate is a union of finitely many orbits $\hat{\mathrm{c}}_1 \cup \hat{\mathrm{c}}_2 \cup \cdots \cup \hat{\mathrm{c}}_m$ in $\mathfrak{l}' \cap \mathfrak{l}''$ (note that z is nilpotent). The inverse image under pr_3 of a point $z \in \hat{\mathrm{c}}_i$ is a product of two varieties of the type considered in (i) for a smaller group (G replaced by L' or L''), thus by the induction hypothesis it has dimension $\leq \frac{1}{2}(\bar{c} - \dim \hat{\mathrm{c}}_i) + \frac{1}{2}(\bar{c} - \dim \hat{\mathrm{c}}_i)$. Hence $\dim \mathrm{pr}_3^{-1}(\hat{\mathrm{c}}_i) \leq \bar{c}$, $\forall\ 1 \leq i \leq m$. Then (3.3) holds. This proves (iii).

We show that (ii) is a consequence of (iii). Let $Z'(\mathrm{c}) = \{(x, P_1, P_2) \in Z' | x \in \mathrm{c}\} \subset Z'$. If $Z'(\mathrm{c})$ is empty then the variety in (ii) is empty and (ii) follows. Hence we may assume that $Z'(\mathrm{c})$ is non-empty. From (iii), we have $\dim Z'(\mathrm{c}) \leq d_0$. Consider the map $Z'(\mathrm{c}) \to \mathrm{c}$, $(x, P_1, P_2) \mapsto x$. Each fiber of this map is a product of two copies of the variety in (ii). It follows that the variety in (ii) has dimension equal to $\frac{1}{2}(\dim Z'(\mathrm{c}) - \dim \mathrm{c}) \leq \frac{1}{2}(d_0 - c) = \nu_G - \bar{\nu} + \frac{\bar{c}}{2} - \frac{c}{2}$. Then (ii) follows.

We show that (i) is a consequence of (ii). Consider the variety $\{(x, P) \in \mathrm{c} \times \mathcal{P} | x \in \pi_{\mathfrak{p}}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}})\}$. By projecting it to the x-coordinate and using (ii), we see that it has dimension $\leq \nu_G - \bar{\nu} + \frac{\bar{c}}{2} + \frac{c}{2}$. If we project it to the P-coordinate, each fiber is isomorphic to the variety $\mathrm{c} \cap \pi_{\mathfrak{p}}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}})$. Hence $\dim(\mathrm{c} \cap \pi_{\mathfrak{p}}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}})) \leq \nu_G - \bar{\nu} + \frac{\bar{c}}{2} + \frac{c}{2} - \dim \mathcal{P} = \frac{c+\bar{c}}{2}$. Now $\pi_{\mathfrak{p}}$ maps $\mathrm{c} \cap \pi_{\mathfrak{p}}^{-1}(\mathrm{c}_{\bar{\mathfrak{p}}})$ onto $\mathrm{c}_{\bar{\mathfrak{p}}}$ and each fiber is the variety in (i). Hence the variety in (i) has dimension $\leq \frac{c+\bar{c}}{2} - \bar{c} = \frac{c-\bar{c}}{2}$. The proposition is proved. □

3.3.2 Let $P \supset B$ be a parabolic subgroup of G with a Levi subgroup L such that $T \subset L$. Let $\mathbf{W}_L = N_L(T)/T$. Then $\bar{\mathbb{Q}}_l[\mathbf{W}_L]$ is in a natural way a subalgebra of $\bar{\mathbb{Q}}_l[\mathbf{W}_G]$.

Recall that we have the map (see 1.6.3)

$$\pi : \widetilde{Y} = \{(x, gT) \in Y \times G/T | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{t}_0\} \to Y,\ (x, gT) \mapsto x,$$

where Y, $\mathfrak{t}_0$ is the set of regular semisimple elements in $\mathfrak{g}$, $\mathfrak{t}$ respectively. Let

$$Y_L = \bigcup_{g \in L} \mathrm{Ad}(g)\mathfrak{t}_0,\ \widetilde{Y}_1 = \{(x, gL) \in \mathfrak{g} \times G/L | \mathrm{Ad}(g^{-1})(x) \in Y_L\}.$$

Then π factors as

$$\widetilde{Y} \xrightarrow{\pi'} \widetilde{Y}_1 \xrightarrow{\pi''} Y,$$

where π' is $(x, gT) \mapsto (x, gL)$ and π'' is $(x, gL) \mapsto x$. The map $\pi' : \widetilde{Y} \to \widetilde{Y}_1$ is a principal bundle with group $\mathbf{W}_L$. It follows that $\mathrm{End}(\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}}) = \bar{\mathbb{Q}}_l[\mathbf{W}_L]$ and that we have a canonical decomposition

$$\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}} = \bigoplus\nolimits_{\rho' \in \mathbf{W}_L^\wedge} (\rho' \otimes (\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_{\rho'}),$$

where $(\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_{\rho'} = \mathrm{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho', \pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})$ is an irreducible local system on $\widetilde{Y}_1$. We have

$$\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}} = \pi''_! (\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}}) = \bigoplus\nolimits_{\rho' \in \mathbf{W}_L^\wedge} (\rho' \otimes \pi''_! ((\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_{\rho'})),$$

hence $\pi''_!((\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_{\rho'}) = \mathrm{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho', \pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}}) = \mathrm{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho', \bigoplus_{\rho \in \mathbf{W}_G^\wedge} (\rho \otimes (\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_\rho))$. (Recall that $\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}} = \bigoplus_{\rho \in \mathbf{W}_G^\wedge} (\rho \otimes (\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_\rho)$, where $(\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_\rho = \mathrm{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_G]}(\rho, \pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}})$ is an irreducible local system on Y.) We see that for any $\rho' \in \mathbf{W}_L^\wedge$,

$$\pi''_!((\pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_{\rho'}) = \bigoplus_{\rho \in \mathbf{W}_G^\wedge} ((\pi_! \bar{\mathbb{Q}}_{l\widetilde{Y}})_\rho \otimes \mathrm{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho', \rho)). \tag{3.4}$$

3.3.3 Recall that we have the map (see 1.6.3)

$$\varphi : X = \{(x, gB) \in \mathfrak{g} \times G/B | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{b}\} \to \mathfrak{g}, (x, gB) \mapsto x.$$

Let

$$X_1 = \{(x, gP) \in \mathfrak{g} \times G/P | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{p}\}.$$

Then φ factors as

$$X \xrightarrow{\varphi'} X_1 \xrightarrow{\varphi''} \mathfrak{g}$$

where φ' is $(x, gB) \mapsto (x, gP)$ and φ'' is $(x, gP) \mapsto x$. The maps φ', φ'' are proper and surjective. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{Y} & \xrightarrow{\pi'} & \widetilde{Y}_1 & \xrightarrow{\pi''} & Y \\ {\scriptstyle j_0}\downarrow & & {\scriptstyle j_1}\downarrow & & {\scriptstyle j_2}\downarrow \\ X & \xrightarrow{\varphi'} & X_1 & \xrightarrow{\varphi''} & \mathfrak{g} \end{array}$$

where j_2 is $x \mapsto x$, j_0 is $(x, gT) \mapsto (x, gB)$ (an isomorphism of $\widetilde{Y}$ with the open subset $\varphi^{-1}(Y)$ of X) and j_1 is $(x, gL) \mapsto (x, gP)$ (an isomorphism onto the open subset $\varphi''^{-1}(Y)$ of X_1). Note also that $\widetilde{Y}_1$ is smooth (since $\widetilde{Y}$ is smooth). We identify $\widetilde{Y}, \widetilde{Y}_1$ with open subsets of X, X_1 via the maps j_0, j_1 respectively. Let

$$X_L = \{(x, g(B \cap L)) \in \mathfrak{l} \times L/(B \cap L) | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{b} \cap \mathfrak{l}\},$$
$$X'' = \{(g_1, x, pB) \in G \times \mathfrak{p} \times P/B | \mathrm{Ad}(p^{-1})(x) \in \mathfrak{b}\}.$$

We have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X'' & \xrightarrow{p_2} & X_L \\ {\scriptstyle \varphi'}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \varphi_L}\downarrow \\ X_1 & \xleftarrow{p_3} & G \times \mathfrak{n}_P \times \mathfrak{l} & \xrightarrow{p_4} & \mathfrak{l} \end{array}$$

where p_1 is $(g_1, x, pB) \mapsto (\mathrm{Ad}(g_1)(x), g_1 pB)$, a principal P-bundle; p_2 is $(g_1, l + n, g'B) \mapsto (l, g'(B \cap L))$ with $l \in \mathfrak{l}, n \in \mathfrak{n}_P, g' \in L$, a principal $G \times \mathfrak{n}_P$-bundle; p_3 is $(g_1, n, l) \mapsto (\mathrm{Ad}(g_1)(l + n), g_1 P)$, a principal P-bundle; p_4 is $(g_1, n, l) \mapsto l$, a principal $G \times \mathfrak{n}_P$-bundle; φ_L is $(x, g(B \cap L)) \mapsto x$; ϕ is $(g_1, l + n, g'B) \mapsto (g_1, n, l)$, with $l \in \mathfrak{l}, n \in \mathfrak{n}_P, g' \in L$.

Let π_L be the map $\widetilde{Y}_L = \{(x, gT) \in \mathfrak{l} \times L/T | \mathrm{Ad}(g^{-1})(x) \in \mathfrak{t}_0\} \to Y_L, (x, gL) \mapsto x$. Since p_3, p_4 are principal bundles with connected groups, we have $p_3^* IC(X_1, \pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}}) = p_4^* IC(\mathfrak{l}, \pi_{L!} \bar{\mathbb{Q}}_{l\widetilde{Y}_L})$ (both can be identified with $IC(G \times \mathfrak{n}_P \times \mathfrak{l}, p_4^* \pi_{L!} \bar{\mathbb{Q}}_{l\widetilde{Y}_L})$). From the commutative diagram above it follows that $p_3^* \varphi'_! \bar{\mathbb{Q}}_{lX} = \phi_! p_1^* \bar{\mathbb{Q}}_{lX} = \phi_! p_2^* \bar{\mathbb{Q}}_{lX_L} = p_4^* \varphi_{L!} \bar{\mathbb{Q}}_{lX_L} = p_4^* IC(\mathfrak{l}, \pi_{L!} \bar{\mathbb{Q}}_{l\widetilde{Y}_L})$ (the last equality comes from Proposition 1.6.5 for L instead of G), hence $p_3^* \varphi'_! \bar{\mathbb{Q}}_{lX} = p_3^* IC(X_1, \pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})$. Since p_3 is a principal P-bundle we see that $\varphi'_! \bar{\mathbb{Q}}_{lX} = IC(X_1, \pi'_! \bar{\mathbb{Q}}_{l\widetilde{Y}})$. It follows that $\mathrm{End}(\varphi'_! \bar{\mathbb{Q}}_{lX}) \cong \bar{\mathbb{Q}}_l[\mathbf{W}_L]$ and

$\varphi'_!\bar{\mathbb{Q}}_{lX} = \bigoplus_{\rho'\in\mathbf{W}_L^\wedge}(\rho'\otimes(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})$ where

$$(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'} = IC(X_1,(\pi'_!\bar{\mathbb{Q}}_{l\tilde{Y}})_{\rho'}). \tag{3.5}$$

Next we show that

$$\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) = IC(\mathfrak{g},\pi''_!((\pi'_!\bar{\mathbb{Q}}_{l\tilde{Y}})_{\rho'})), \text{ for any } \rho'\in\mathbf{W}_L^\wedge. \tag{3.6}$$

From (3.5) we see that the restriction of $\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})$ to Y is the local system $\pi''_!(\pi'_!(\bar{\mathbb{Q}}_{l\tilde{Y}})_{\rho'})$. Since φ'' is proper, (3.6) is a consequence of (3.5) and the following assertion:

$$\text{For any } i>0, \dim\operatorname{supp}\mathcal{H}^i(\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})) < \dim\mathfrak{g}-i. \tag{3.7}$$

We have $\operatorname{supp}\mathcal{H}^i(\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}))\subset\operatorname{supp}\mathcal{H}^i(\varphi''_!(\varphi'_!\bar{\mathbb{Q}}_{lX}))=\operatorname{supp}\mathcal{H}^i(\varphi_!\bar{\mathbb{Q}}_{lX})$, thus (3.7) follows from the proof of Proposition 1.6.5. Hence (3.6) is verified. Combining (3.6) with (3.4), we see that for any $\rho'\in\mathbf{W}_L^\wedge$,

$$\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) \cong \bigoplus_{\rho\in\mathbf{W}_G^\wedge}((\varphi_!\bar{\mathbb{Q}}_{lX})_\rho\otimes\operatorname{Hom}_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho',\rho)). \tag{3.8}$$

(Recall that we have $\varphi_!\bar{\mathbb{Q}}_{lX} = \bigoplus_{\rho\in\mathbf{W}_G^\wedge}(\rho\otimes(\varphi_!\bar{\mathbb{Q}}_{lX})_\rho)$, $(\varphi_!\bar{\mathbb{Q}}_{lX})_\rho = IC(\mathfrak{g},(\pi_!\bar{\mathbb{Q}}_{l\tilde{Y}})_\rho)$.)

3.3.4 Let $(\mathrm{c},\mathcal{F})\in\mathfrak{A}_\mathfrak{g}$, $(\mathrm{c}',\mathcal{F}')\in\mathfrak{A}_\mathfrak{l}$ and $\rho=\gamma_\mathfrak{g}((\mathrm{c},\mathcal{F}))\in\mathbf{W}_G^\wedge$, $\rho'=\gamma_\mathfrak{l}((\mathrm{c}',\mathcal{F}'))\in\mathbf{W}_L^\wedge$, where $\gamma_\mathfrak{g}$ and $\gamma_\mathfrak{l}$ are the Springer correspondence maps for $\mathfrak{g}$ and $\mathfrak{l}$ respectively. Let $X_1^\omega=\{(x,gP)\in X_1|x \text{ nilpotent}\}$ and

$$R=\{(x,gP)\in\mathfrak{g}\times(G/P)|\operatorname{Ad}(g^{-1})(x)\in\bar{\mathrm{c}}'+\mathfrak{n}_P\}\subset X_1^\omega.$$

We show that

$$\operatorname{supp}(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}\cap X_1^\omega\subset R. \tag{3.9}$$

Let $(x,gP)\in\operatorname{supp}(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}\cap X_1^\omega$. The isomorphism $p_3^*\varphi'_!\bar{\mathbb{Q}}_{lX}=p_4^*\varphi_{L!}\bar{\mathbb{Q}}_{lX_L}$ is compatible with the action of $\mathbf{W}_L$. Thus $p_3^*(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}=p_4^*(\varphi_{L!}\bar{\mathbb{Q}}_{lX_L})_{\rho'}$ and $p_3^{-1}(\operatorname{supp}(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})=p_4^{-1}(\operatorname{supp}(\varphi_{L!}\bar{\mathbb{Q}}_{lX_L})_{\rho'})$. Hence there exists $(g_1,n,l)\in G\times\mathfrak{n}_P\times\mathfrak{l}$ such that $(x,gP)=(\operatorname{Ad}(g_1)(n+l),g_1P)$ and $l\in\operatorname{supp}(\varphi_{L!}\bar{\mathbb{Q}}_{lX_L})_{\rho'}$. Since x is nilpotent, $n+l$ is nilpotent and thus l is nilpotent. Hence $l \in \bar{c}'$ since by Proposition 1.6.5 (for L instead of G),

$$(\varphi_{L!}\bar{\mathbb{Q}}_{lX_L})_{\rho'}|_{\mathcal{N}_L} \text{ is } IC(\bar{c}', \mathcal{F}')[\dim c' - 2\nu_L](\text{ extend by zero outside } \bar{c}'), \quad (3.10)$$

where $\mathcal{N}_L$ is the nilpotent variety of $\mathfrak{l}$. We have $g = g_1 p$ for some $p \in P$ and $x = \mathrm{Ad}(g_1)(n + l)$, hence $\mathrm{Ad}(g^{-1})(x) = \mathrm{Ad}(p^{-1})(n + l) \in \bar{c}' + \mathfrak{n}_P$ and $(x, gP) \in R$. This proves (3.9).

We have a partition $R = \cup_{\tilde{c}'} R_{\tilde{c}'}$, where $\tilde{c}'$ runs over the nilpotent L-orbits in $\bar{c}'$ and $R_{\tilde{c}'} = \{(x, gP) \in \mathfrak{g} \times (G/P) | \mathrm{Ad}(g^{-1})(x) \in \tilde{c}' + \mathfrak{n}_P\}$. Then $R' = R_{c'}$ is open in R. It is clear that $p_3^{-1}(R) = p_4^{-1}(\bar{c}') = G \times \mathfrak{n}_P \times \bar{c}'$ and $p_3^{-1}(R_{\tilde{c}'}) = p_4^{-1}(\tilde{c}') = G \times \mathfrak{n}_P \times \tilde{c}'$.

Let $\tilde{\mathcal{F}}'$ be the local system on R' whose inverse image under $p_3 : G \times \mathfrak{n}_P \times c' \to R'$ equals the inverse image of $\mathcal{F}'$ under $p_4 : G \times \mathfrak{n}_P \times c' \to c'$. Since p_3, p_4 are principal bundles with connected groups, it follows that the inverse image of $IC(R, \tilde{\mathcal{F}}')$ under $p_3 : G \times \mathfrak{n}_P \times \bar{c}' \to R$ equals the inverse image of $IC(\bar{c}', \mathcal{F}')$ under $p_4 : G \times \mathfrak{n}_P \times \bar{c}' \to \bar{c}'$. It follows that (using p_3^* this is reduced to (3.10))

$$(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}|_{X_1^{\omega}} = IC(R, \tilde{\mathcal{F}}')[\dim c' - 2\nu_L](\text{ extend by zero outside } R). \quad (3.11)$$

For any subvariety S of X_1, we denote ${}_S\varphi'' : S \to \mathfrak{g}$ the restriction of $\varphi'' : X_1 \to \mathfrak{g}$ to S.

Proposition 3.3.2. *Let* $d = \nu_G - \frac{1}{2}\dim c$, $d' = \frac{1}{2}(\dim c - \dim c')$ *and* $d'' = \nu_G - \nu_L - d'$. *The following five numbers coincide:*

(i) $\dim Hom_{\bar{\mathbb{Q}}_l[\mathbf{W}_L]}(\rho', \rho)$;

(ii) *the multiplicity of* $\mathcal{F}$ *in the local system* $\mathcal{L}_1 = \mathcal{H}^{2d}(\varphi''_!(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})|_c$;

(iii) *the multiplicity of* $\mathcal{F}$ *in the local system* $\mathcal{L}_2 = \mathcal{H}^{2d''}({}_R\varphi''_! IC(R, \tilde{\mathcal{F}}'))|_c$;

(iv) *the multiplicity of* $\mathcal{F}$ *in the local system* $\mathcal{L}_3 = \mathcal{H}^{2d''}({}_{R'}\varphi''_! IC(R, \tilde{\mathcal{F}}'))|_c = \mathcal{H}^{2d''}({}_{R'}\varphi''_! \tilde{\mathcal{F}}')|_c$;

(v) *the multiplicity of* $\mathcal{F}'$ *in the local system* $\mathcal{H}^{2d'} f_!(\mathcal{F})$ *on* c', *where* $f : \pi_{\mathfrak{p}}^{-1}(c') \cap c \to c'$ *is the restriction of* $\pi_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{l}$.

Proof. For $\tilde{\rho} \in \mathbf{W}_G^{\wedge}$, the multiplicity of $\mathcal{F}$ in $\mathcal{H}^{2d}((\varphi_!\bar{\mathbb{Q}}_{lX})_{\tilde{\rho}})|_c$ is 1 if $\tilde{\rho} = \rho$ and is 0 if

$\tilde\rho \neq \rho$. Hence it follows from (3.8) that the numbers in (i)(ii) are equal.

We show that $\mathcal{L}_1 = \mathcal{L}_2$. By (3.11), we have $\mathcal{L}_2 = \mathcal{H}^{2d}({}_R\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}|_R))|_{\mathrm{c}}$. It suffices to show that ${}_{(X_1-R)}\varphi''_!((\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}|_{X_1-R})|_{\mathrm{c}} = 0$. Assume this is not true. Then there exists $(x, gP) \in X_1 - R$ such that $x \in \mathrm{c}$ and $(x, gP) \in \operatorname{supp}(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}$. Since x is nilpotent, this contradicts (3.9).

We show that $\mathcal{L}_2 = \mathcal{L}_3$. For any $x \in \mathrm{c}$ we consider the natural exact sequence $H_c^{2d-1}(\varphi''^{-1}(x) \cap (R - R'), (\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) \xrightarrow{a} H_c^{2d}(\varphi''^{-1}(x) \cap R', (\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) \to H_c^{2d}(\varphi''^{-1}(x) \cap R, (\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) \to H_c^{2d}(\varphi''^{-1}(x) \cap (R - R'), (\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'})$. It is enough to show that $H_c^{2d}(\varphi''^{-1}(x) \cap (R - R'), (\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}) = 0$ and that $a = 0$. By (3.11), we can replace $(\varphi'_!\bar{\mathbb{Q}}_{lX})_{\rho'}|_{X_1^{\varphi}}$ by $IC(R, \tilde{\mathcal{F}}')[\dim \mathrm{c}' - 2\nu_L]$. It is enough to show

$$H_c^{2d''}(\varphi''^{-1}(x) \cap (R - R'), IC(R, \tilde{\mathcal{F}}')) = 0, \tag{3.12}$$

$$H_c^{2d''-1}(\varphi''^{-1}(x) \cap (R - R'), IC(R, \tilde{\mathcal{F}}')) \xrightarrow{a} H_c^{2d''}(\varphi''^{-1}(x) \cap R', IC(R, \tilde{\mathcal{F}}'))\ \text{is zero.} \tag{3.13}$$

From Proposition 3.3.1, we see that for any L-orbit $\tilde{\mathrm{c}}'$ in $\bar{\mathrm{c}}'$,

$$\dim(\varphi''^{-1}(x) \cap R_{\tilde{\mathrm{c}}'}) \leq (\nu_G - \frac{1}{2}\dim \mathrm{c}) - (\nu_L - \frac{1}{2}\dim \tilde{\mathrm{c}}'). \tag{3.14}$$

If (3.12) is not true, then using the partition

$$\varphi''^{-1}(x) \cap (R - R') = \bigcup_{\tilde{\mathrm{c}}' \neq \mathrm{c}'} (\varphi''^{-1}(x) \cap R_{\tilde{\mathrm{c}}'}), \tag{3.15}$$

we see that $H_c^{2d''}(\varphi''^{-1}(x) \cap R_{\tilde{\mathrm{c}}'}, IC(R, \tilde{\mathcal{F}}')) \neq 0$ for some $\tilde{\mathrm{c}}' \neq \mathrm{c}'$. Hence there exist i, j such that $2d'' = i + j$ and $H_c^i(\varphi''^{-1}(x) \cap R_{\tilde{\mathrm{c}}'}, \mathcal{H}^j(IC(R, \tilde{\mathcal{F}}'))) \neq 0$. It follows that $i \leq 2\dim(\varphi''^{-1}(x) \cap R_{\tilde{\mathrm{c}}'}) \leq 2\nu_G - \dim \mathrm{c} - 2\nu_L + \dim \tilde{\mathrm{c}}'$ (we use (3.14)). The local system $\mathcal{H}^j(IC(R, \tilde{\mathcal{F}}')) \neq 0$ so that $R_{\tilde{\mathrm{c}}'} \subset \operatorname{supp} \mathcal{H}^j(IC(R, \tilde{\mathcal{F}}'))$ and $\dim R_{\tilde{\mathrm{c}}'} < \dim R - j$. It follows that $j < \dim R - \dim R_{\tilde{\mathrm{c}}'} = \dim \mathrm{c}' - \dim \tilde{\mathrm{c}}'$ and $i + j < 2d''$ in contradiction to $i + j = 2d''$. This proves (3.12).

To prove (3.13), we can assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$, that G has a fixed $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$, that P, B, L, T (hence

$X_1, \varphi'')$ are defined over $\mathbf{F}_q$, that any $\tilde{c}'$ as above is defined over $\mathbf{F}_q$, that $F(x) = x$ and that we have an isomorphism $F^*\mathcal{F}' \to \mathcal{F}'$ which makes $\mathcal{F}'$ into a local system of pure weight 0. Then we have natural (Frobenius) endomorphisms of $H_c^{2d''-1}(\varphi''^{-1}(x) \cap (R - R'), IC(R, \tilde{\mathcal{F}}'))$ and $H_c^{2d''}(\varphi''^{-1}(x) \cap R', IC(R, \tilde{\mathcal{F}}')) = H_c^{2d''}(\varphi''^{-1}(x) \cap R', \tilde{\mathcal{F}}')$ compatible with a. To show that $a = 0$, it is enough to show that

$$H_c^{2d''}(\varphi''^{-1}(x) \cap R', IC(R, \tilde{\mathcal{F}}')) \text{ is pure of weight } 2d''; \tag{3.16}$$

$$H_c^{2d''-1}(\varphi''^{-1}(x) \cap (R - R'), IC(R, \tilde{\mathcal{F}}')) \text{ is mixed of weight } \leq 2d'' - 1. \tag{3.17}$$

Since $\dim(\varphi''^{-1}(x) \cap R') \leq d''$ (see(3.14)), (3.16) is clear. Using the partition (3.15), we see that to prove (3.17), it is enough to prove that $H_c^{2d''-1}(\varphi''^{-1}(x) \cap R_{\tilde{c}'}, IC(R, \tilde{\mathcal{F}}'))$ is mixed of weight $\leq 2d'' - 1$ for any $\tilde{c}' \neq c'$.

Using the hypercohomology spectral sequence we see that it is enough to prove if i, j are such that $2d'' - 1 = i + j$, then $H_c^i(\varphi''^{-1}(x) \cap R_{\tilde{c}'}, \mathcal{H}^j(IC(R, \tilde{\mathcal{F}}')))$ is mixed of weight $\leq 2d'' - 1$ for any $\tilde{c}'$. By Gabber's theorem [BBD, 5.3.2], the local system $\mathcal{H}^j(IC(R, \tilde{\mathcal{F}}'))$ is mixed of weight $\leq j$. Then by Deligne's theorem [BBD, 5.1.14(i)], $H_c^i(\varphi''^{-1}(x) \cap R_{\tilde{c}'}, \mathcal{H}^j(IC(R, \tilde{\mathcal{F}}')))$ is mixed of weight $\leq i + j = 2d'' - 1$. This proves (3.17). Hence $\mathcal{L}_2 = \mathcal{L}_3$ is proved.

Now consider the diagram $\mathbf{V} \xleftarrow{f_2} \mathbf{V}' \xrightarrow{f_1} c$, where $\mathbf{V}' = \varphi''^{-1}(c) \cap R' = \{(x, gP) \in c \times (G/P) | \mathrm{Ad}(g^{-1})(x) \in c' + \mathfrak{n}_P\}$, $\mathbf{V} = P \backslash (c' \times G)$ with P acting by $p : (x, g) \mapsto (\mathrm{Ad}(\pi(p))(x), gp^{-1})$, $\pi : P \to L$ the natural projection, $f_2(x, gP) = P$-orbit of $(\pi_{\mathfrak{p}}(\mathrm{Ad}(g^{-1})(x)), g)$, $f_1(x, gP) = x$. We have G-actions on $\mathbf{V}$ by $g' : (x, g) \mapsto (x, g'g)$, on $\mathbf{V}'$ by $g' : (x, gP) \mapsto (\mathrm{Ad}(g')(x), g'gP)$ and on c by $g' : x \mapsto \mathrm{Ad}(g')(x)$. Then f_1 and f_2 are G-equivariant and G acts transitively on $\mathbf{V}$ and c.

Note all fibers of f_1 have dimension $\leq d''$ and all fibers of f_2 have dimension $\leq d'$. Applying [L5, 8.4(a)] with $\mathcal{E}_1 = \mathcal{F}$ and with $\mathcal{E}_2$ the local system on $\mathbf{V}$ whose inverse image under the natural map $c' \times G \to \mathbf{V}$ is $\mathcal{F}' \boxtimes \bar{\mathbb{Q}}_l$, we see that the numbers (iv) and (v) are equal. This completes the proof of the proposition. □

3.3.5 Now we are ready to prove the restriction formula (**R**). Let the notation be as in 3.2.6. Let c be the G-orbit of x and c′ be the L-orbit of x'. Let $\tau : G/Z_G^0(x) \to$

$G/Z_G(x) \simeq \mathrm{c}$ be a covering of c with group $A_G(x)$. We have the following commutative diagram

$$\begin{array}{ccc} Y_{x,x'} & \xrightarrow{a} & Y_{x,x'}/Z_G^0(x) \\ {\scriptstyle b}\downarrow & & \wr\downarrow \\ (x'+\mathfrak{n}_P)\cap \mathrm{c} & \xleftarrow{\tau} & \tau^{-1}((x'+\mathfrak{n}_P)\cap \mathrm{c}), \end{array}$$

where a is the natural projection and b is given by $g \mapsto \mathrm{Ad}(g^{-1})(x)$. Then a induces an $A_G(x)$-equivariant bijection between $S_{x,x'}$ and the set of irreducible components of $\tau^{-1}((x'+\mathfrak{n}_P)\cap \mathrm{c})$ of dimension $d' = \frac{1}{2}(\dim \mathrm{c} - \dim \mathrm{c}')$ (note that $\dim(x'+\mathfrak{n}_P)\cap \mathrm{c} \leq d'$ by Proposition 3.3.1, (i)).

Assume $\mathcal{F}$ corresponds to $\phi \in A_G(x)^\wedge$ and $\mathcal{F}'$ corresponds to $\phi' \in A_L(x')^\wedge$. We have $\mathcal{F} \simeq \mathrm{Hom}_{A_G(x)}(\phi, \tau_*\bar{\mathbb{Q}}_l)$ and thus $H_c^{2d'}((x'+\mathfrak{n}_P)\cap \mathrm{c}, \mathcal{F}) \cong (H_c^{2d'}((x'+\mathfrak{n}_P)\cap \mathrm{c}, \tau_*\bar{\mathbb{Q}}_l) \otimes \phi^\wedge)^{A_G(x)} \cong (H_c^{2d'}(\tau^{-1}((x'+\mathfrak{n}_P)\cap \mathrm{c}), \bar{\mathbb{Q}}_l) \otimes \phi^\wedge)^{A_G(x)}$. Then the number (v) in Proposition 3.3.2 is equal to $\langle \phi', H_c^{2d'}(f^{-1}(x'), \mathcal{F})\rangle_{A_L(x')} = \langle \phi', H_c^{2d'}((x'+\mathfrak{n}_P)\cap \mathrm{c}, \mathcal{F})\rangle_{A_L(x')} = \langle \phi\otimes\phi', \varepsilon_{x,x'}\rangle_{A_G(x)\times A_L(x')}$. Hence the restriction formula (**R**) follows from Proposition 3.3.2 ((i)=(v)).

3.4 Orthogonal Lie algebras

In this section we assume $G = SO(N)$. Let $\tilde{G} = O(N)$. Let $x \in \mathcal{N}_\mathfrak{g}$.

3.4.1 The $\tilde{G}$-orbit c of x is characterized by the following data ([H]):

(d1) The sizes of the Jordan blocks of x give rise to a partition λ of $2n$, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s$.

(d2) For each λ_i, $i = 1, \ldots, s$, there is an integer $\chi(\lambda_i)$ satisfy $\frac{\lambda_i}{2} \leq \chi(\lambda_i) \leq \lambda_i$. Moreover, $\chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1})$, $i = 2, \ldots, s$.

Denote $m(\lambda_i)$ the multiplicity of λ_i in the partition λ. If N is even, then $m(\lambda_i)$ is even for each $\lambda_i > 0$. If N is odd, then the set $\{\lambda_i > 0 | m(\lambda_i) \text{ is odd}\}$ is $\{a, a-1\} \cap \mathbb{Z}_{>0}$ for some positive integer a.

We write

$$x \text{ (or c)} = (\lambda, \chi) = (\lambda_s)_{\chi(\lambda_s)} \cdots (\lambda_1)_{\chi(\lambda_1)}.$$

The component groups $\tilde{A}(x) = Z_{\tilde{G}}(x)/Z_{\tilde{G}}^0(x)$ and $A(x) = Z_G(x)/Z_G^0(x)$ can be described as follows (see Proposition 1.7.1). Let ϵ_i correspond to λ_i, $i = 1, \ldots, s$. Then $\tilde{A}(x)$ is isomorphic to the abelian group generated by $\{\epsilon_i, 1 \leq i \leq s | \chi(\lambda_i) \neq \lambda_i/2\}$ with relations

(r1) $\epsilon_i^2 = 1$,

(r2) $\epsilon_i = \epsilon_{i+1}$ if $\chi(\lambda_i) + \chi(\lambda_{i+1}) > \lambda_{i+1}$,

(r3) $\epsilon_i = 1$ if $m(\lambda_i)$ is odd.

If N is even, $A(x)$ is the subgroup of $\tilde{A}(x)$ consisting of those elements that can be written as a product of even number of generators.

3.4.2 Let $c' = (\lambda', \chi') \in f_x(\tilde{\mathcal{P}}_x)$, $\mathbf{Y} = f_x^{-1}(c')$ and $\mathbf{X} = \varrho_x^{-1}(\mathbf{Y})$ (see 3.2.8). Spaltenstein [Spal] has described the necessary and sufficient conditions for $\dim \mathbf{X} = \dim \mathcal{B}_x$ as follows.

Proposition 3.4.1 ([Spal]). *We have* $\dim \mathbf{X} = \dim \mathcal{B}_x$ *if and only if* (λ', χ') *satisfy (a) or (b):*

(a) Assume that $\lambda_i \neq \lambda_{i+1} \neq \lambda_{i+2}$ *and* $\chi(\lambda_{i+2}) = \lambda_{i+2}$. $\lambda'_j = \lambda_j$, $j \neq i+2, i+1$, $\lambda'_{i+2} = \lambda_{i+2} - 1, \lambda'_{i+1} = \lambda_{i+1} - 1$, $\chi'(\lambda'_j) = \chi(\lambda_j)$ *if* $j > i+2$, $\chi'(\lambda'_j) = \lambda'_j$ *if* $j \leq i+2$. *In this case,* $\dim \mathbf{Y} = s - i - 2$.

(b) Assume that $\lambda_{i+1} = \lambda_i > \lambda_{i-1}$. $\lambda'_j = \lambda_j$, $j \neq i+1, i$, $\lambda'_{i+1} = \lambda_{i+1} - 1, \lambda'_i = \lambda_i - 1$, $\chi'(\lambda'_j) = \chi(\lambda_j)$, $j \neq i, i+1$ *and* $\chi'(\lambda'_i) = \chi'(\lambda'_{i+1}) \in \{\chi(\lambda_i), \chi(\lambda_i) - 1\}$ *satisfies* $\lambda'_i/2 \leq \chi'(\lambda'_i) \leq \lambda'_i$, $\chi(\lambda_{i-1}) \leq \chi'(\lambda'_i) \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_{i-1} - 1$. *In this case,* $\dim \mathbf{Y} = s - i$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i)$ *and* $\dim \mathbf{Y} = s - i - 1$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$.

3.4.3 From now on let c' be as in Proposition 3.4.1. Let $\tilde{A}_P$ and $\tilde{A}'_P$ be defined as in 3.2.8.

Proposition 3.4.2. *The group* $Z_{\tilde{G}}(x)$ *acts transitively on* $\mathbf{Y}$. *The group* $\tilde{A}_P$ *is the subgroup of* $\tilde{A}(x)$ *generated by the elements* ϵ_i *which appear both in the generators of* $\tilde{A}(x)$ *and of* $\tilde{A}'(x')$. *The group* $\tilde{A}'_P$ *is the smallest subgroup of* $\tilde{A}'(x')$ *such that the map* $\tilde{A}_P \to \tilde{A}'(x')/\tilde{A}'_P$ *given by* $\epsilon_i \mapsto \epsilon'_i$ *is a morphism.*

Corollary. (i) *The variety* $\mathbf{Y}$ *has two irreducible components (and* $|\tilde{A}(x)/\tilde{A}_P| = 2$*) if* c' *is as in Proposition 3.4.1 (b) with* $\chi(\lambda_i) = \frac{\lambda_i+1}{2}, \lambda_{i+2} - \chi(\lambda_{i+2}) \geq \frac{\lambda_i+1}{2}$ *and*

$\chi'(\lambda_i') = \chi(\lambda_i) - 1$. *In this case, suppose* $D = \{1, \epsilon_i\} = \{1, \epsilon_{i+1}\} \subset \tilde{A}(x)$, *then* $\tilde{A}(x) = D \times \tilde{A}_P$. *In the other cases,* $\mathbf{Y}$ *is irreducible and* $\tilde{A}_P = \tilde{A}(x)$.

(ii) *The group* $\tilde{A}'_P$ *is trivial, except in the following cases where it has order 2:*
(a) $\tilde{A}'_P = \{1, \epsilon'_{i+3}\} \subset \tilde{A}'(x')$ *if* c′ *is as in Proposition 3.4.1 (a) with* $\lambda_{i+2} + \chi(\lambda_{i+3}) = \lambda_{i+3} + 1$.
(b) $\tilde{A}'_P = \{1, \epsilon'_{i+1}\epsilon'_{i+2}\} \subset \tilde{A}'(x')$ *if* c′ *is as in Proposition 3.4.1 (b) with* $\chi(\lambda_i) \neq \frac{\lambda_i+1}{2}, \chi(\lambda_{i+2}) + \chi(\lambda_i) = \lambda_{i+2} + 1$ *and* $\chi'(\lambda_i') = \chi(\lambda_i) - 1$.

3.4.4 Assume $G = O(2n+1) = O(V, \alpha)$ and x corresponds to the form module

$$V = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_k}(\lambda_k) \oplus D(\lambda_{k+1}) \oplus W_{\lambda_{k+2}}(\lambda_{k+2}) \cdots \oplus W_{\lambda_s}(\lambda_s),$$

where $l_i = \chi(\lambda_i), i = 1, \ldots, k$. (Note λ_i are different from those in 3.4.1. We use notations from Proposition 1.2.2.) We describe the orbits c′ and the corresponding set $\mathbf{Y}$.

We view V as an $A = k[[t]]$-module by $\sum a_i t^i v = \sum a_i (x^i v)$. For all $i \geq 1$, let $W_i = \ker t \cap \operatorname{Im}(t^{i-1})$. Denote $\mathbb{P}(W)$ the set of all lines in the space W. We identify $\mathcal{P}_x$ with $\mathbb{P}(\ker x \cap \alpha^{-1}(0))$. Let $\mathbf{Y}$ be as in 3.4.3. There exists a unique i_0 such that $\mathbf{Y} \subset \mathbb{P}(W_{i_0}) - \mathbb{P}(W_{i_0+1})$. Then $i_0 = \lambda_j$ for some $j = 1, \ldots, s$ or $i_0 = \lambda_{k+1} - 1$. Write $V' = \Sigma^\perp / \Sigma$. We have the following cases.

(i) Assume $i_0 = \lambda_j$, $1 \leq j \leq k$, $\lambda_j - 1 \geq \lambda_{j+1}$ and $\lambda_j - l_j - 1 \geq \lambda_{j+1} - l_{j+1}$.

$$\mathrm{c}' = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_j}(\lambda_j - 1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{\lambda_s}(\lambda_s).$$

$$\mathbf{Y} = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \operatorname{Im} t\}, \text{ if } l_j = \lambda_j/2 \text{ or } l_j = l_{j+1};$$

$$\mathbf{Y} = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \operatorname{Im} t, \alpha(t^{l_j - 1}w) \neq 0\}, \text{ otherwise.}$$

$$\dim \mathbf{Y} = 2j - 1.$$

(ii) Assume $i_0 = \lambda_j$, $1 \leq j \leq k$, $\lambda_j - 1 \geq \lambda_{j+1}$, $l_j - 1 \geq l_{j+1}$ and $l_j - 1 \geq [\lambda_j/2]$. Let $\mathbf{Y}' = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \operatorname{Im} t, \alpha(t^{l_j - 1}w) = 0\}$.

$$\mathrm{c}' = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_j - 1}(\lambda_j - 1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{\lambda_s}(\lambda_s).$$

$\mathbf{Y} = \mathbf{Y}'$ except if $\lambda_a - l_a > \lambda_{j-1} - l_{j-1}$, for all $\lambda_a > \lambda_{j-1}$, and $l_{j-1} = l_j > \frac{\lambda_{j-1}+1}{2}$, then $\mathbf{Y} = \mathbf{Y}' - \{\Sigma \in \mathbf{Y}' | \chi_{V'}(\lambda_{j-1}) = l_{j-1} - 1\}$(an open dense subset in $\mathbf{Y}'$).

$$\dim \mathbf{Y} = 2j - 2.$$

(iii) Assume $i_0 = \lambda_j$, $j \geq k+2$ and $\lambda_j \geq \lambda_{j+1} + 1$.

$$\mathrm{c}' = W_{l_1}(\lambda_1) \oplus \cdots \oplus D(\lambda_{k+1}) \oplus \cdots \oplus W_{\lambda_j - 1}(\lambda_j - 1) \oplus \cdots \oplus W_{\lambda_s}(\lambda_s).$$

$$\mathbf{Y} = \{\mathbf{k}t^{\lambda_j-1}w | t^{\lambda_j}w = 0, w \notin \text{Im } t, \alpha(t^{\lambda_j-1}w) = 0\},$$
$$\dim \mathbf{Y} = 2j-2.$$

(iv) Assume $i_0 = \lambda_{k+1}$ and $l_k = \lambda_k$.

$$\mathrm{c}' = W_{l_1}(\lambda_1) \oplus \cdots \oplus D(\lambda_k) \oplus W_{\lambda_{k+1}-1}(\lambda_{k+1}-1) \oplus \cdots \oplus W_{\lambda_s}(\lambda_s).$$
$$\mathbf{Y} = \{\mathbf{k}(t^{\lambda_{k+1}-1}w + t^{\lambda_k-1}w') | t^{\lambda_{k+1}-1}w \text{ spans } V^{\perp}, t^{\lambda_k}w' = 0, w' \notin \text{Im } t,$$
$$\alpha(t^{\lambda_k-1}w') = \alpha(t^{\lambda_{k+1}-1}w)\},$$
$$\dim \mathbf{Y} = 2k-1.$$

(v) Assume $i_0 = \lambda_{k+1}-1$ and $\lambda_{k+1}-2 \geq \lambda_{k+2}$. Let $\mathbf{Y}' = \{\mathbf{k}t^{\lambda_{k+1}-2}w | t^{\lambda_{k+1}-1}w = 0, w \notin \text{Im } t, \alpha(t^{\lambda_{k+1}-2}w) = 0\}$.

$$\mathrm{c}' = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_k}(\lambda_k) \oplus D(\lambda_{k+1}-1) \oplus W_{\lambda_{k+2}}(\lambda_{k+2}) \oplus \cdots \oplus W_{\lambda_s}(\lambda_s).$$

$\mathbf{Y} = \mathbf{Y}'$ except if $\lambda_a - l_a > \lambda_k - l_k$, for all $\lambda_a > \lambda_k$, and $l_k = \lambda_{k+1} > \frac{\lambda_k+1}{2}$, then $\mathbf{Y} = \mathbf{Y}' - \{\Sigma \in \mathbf{Y}' | \chi_{V'}(\lambda_k) = l_k - 1\}$(an open dense subset in $\mathbf{Y}'$),

$$\dim \mathbf{Y} = 2k.$$

3.4.5 Let $x, \mathrm{c}', \mathbf{Y}, \mathbf{X}$ be as in 3.4.4. Assume $\Sigma \in \mathbf{Y} \subset \mathbb{P}(W_{i_0}) - \mathbb{P}(W_{i_0+1})$. Let $X(\Sigma)$ be the set of nondegenerate submodules M of V satisfying the following conditions:

c1) $\Sigma \subset M$ and M has no proper submodule containing Σ,

c2) $\chi_M(i_0) = \chi_V(i_0)$. Moreover, in case (v) of 3.4.4, $\chi_M(\lambda_k) = \chi_V(\lambda_k)$.

We describe the set $X(\Sigma)$ in the cases (i)-(v) of 3.4.4 in the following.

(i) Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = t^{\lambda_j-1}w$. There exists $v' \in W_{\lambda_j} - W_{\lambda_{j+1}}$, such that $\beta(v', w) \neq 0$. Take w' such that $v' = t^{\lambda_j-1}w'$. Then $M = Aw \oplus Aw' \in X(\Sigma)$ and every module in $X(\Sigma)$ is obtained in this way. It is easily seen that $M = W_{l_j}(\lambda_j)$.

(ii) Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = t^{\lambda_j-1}w$. There exist $v' = t^{\lambda_j-1}w' \in W_{\lambda_j} - W_{\lambda_{j+1}}$, such that $\beta(v', w) \neq 0$ and $\alpha(t^{l_j-1}w') \neq 0$. Then $M = Aw \oplus Aw' \in X(\Sigma)$ and every module in $X(\Sigma)$ is obtained in this way. It is easily seen that $M = W_{l_j}(\lambda_j)$.

(iii) Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = t^{\lambda_j-1}w$. There exists $v' = t^{\lambda_j-1}w' \in W_{\lambda_j} - W_{\lambda_{j+1}}$, such that $\beta(v', w) \neq 0$ and $\alpha(t^{\lambda_j-1}w') \neq 0$. Then $M = Aw \oplus Aw' \in X(\Sigma)$ and every module in $X(\Sigma)$ is obtained in this way. It is easily seen that $M = W_{\lambda_j}(\lambda_j)$.

(iv) Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = t^{\lambda_{k+1}-1}w + t^{\lambda_k-1}w'$. There exists $v_1 = t^{\lambda_{k+1}-2}w_1 \in W_{\lambda_{k+1}-1} - W_{\lambda_{k+1}}$ such that $\beta(w, v_1) \neq 0$ and $v_1' = t^{\lambda_k-1}w_1' \in W_{\lambda_k} - W_{\lambda_k+1}$ such that $\beta(v_1', t^{\lambda_k-1}w') \neq 0$. Then $M = Aw \oplus Aw_1 \oplus Aw' \oplus Aw_1' \in X(\Sigma)$ and every module in

$X(\Sigma)$ is obtained in this way. It is easily seen that $M = W_{\lambda_k}(\lambda_k) \oplus D(\lambda_{k+1})$.

(v) Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = t^{\lambda_{k+1}-2}w$. There exists $v' = t^{\lambda_{k+1}-1}w' \in W_{\lambda_{k+1}} - W_{\lambda_{k+1}+1}$ such that $\beta(w', v) \neq 0$. Then $M = Aw \oplus Aw' \in X(\Sigma)$ and every module in $X(\Sigma)$ is obtained in this way. It is easily seen that $M = D(\lambda_{k+1})$.

3.4.6 Let $M \in X(\Sigma)$ and $M^{\perp} = \{v \in V | \beta(v, M) = 0\}$. Then $M^{\perp}$ is a non-degenerate submodule of V. In cases (i)-(iii) of 3.4.4, we have that $V = M \oplus M^{\perp}$. In cases (iv)-(v), we have $V = M + M^{\perp}$ and $M \cap M^{\perp} = V^{\perp}$. The nondegenerate submodule $M^{\perp}$ has orthogonal decomposition $M^{\perp} = M' \oplus D(1)$, where M' is a non-defective submodule. Hence $V = M' \oplus M$ (direct sum of orthogonal submodules). Now the map $t' : \Sigma^{\perp}/\Sigma \to \Sigma^{\perp}/\Sigma$ induced by t is given by the form module $\frac{\Sigma^{\perp} \cap M}{\Sigma} \oplus M'$, where M' is defined as above in cases (iv)-(v) and $M' = M^{\perp}$ in cases (i)-(iii). We write $\tilde{M} = \frac{\Sigma^{\perp} \cap M}{\Sigma}$.

We explain case (ii) of 3.4.4 in detail and the other cases are similar. In this case $M = W_{l_j}(\lambda_j), \tilde{M} = W_{l_j-1}(\lambda_j - 1)$. Recall that $\chi_{W_{l_j}}(\lambda_j) = [\lambda_j : l_j]$, where $[m : l] : \mathbb{N} \to \mathbb{N}$ is defined by $[m; l](k) = \max\{0, \min\{k - m + l, l\}\}$. We have that $\chi(\lambda_i) = \max\{\chi_{M'}(\lambda_i), \chi_M(\lambda_i)\}$ and $\chi'(\lambda_i) = \max\{\chi_{M'}(\lambda_i), \chi_{\tilde{M}}(\lambda_i)\}$. One easily check that $\chi(\lambda_i) = \chi'(\lambda_i)$ for $i \geq j + 1$, $\chi(\lambda_j - 1) = \chi'(\lambda_j - 1) = l_j - 1$, and $\chi(\lambda_i) = l_i = \max\{\chi_{M'}(\lambda_i), l_j\}, \chi'(\lambda_i) = \max\{\chi_{M'}(\lambda_i), l_j - 1\}$ for $i \leq j - 1$.

If $l_{j-1} > l_j$, then $l_i > l_j$, $\forall\, i \leq j - 1$. It follows that $\chi_{M'}(\lambda_i) = l_i$ and thus $\chi'(\lambda_i) = l_i$, $\forall\, i \leq j - 1$. Assume $l_{j-1} = l_j$ and there exists some $\lambda_i > \lambda_{j-1}$ such that $\lambda_i - l_i = \lambda_{j-1} - l_{j-1}$, then $l_i > l_j$ and $\chi_{M'}(\lambda_i) = l_i$. It follows that $\chi_{M'}(\lambda_{j-1}) = \lambda_{j-1} - \lambda_i + l_i = l_{j-1}$ and thus $\chi_{M'}(\lambda_i) = l_i, \forall\, i \leq j-1$. Assume $l_{j-1} = l_j \geq [(\lambda_{j-1}+1)/2]+1$ and for all $\lambda_i > \lambda_{j-1}$, $\lambda_i - l_i > \lambda_{j-1} - l_{j-1}$. Since we require $\chi_{V'}(\lambda_{j-1}) = l_{j-1}$, $\chi_{M'}(\lambda_{j-1}) = l_{j-1}$ and thus $\chi_{M'}(\lambda_i) = l_i, \forall\, i \leq j - 1$. In any case, $\chi'(\lambda_i) = l_i$, $\forall\, i \leq j - 1$. Hence c$'$ is of the form as stated.

3.4.7 The form modules $(\Sigma^{\perp} \cap M)/\Sigma$ are described in the following.

(1) Assume $x = W_m(2m)$, $m \geq 1$. Then $\tilde{\mathcal{P}}_x = \mathbb{P}(\ker x)$ and $f_x(\tilde{\mathcal{P}}_x) = W_m(2m-1)$.

(2) Assume $x = W_{m+1}(2m+1)$, $m \geq 1$. Then $\tilde{\mathcal{P}}_x = \mathbb{P}(\ker x)$, $\mathbf{Y}_1 = f_x^{-1}(W_m(2m))$ consists of two points and $\mathbf{Y}_2 = f_x^{-1}(W_{m+1}(2m)) = \tilde{\mathcal{P}}_x - \mathbf{Y}_1$.

(3) Assume $x = W_l(m)$, $(m + 1)/2 < l < m$. Then $\tilde{\mathcal{P}}_x = \mathbb{P}(\ker x)$, $\mathbf{Y}_1 =$

$f_x^{-1}(W_{l-1}(m-1))$ consists of one point and $\mathbf{Y}_2 = f_x^{-1}(W_l(m-1)) = \tilde{\mathcal{P}}_x - \mathbf{Y}_1$.

(4) Assume $x = W_m(m)$, $m \geq 2$. Then $\tilde{\mathcal{P}}_x$ consists one point and $f_x(\tilde{\mathcal{P}}_x) = W_{m-1}(m-1)$.

(5) Assume $x = W_1(1)$. Then $\tilde{\mathcal{P}}_x$ consists of two points and $f_x(\tilde{\mathcal{P}}_x) = \{0\}$.

(6) Assume $x = W_m(m) \oplus D(k)$, $m \geq k \geq 1$. Then $\tilde{\mathcal{P}}_x = \mathbb{P}(\ker x \cap \alpha^{-1}(0))$ and $f_x^{-1}(D(m) \oplus W_{k-1}(k-1)) = \mathbb{P}((W_k - W_{k+1}) \cap \alpha^{-1}(0))$.

(7) Assume $x = D(m)$, $m \geq 2$. Then $\tilde{\mathcal{P}}_x$ consists of one point and $f_x(\tilde{\mathcal{P}}_x) = D(m-1)$.

3.4.8 We prove Proposition 3.4.2 for $O(2n+1)$. The proof for $O(2n)$ is entirely similar and simpler. We use similar ideas as in [Spa2]. We first show that $Z_G(x)$ acts transitively on $\mathbf{Y}$. Consider $\widetilde{\mathbf{Y}}^* = \{(\Sigma, M) | \Sigma \in \mathbf{Y}, M \in X(\Sigma)^*\}$, where $X(\Sigma)^*$ is the nonempty subset $\{M \in X(\Sigma) | \chi_{M^\perp}(\lambda_a) = \chi_V(\lambda_a), \forall a \neq j$ in cases (i)-(iii), $a \neq k, k+1$ in case (iv) and $a \neq k+1$ in case (v)$\}$ in $X(\Sigma)$. For $M \in pr_2(\widetilde{\mathbf{Y}}^*)$, the equivalence classes of $M, M^\perp$ do not depend on the choice of $\Sigma \in \mathbf{Y}$ such that $(\Sigma, M) \in \widetilde{\mathbf{Y}}^*$. It follows that $Z_G(x)$ acts transitively on $pr_2(\widetilde{\mathbf{Y}}^*)$.

Fix $\Sigma \in \mathbf{Y}$ and $M \in X(\Sigma)^*$. Let Z_M be the stabilizer of M in $Z_G(x)$. The quadratic form α on V restricts to nondegenerate quadratic forms on M, $M^\perp$ (or M', if $M' \neq M^\perp$). Let $G(M)$, $G(M^\perp)$ (or $G(M')$) be the groups preserving the respective quadratic forms and $\mathfrak{g}(M)$, $\mathfrak{g}(M^\perp)$ (or $\mathfrak{g}(M')$) the Lie algebras. Let x_M, $x_{M^\perp}$ (or $x_{M'}$) be the restriction of x on M, $M^\perp$ (or M') respectively. Then $x_M \in \mathfrak{g}(M)$, $x_{M^\perp} \in \mathfrak{g}(M^\perp)$ (or $x_{M'} \in \mathfrak{g}(M')$). We have that Z_M is isomorphic to $Z_{G(M)}(x_M) \times Z_{G(M^\perp)}(x_{M^\perp})$. Set $\widetilde{\mathbf{Y}}^*_M = pr_2^{-1}(M) = \{\Sigma \in \mathbf{Y} | M \in X(\Sigma)^*\}$. By examining the cases (1)-(7) from 3.4.7 we see that Z_M acts transitively on $\widetilde{\mathbf{Y}}^*_M$. Thus $Z_G(x)$ acts transitively on $\widetilde{\mathbf{Y}}^*$ and hence acts transitively on $\mathbf{Y} = pr_1(\widetilde{\mathbf{Y}}^*)$.

Let Z_Σ be the stabilizer of Σ in $Z_G(x)$. The morphism $A_P \to A'(x')/A'_P$ is induced by the natural morphism $Z_\Sigma/Z_\Sigma^0 \to A'(x')$. Since $X(\Sigma)^*$ is irreducible, $Z_{\Sigma,M} = Z_\Sigma \cap Z_M$ meets all the irreducible components of Z_Σ. Thus to study the morphism $A_P \to A'(x')/A'_P$, it suffices to study the natural morphism $Z_{\Sigma,M}/Z^0_{\Sigma,M} \to A'(x')$.

Let $x_{\tilde{M}}$ be the endomorphism of $\tilde{M} = (\Sigma^\perp \cap M)/\Sigma$ induced by x_M. Then $x_{\tilde{M}} \in \mathfrak{g}(\tilde{M})$. Let $A'(x_{\tilde{M}}) = Z_{G(\tilde{M})}(x_{\tilde{M}})/Z^0_{G(\tilde{M})}(x_{\tilde{M}})$. Let $Z = \{z \in Z_{G(M)}(x_M) | z\Sigma = \Sigma\}$.

We have a natural isomorphism $Z_{\Sigma,M} \cong Z \times Z_{G(M^\perp)}(x_{M^\perp})$ and $Z_{\Sigma,M}/Z^0_{\Sigma,M} \cong Z/Z^0 \times A(x_{M^\perp})$. The morphism $A(x_{M^\perp}) \to A'(x')$ is the one obtained as follows. Note that $A(x_{M^\perp})$ is naturally isomorphic to $A(x_{M'})$. The system of generators of $A(x)$ is the union of the generators of $A(x_M)$ and $A(x_{M^\perp})$ and the morphism $A(x_M)\times A(x_{M^\perp}) \to A(x)$ is equal to the one induced by $Z_{G(M)}(x_M) \times Z_{G(M^\perp)}(x_{M^\perp}) \cong Z_M \subset Z_G(x)$. On the other hand, we have a morphism $A'(x_{\tilde{M}})\times A(x_{M'}) \to A'(x')$ which comes from the isomorphism $\Sigma^\perp/\Sigma \cong \tilde{M} \oplus M'$ and it is given by the system of generators. Hence the map $A(x_{M^\perp}) \hookrightarrow Z_{\Sigma,M}/Z^0_{\Sigma,M} \to A'(x')$ is given by generators. It remains to identify the morphism $Z/Z^0 \to A'(x')$.

We can show by explicit calculation on the cases (1)-(7) in 3.4.7 that the natural morphism $Z/Z^0 \to A(x_M)$ is injective and the image is generated by $\{\epsilon_j | \lambda'_j \neq \lambda_j, \epsilon_j$ belongs to the system of generators of $A(x_M)$ and $A'(x_{\tilde{M}})\}$. Using this description of Z/Z^0 and the above description of the morphism $A'(x_{\tilde{M}}) \times A(x_{M'}) \to A'(x')$, we see that the morphism $Z/Z^0 \to A'(x')$ is given by the system of generators. So we have obtained a complete description of the morphism $Z_{\Sigma,M}/Z^0_{\Sigma,M} \to A'(x')$ and we deduce easily that A'_P and the homomorphism $A_P \to A'(x')/A'_P$ are as in Proposition 3.4.2.

3.5 Dual of symplectic Lie algebras

Assume $G = Sp(V)$ in this section. Let $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ and let α_ξ , T_ξ be defined for ξ as in subsection 2.2.1.

3.5.1 The G-orbit c of ξ is characterized by the following data (see section 2.2):

(d1) The sizes of the Jordan blocks of T_ξ give rise to a partition of $2n$. We write it as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2s+1}$, where $\lambda_1 = 0$.

(d2) For each λ_i, there is an integer $\chi(\lambda_i)$ satisfy $\frac{\lambda_i - 1}{2} \leq \chi(\lambda_i) \leq \lambda_i$. Moreover, $\chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1})$, $i = 2, \ldots, 2s+1$.
Then $m(\lambda_i)$ is even for each $\lambda_i > 0$. We write

$$\xi \text{ (or c)} = (\lambda, \chi) = (\lambda_{2s+1})_{\chi(\lambda_{2s+1})} \cdots (\lambda_1)_{\chi(\lambda_1)}.$$

The component group $A(\xi) = Z_G(\xi)/Z^0_G(\xi)$ can be described as follows (see subsection 2.6.1). Let ϵ_i correspond to λ_i. Then $A(\xi)$ is isomorphic to the abelian group generated by $\{\epsilon_i|\chi(\lambda_i) \neq (\lambda_i - 1)/2\}$ with relations

(r1) $\epsilon_i^2 = 1$,

(r2) $\epsilon_i = \epsilon_{i+1}$ if $\chi(\lambda_i) + \chi(\lambda_{i+1}) \geq \lambda_{i+1}$,

(r3) $\epsilon_i = 1$, if $\lambda_i = 0$.

3.5.2 Let P be the stabilizer of a line $\Sigma = \{\mathbf{k}v\} \subset V$ in G.

Lemma 3.5.1. *$\xi \in \mathfrak{p}'$ if and only if $\alpha_\xi(v) = 0$ and $T_\xi(v) = 0$.*

Proof. P is the stabilizer of the flag $\{0 \subset \{\mathbf{k}v\} \subset \{\mathbf{k}v\}^\perp \subset V\}$. Write $v_1 = v$. There exists vectors v_i, $i = 2, \ldots, 2n$ such that v_i, $i = 1, \ldots, 2n$ span V and $\beta(v_i, v_j) = \delta_{j,i+n}$, $i \leq j$. Let $x \in \mathfrak{n}_P$. We have $xv_1 = 0$, $xv_i = a_i v_1, i \neq 1, n+1$ and $xv_{n+1} = bv_1 + \sum_{i=2}^n a_{n+i}v_i + \sum_{i=2}^n a_i v_{n+i}$. Assume $\xi(x') = \mathrm{tr}(Xx')$ for any $x' \in \mathfrak{g}$. A straightforward calculation shows that $\mathrm{tr}(Xx) = \sum_{i=2}^n a_i\beta_\xi(v_1, v_{n+i}) + \sum_{i=2}^n a_{n+i}\beta_\xi(v_1, v_i) + b\alpha_\xi(v_1)$. Moreover, $T_\xi(v_1) = \sum_{j=1}^n \beta_\xi(v_1, v_{n+j})v_j + \sum_{j=1}^n \beta_\xi(v_1, v_j)v_{n+j}$.

We have $\xi \in \mathfrak{p}'$ if and only if $\xi(x) = 0$ for any $x \in \mathfrak{n}_P$ if and only if $\beta_\xi(v_1, v_i) = \beta_\xi(v_1, v_{n+i}) = 0, i = 2, \ldots, n$ and $\alpha_\xi(v_1) = 0$. Thus $\xi \in \mathfrak{p}'$ if and only if $\alpha_\xi(v_1) = 0$ and $T_\xi(v_1) = av_1$ for some $a \in k$. Since T_ξ is nilpotent, $T_\xi(v_1) = av_1$ if and only if $a = 0$. The lemma is proved. ∎

3.5.3 Assume $\mathrm{c}' = (\lambda', \chi') \in f_\xi(\mathcal{P}_\xi)$, $\mathbf{Y} = f_\xi^{-1}(\mathrm{c}')$ and $\mathbf{X} = \varrho_\xi^{-1}(\mathbf{Y})$ (see 3.2.10).

Proposition 3.5.2. *We have* $\dim \mathbf{X} = \dim \mathcal{B}_\xi$ *if and only if* (λ', χ') *satisfies:*

Assume $\lambda_{i+1} = \lambda_i > \lambda_{i-1}$. $\lambda'_j = \lambda_j$, $j \neq i+1, i$, $\lambda'_{i+1} = \lambda_{i+1} - 1, \lambda'_i = \lambda_i - 1$, $\chi'(\lambda'_j) = \chi(\lambda_j)$, $j \neq i, i+1$ *and* $\chi'(\lambda'_i) = \chi'(\lambda'_{i+1}) \in \{\chi(\lambda_i), \chi(\lambda_i) - 1\}$ *satisfies* $[\lambda'_i/2] \leq \chi'(\lambda'_i) \leq \lambda'_i$, $\chi(\lambda_{i-1}) \leq \chi'(\lambda'_i) \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_{i-1} - 1$. *We have* $\dim \mathbf{Y} = 2s - i + 1$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i)$ *and* $\dim \mathbf{Y} = 2s - i$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$.

From now on let c′ be as in Proposition 3.5.2. Let A_P and A'_P be as in 3.2.10.

Proposition 3.5.3. *The group $Z_G(\xi)$ acts transitively on $\mathbf{Y}$. The group A_P is the subgroup of $A(\xi)$ generated by the elements ϵ_i which appear both in the generators of $A(\xi)$ and of $A'(\xi')$. The group A'_P is the smallest subgroup of $A'(\xi')$ such that the map $A_P \to A'(\xi')/A'_P$ given by $\epsilon_i \mapsto \epsilon'_i$ is a morphism.*

Corollary. (i) *The variety* $\mathbf{Y}$ *has two irreducible components (and* $|A(\xi)/A_P| = 2$*) if* c' *is as in Proposition with* $\chi(\lambda_i) = \frac{\lambda_i}{2}$, $\lambda_{i+2} - \chi(\lambda_{i+2}) > \lambda_i/2$ *and* $\chi'(\lambda_i') = \chi(\lambda_i) - 1$. *In this case, suppose* $D = \{1, \epsilon_i\} = \{1, \epsilon_{i+1}\} \subset A(\xi)$, *then* $A(\xi) = D \times A_P$. *In the other cases,* $\mathbf{Y}$ *is irreducible and* $A_P = A(\xi)$.

(ii) *The group* A'_P *is trivial, except if* c' *is as in Proposition with* $\chi(\lambda_i) \neq \frac{\lambda_i}{2}$, $\chi(\lambda_{i+2}) + \chi(\lambda_i) = \lambda_{i+2}$ *and* $\chi'(\lambda_i') = \chi(\lambda_i) - 1$. *In this case, we have* $A'_P = \{1, \epsilon'_{i+1}\epsilon'_{i+2}\} \subset A'(\xi')$.

Propositions 3.5.2 and 3.5.3 are proved entirely similarly as in the orthogonal Lie algebra case. We describe the orbits c' and the varieties $\mathbf{Y}$. The detail is omitted. Assume ξ corresponds to the form module $V = {}^*W_{l_1}(\lambda_1) \oplus \cdots \oplus {}^*W_{l_s}(\lambda_s)$, where $l_i = \chi(\lambda_i)$ (notations are as in Proposition 2.2.7).

We regard V as an $A = \mathbf{k}[[t]]$-module by $\sum a_i t^i v = \sum a_i T_\xi^i v$. By Lemma 3.5.1, we can identify $\mathcal{P}_\xi$ with $\mathbb{P}(W)$, where $W = \{v \in \ker t | \alpha_\xi(v) = 0\}$. Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$ and $\Sigma^\perp = \{v' \in V | \beta(v', \Sigma) = 0\}$. The quadratic form α_ξ induces a well-defined quadratic form $\bar{\alpha}_\xi : \Sigma^\perp/\Sigma \to \Sigma^\perp/\Sigma$ (note that $\beta_\xi(\Sigma^\perp, \Sigma) = 0$) and T_ξ induces a linear map $\bar{T}_\xi : \Sigma^\perp/\Sigma \to \Sigma^\perp/\Sigma$. Then $\bar{\alpha}_\xi$ defines an element $\xi' \in \mathfrak{sp}(\Sigma^\perp/\Sigma)^* = \mathfrak{l}^*$. Moreover, $\xi' \in c'$, $\alpha_{\xi'} = \bar{\alpha}_\xi$ and $T_{\xi'} = \bar{T}_\xi$. We have the following cases.

(i) Assume $1 \leq j \leq s$, $\lambda_j - 1 \geq \lambda_{j+1}$ and $\lambda_j - l_j - 1 \geq \lambda_{j+1} - l_{j+1}$.
$c' = {}^*W_{l_1}(\lambda_1) \oplus \cdots \oplus {}^*W_{l_j}(\lambda_j - 1) \oplus \cdots \oplus {}^*W_{\lambda_s}(\lambda_s)$. $\mathbf{Y} = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \text{Im } t\}$ if $l_j = (\lambda_j - 1)/2$ or $l_{j+1} = l_j$, $\mathbf{Y} = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \text{Im } t, \alpha_\xi(t^{l_j - 1}w) \neq 0\}$, otherwise. $\dim \mathbf{Y} = 2j - 1$.

(ii) Assume $1 \leq j \leq s$, $\lambda_j - 1 \geq \lambda_{j+1}$, $l_j - 1 \geq l_{j+1}$ and $l_j - 1 \geq [(\lambda_j - 1)/2]$.
Let $\mathbf{Y}' = \{\mathbf{k}t^{\lambda_j - 1}w | t^{\lambda_j}w = 0, w \notin \text{Im } t, \alpha_\xi(t^{l_j - 1}w) = 0\}$. $c' = {}^*W_{l_1}(\lambda_1) \oplus \cdots \oplus {}^*W_{l_j - 1}(\lambda_j - 1) \oplus \cdots \oplus {}^*W_{l_s}(\lambda_s)$. $\mathbf{Y} = \mathbf{Y}'$ except if $\lambda_a - l_a > \lambda_{j-1} - l_{j-1}$, for all $\lambda_a > \lambda_{j-1}$, and $l_{j-1} = l_j > \frac{\lambda_{j-1}}{2}$, then $\mathbf{Y} = \mathbf{Y}' - \{\Sigma \in \mathbf{Y}' | \chi_{V'}(\lambda_{j-1}) = l_{j-1} - 1\}$ (an open dense subset in $\mathbf{Y}'$). $\dim \mathbf{Y} = 2j - 2$.

3.6 Dual of odd orthogonal Lie algebras

Let $G = O(2n+1) = O(V, \alpha)$. Let $\xi \in \mathcal{N}_{\mathfrak{g}^*}$. Let $V = V_{2m+1} \oplus W$ be a normal form of ξ (see subsection 2.3.3), β_ξ and $T_\xi : W \to W$ defined for ξ as in section 2.3.

3.6.1 The orbit c of ξ is characterized by the following data (see section 2.3):

(d1) An integer $0 \leq m \leq n$.

(d2) The sizes of the Jordan blocks of T_ξ give rise to a partition of $2n-2m$. We write it as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2s}$.

(d3) For each λ_i, there is an integer $\chi(\lambda_i)$ satisfy $\frac{\lambda_i}{2} \leq \chi(\lambda_i) \leq \lambda_i$. Moreover, $\chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1})$, $i = 2, \ldots, 2s$.

(d4) $m \geq \lambda_{2s} - \chi(\lambda_{2s})$.

Then $m(\lambda_i)$ is even for each $\lambda_i > 0$. We write

$$\xi \text{ (or c)} = (m; \lambda, \chi) = (m; (\lambda_{2s})_{\chi(\lambda_{2s})} \cdots (\lambda_1)_{\chi(\lambda_1)}).$$

The component group $A(\xi) = Z_G(\xi)/Z_G^0(\xi)$ can be described as follows (see subsection 2.6.2). Let ϵ_i correspond to λ_i, $i = 1\ldots, 2s$. Then $A(\xi)$ is isomorphic to the abelian group generated by $\{\epsilon_i | \chi(\lambda_i) \neq \lambda_i/2\}$ with relations

(r1) $\epsilon_i^2 = 1$,

(r2) $\epsilon_i = \epsilon_{i+1}$ if $\chi(\lambda_i) + \chi(\lambda_{i+1}) > \lambda_{i+1}$,

(r3) $\epsilon_{2s} = 1$ if $\chi(\lambda_{2s}) \geq m$.

3.6.2 Let P be the stabilizer of a line $\Sigma = \{\mathbf{k}v\} \subset V$ in G, where $\alpha(v) = 0$.

Lemma 3.6.1. *$\xi \in \mathfrak{p}'$ if and only if $\beta_\xi(v, v') = 0$ for any $v' \in V$.*

Proof. P is the stabilizer of the flag $\{0 \subset \{\mathbf{k}v\} \subset \{\mathbf{k}v\}^\perp \subset V\}$. Write $v_1 = v$. There exists vectors v_i, $i = 2, \ldots, 2n+1$ such that v_i, $i = 1, \ldots, 2n+1$ span V and $\beta(v_i, v_j) = \delta_{j,i+n}$, $1 \leq i \leq j \leq 2n$, $\beta(v_i, v_{2n+1}) = 0$, $i = 1, \ldots, 2n+1$, $\alpha(v_i) = 0$, $i = 1, \ldots, 2n$, $\alpha(v_{2n+1}) = 1$. Let $x \in \mathfrak{n}_P$. We have $xv_1 = 0$, $xv_i = a_i v_1, i \neq 1, n+1, 2n+1$ and $xv_{n+1} = \sum_{i=2}^n a_{n+i} v_i + \sum_{i=2}^n a_i v_{n+i} + b v_{2n+1}$, $xv_{2n+1} = 0$. Assume $\xi(x') = \mathrm{tr}(Xx')$ for any $x' \in \mathfrak{g}$. A straightforward calculation shows that $\mathrm{tr}(Xx) = \sum_{i=2}^n a_i \beta_\xi(v_1, v_{n+i}) + \sum_{i=2}^n a_{n+i} \beta_\xi(v_1, v_i) + b\beta_\xi(v_1, v_{2n+1})$. Thus if $\xi \in \mathfrak{p}'$ then $\beta_\xi(v_1, v_i) = 0, i \neq n+1$.

Now let W be the subspace of V spanned by $v_i, i = 1, \ldots, 2n$. Then β is nondegenerate on W. We define a map $T : W \to W$ by $\beta(Tw, w) = \beta_\xi(w, w')$, $w, w' \in W$. Then similar argument as in Lemma 2.3.12 shows that T is nilpotent. One easily shows that $Tv_1 = \sum_{j=1}^n \beta_\xi(v_1, v_{n+j}) v_j + \sum_{j=1}^n \beta_\xi(v_1, v_j) v_{n+j}$. It follows that $Tv_1 = \beta_\xi(v_1, v_{n+1}) v_1$ and thus $\beta_\xi(v_1, v_{n+1}) = 0$. The lemma follows. $\square$

3.6.3 Let $c' = (m'; \lambda', \chi') \in f_\xi(\mathcal{P}_\xi)$, $\mathbf{Y} = f_\xi^{-1}(c')$ and $\mathbf{X} = \rho_\xi^{-1}(\mathbf{Y})$ (see 3.2.10).

Proposition 3.6.2. *We have* $\dim \mathbf{X} = \dim \mathcal{B}_\xi$ *if and only if* (λ', χ') *and* m' *satisfy (a) or (b):*

(a) Assume $m - 1 \geq \lambda_{2s} - \chi(\lambda_{2s})$. $m' = m - 1$, $\lambda'_i = \lambda_i$ *and* $\chi'(\lambda'_i) = \chi(\lambda_i)$, $i = 1, \ldots, 2s$. *We have* $\dim \mathbf{Y} = 0$*;*

(b) Assume that $\lambda_{i+1} = \lambda_i > \lambda_{i-1}$. $m' = m$, $\lambda'_j = \lambda_j$, $j \neq i+1, i$, $\lambda'_{i+1} = \lambda_{i+1} - 1, \lambda'_i = \lambda_i - 1$, $\chi'(\lambda'_j) = \chi(\lambda_j)$, $j \neq i, i+1$ *and* $\chi'(\lambda'_i) = \chi'(\lambda'_{i+1}) \in \{\chi(\lambda_i), \chi(\lambda_i) - 1\}$ *satisfies* $\lambda'_i/2 \leq \chi'(\lambda'_i) \leq \lambda'_i$, $\chi(\lambda_{i-1}) \leq \chi'(\lambda'_i) \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_{i-1} - 1$. *We have* $\dim \mathbf{Y} = 2s - i + 1$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i)$ *and* $\dim \mathbf{Y} = 2s - i$ *if* $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$.

From now on let c' be as in Proposition 3.6.2 and A_P, A'_P defined as in 3.2.10.

Proposition 3.6.3. *The group* $Z_G(\xi)$ *acts transitively on* $\mathbf{Y}$. *The group* A_P *is the subgroup of* $A(\xi)$ *generated by the elements* ϵ_i *which appear both in the generators of* $A(\xi)$ *and of* $A'(\xi')$. *The group* A'_P *is the smallest subgroup of* $A'(\xi')$ *such that the map* $A_P \to A'(\xi')/A'_P$ *given by* $\epsilon_i \mapsto \epsilon'_i$ *is a morphism.*

Corollary. (i) *The variety* $\mathbf{Y}$ *has two irreducible components (and* $|A(\xi)/A_P| = 2$*) if* c' *is as in Proposition 3.6.2 (b) with* $\chi(\lambda_i) = \frac{\lambda_i+1}{2}$, $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$, *and* $\lambda_{i+2} - \chi(\lambda_{i+2}) \geq (\lambda_i + 1)/2$ *if* $i < 2s - 1$, $m \geq (\lambda_i + 1)/2$ *if* $i = 2s - 1$. *In this case, suppose* $D = \{1, \epsilon_i\} = \{1, \epsilon_{i+1}\} \subset A(\xi)$, *then* $A(\xi) = D \times A_P$. *In the other cases,* $\mathbf{Y}$ *is irreducible and* $A_P = A(\xi)$.

(ii) *The group* A'_P *is trivial, except if* c' *is as in Proposition 3.6.2 (b) with* $\chi(\lambda_i) \neq \frac{\lambda_i+1}{2}$, $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$, *and* $\chi(\lambda_{i+2}) + \chi(\lambda_i) = \lambda_{i+2} + 1$ *if* $i < 2s - 1$, $\chi(\lambda_i) = m + 1$ *if* $i = 2s - 1$. *We have* $A'_P = \{1, \epsilon'_{i+1}\epsilon'_{i+2}\} \subset A'(\xi')$ *if* $i < 2s - 1$ *and* $A'_P = \{1, \epsilon'_{2s}\} \subset A'(\xi')$ *if* $i = 2s - 1$.

Write $\xi = V_{2m+1} \oplus W$, where $W = W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_s}(\lambda_s)$, $l_i = \chi(\lambda_i)$, $\lambda_i \geq \lambda_{i+1}$ (notation as in section 2.3). Let $\{v_i, i = 0, \ldots, m\}$ be the set of vectors as in Lemma 2.3.5. We view W as a $\mathbf{k}[[t]]$ module by $\sum a_i t^i w = \sum a_i T_\xi^i w$. It follows from Lemma 3.6.1 that $\mathcal{P}_\xi$ is identified with $\mathbb{P}((\mathbf{k}v_0 \oplus \ker t) \cap \alpha^{-1}(0))$ (α is the non-degenerate quadratic form on V). Let $\Sigma \in \mathbf{Y}$ and $\Sigma^\perp = \{v' \in V | \beta(v', \Sigma) = 0\}$. The bilinear form β_ξ induces a bilinear form $\bar{\beta}_\xi$ on $\Sigma^\perp/\Sigma$. Then $\bar{\beta}_\xi$ defines an element $\xi' \in \mathfrak{o}(\Sigma^\perp/\Sigma)^* \cong \mathfrak{l}^*$. We have that $\xi' \in \mathrm{c}'$ and $\beta_{\xi'} = \bar{\beta}_\xi$. The variety $\mathbf{Y}$ in various cases is described in the following.

(i) Assume $m \geq 1$ and $m-1 \geq \lambda_1 - l_1$.

$\xi' = V_{2m-1} \oplus W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_s}(\lambda_s)$, $\mathbf{Y} = \{\mathbf{k}v_0\}$ consists of one point.

(ii) Assume $\lambda_j - l_j - 1 \geq \lambda_{j+1} - l_{j+1}, \lambda_j \geq \lambda_{j+1} + 1$. Then $m \geq 1$.

$\xi' = V_{2m+1} \oplus W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_j}(\lambda_j - 1) \oplus \cdots \oplus W_{l_s}(\lambda_s)$. $\mathbf{Y} = \{\mathbf{k}v | v = av_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW\}$ if $l_j = \lambda_j/2$ or $l_{j+1} = l_j$; $\mathbf{Y} = \{\mathbf{k}v | v = av_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW, \alpha(t^{l_j - 1}w) \neq a^2\delta_{m,\lambda_j - l_j}\}$ otherwise. $\dim \mathbf{Y} = 2j$.

(iii) Assume $l_j - 1 \geq l_{j+1}, l_j \geq [\lambda_j/2] + 1, \lambda_j \geq \lambda_{j+1} + 1$.

$\xi' = V_{2m+1} \oplus W_{l_1}(\lambda_1) \oplus \cdots \oplus W_{l_j - 1}(\lambda_j - 1) \oplus \cdots \oplus W_{l_s}(\lambda_s)$. $\mathbf{Y} \subset \mathbf{Y}' := \{\mathbf{k}v | v = av_0 + t^{\lambda_j - 1}w, w \in W, t^{\lambda_j}w = 0, w \notin tW, \alpha(t^{l_j - 1}w) = a^2\delta_{m,\lambda_j - l_j}\}$ (for a description of $\mathbf{Y}$ see below). $\dim \mathbf{Y} = 2j - 1$.

Case (i) is clear. We explain case (iii) in detail. Case (ii) is similar. Let $\Sigma = \mathbf{k}v \in \mathbf{Y}$, where $v = av_0 + t^{\lambda_j - 1}w$. Let $\{u_i \in V_{2m+1}, i = 0, \ldots, m-1\}$ be a set of vectors as in Lemma 2.3.6. Assume $a \neq 0$. There exists $w_0 \in W$ such that $\beta(w_0, t^{\lambda_j - 1}w) = 1$ and $\alpha(w_0) = 0$. Let $\tilde{u_0} = aw_0 + u_0$ and we define $\tilde{V}_{2m+1}$, $\tilde{W}$ as in Remark 2.3.9. Then $V = \tilde{V}_{2m+1} \oplus \tilde{W}$ (we can choose w_0 such that $\chi_{\tilde{W}}(\lambda_i) = l_i$) and $\Sigma \subset \tilde{W}$. Note that $v = t^{\lambda_j - 1}(\tilde{w})$, where $\tilde{w} = w + \sum_{i=0}^m \beta(aw_0, t^i w)v_i \in \tilde{W}$ and $\alpha(t^{l_j - 1}\tilde{w}) = \alpha(t^{l_j - 1}w) + a^2\delta_{m,\lambda_j - l_j}$.

Now we can assume $V = V_{2m+1} \oplus W$ is a normal form of ξ, with $\Sigma = \mathbf{k}v \subset W$ and $v = t^{\lambda_j - 1}w, w \in W$. Then $\Sigma^\perp/\Sigma = V_{2m+1} \oplus (\Sigma^\perp \cap W)/\Sigma$. We apply the results for orthogonal Lie algebras to $(\Sigma^\perp \cap W)/\Sigma$ (see 3.4.4). Write $W' = (\Sigma^\perp \cap W)/\Sigma$. The set $\mathbf{Y} = \mathbf{Y}'$, except if $l_{j-1} = l_j > \frac{\lambda_{j-1}+1}{2}$, $m > \lambda_{j-1} - l_{j-1}$ and for all $\lambda_a > \lambda_{j-1}$, $\lambda_a - l_a > \lambda_j - l_j$, then $\mathbf{Y}$ consists of those v such that $\chi_{W'}(\lambda_{j-1}) = l_{j-1}$.

3.6.4 We prove Proposition 3.6.3. In case (i), we have $L = \{\mathbf{k}v_0\} \subset V_{2m+1}$. For any $g \in Z_G(\xi)$, we have that $gv_0 = v_0$. Hence $H = Z_P(\xi) = Z_G(\xi)$, $K = Z_P(\xi) \cap Z_G^0(\xi) = Z_G^0(\xi)$ and $A_P = A(\xi), A'_P = 1$.

In cases (ii) and (iii), we can find a normal form $V = V_{2m+1} \oplus W$ such that $\Sigma \subset W$ (see subsection 3.6.3). Let $X(\Sigma)$ be the set of all such W. We first show that $Z_G(\xi)$

acts transitively on $\mathbf{Y}$. Let $\widetilde{\mathbf{Y}} = \{(\Sigma, W)|\Sigma \in \mathbf{Y}, W \in X(\Sigma)\}$. Then $Z_G(\xi)$ acts transitively on $pr_2(\widetilde{\mathbf{Y}})$. Set $\widetilde{\mathbf{Y}}_W = pr_2^{-1}(W) = \{\Sigma \in \mathbf{Y}|W \in X(\Sigma)\}$. It follows from the results in the orthogonal Lie algebra case that Z_W acts transitively on $\widetilde{\mathbf{Y}}_W$ (see Proposition 3.4.2). Then $Z_G(\xi)$ acts transitively on $\widetilde{\mathbf{Y}}$ and hence acts transitively on $\mathbf{Y} = pr_1(\widetilde{\mathbf{Y}})$.

Fix $\Sigma \in \mathbf{Y}$ and $W \in X(\Sigma)$. Let Z_W and Z_Σ be the stabilizer of W and Σ in $Z_G(\xi)$ respectively. The morphism $A_P \to A'(\xi')/A'_P$ is induced by the natural morphism $Z_\Sigma/Z_\Sigma^0 \to A'(\xi')$. Since $X(\Sigma)$ is irreducible, $Z_{\Sigma,W} = Z_\Sigma \cap Z_W$ meets all the irreducible components of Z_Σ. Thus to study the morphism $A_P \to A'(\xi')/A'_P$, it suffices to study the natural morphism $Z_{\Sigma,W}/Z^0_{\Sigma,W} \to A'(\xi')$.

The quadratic form α on V restricts to nondegenerate quadratic forms on W and $W^\perp$. Let $G(W)$, $G(W^\perp)$ be the groups preserving the respective quadratic forms and $\mathfrak{g}(W)$, $\mathfrak{g}(W^\perp)$ the Lie algebras. The bilinear form β_ξ on V restricts to bilinear forms on W and $W^\perp$. Let ξ_W and $\xi_{W^\perp}$ be the corresponding elements in $\mathfrak{g}(W)^*$ and $\mathfrak{g}(W^\perp)^*$ respectively. Moreover, the bilinear form β_ξ induces a bilinear form on $\tilde{W} = (\Sigma^\perp \cap W)/\Sigma$. Let $\xi_{\tilde{W}}$ be the corresponding element in $\mathfrak{g}(\tilde{W})^*$ and $A'(\xi_{\tilde{W}}) = Z_{G(\tilde{W})}(\xi_{\tilde{W}})/Z^0_{G(\tilde{W})}(\xi_{\tilde{W}})$.

Let $Z = \{z \in Z_{G(W)}(\xi_W)|z\Sigma = \Sigma\}$. Since $Z_W \cong Z_{G(W)}(\xi_W) \times Z_{G(W^\perp)}(\xi_{W^\perp})$, we have natural isomorphisms $Z_{\Sigma,W} \cong Z \times Z_{G(W^\perp)}(\xi_{W^\perp})$ and $Z_{\Sigma,W}/Z^0_{\Sigma,W} \cong Z/Z^0 \times A(\xi_{W^\perp})$. Note that $A(\xi_{W^\perp}) = \{1\}$. On the other hand, we have a morphism $A'(\xi_{\tilde{W}}) \times A(\xi_{W^\perp}) \to A'(\xi')$ which comes from the isomorphism $\Sigma^\perp/\Sigma \cong \tilde{W} \oplus W^\perp$ and it is given by the system of generators. It follows form the results for orthogonal Lie algebras that the morphism $Z/Z^0 \to A'(\xi_{\tilde{W}})$ is given by generators (see Proposition 3.4.2). We then deduce easily that A'_P and the morphism $A_P \to A'(\xi')/A'_P$ are as in Proposition 3.6.3.

3.7 Some combinatorics

In this section we recall some combinatorics from [L3, LS2]. The combinatorics goes back to [L1], where it is used to parametrize unipotent representations of classical groups. We will use the same kind of combinatorial objects to describe the Springer correspondence for classical Lie algebras and their duals in characteristic 2.

3.7.1 Let $r, s, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $d \in \mathbb{Z}$, $e = [\frac{d}{2}] \in \mathbb{Z}$ ($[-]$ means the integer part). Let $\tilde{X}^{r,s}_{n,d}$ be the set of all ordered pairs (A, B) of finite sequences of natural integers $A = (a_1, a_2, \ldots, a_{m+d})$ and $B = (b_1, b_2, \ldots, b_m)$ (for some m) satisfying the following conditions: $a_{i+1} - a_i \geq r + s$, $i = 1, \ldots, m + d - 1$, $b_{i+1} - b_i \geq r + s$, $i = 1, \ldots, m - 1, b_1 \geq s$, $\sum a_i + \sum b_i = n + r(m+e)(m+d-e-1) + s(m+e)(m+d-e)$.

The set $\tilde{X}^{r,s}_{n,d}$ is equipped with a shift $\sigma_{r,s}$. If (A, B) is as above, then $\sigma_{r,s}(A, B) = (A', B')$, $A' = (0, a_1 + r + s, \ldots, a_{m+d} + r + s)$, $B' = (s, b_1 + r + s, \ldots, b_m + r + s)$. Let $X^{r,s}_{n,d}$ be the quotient of $\tilde{X}^{r,s}_{n,d}$ by the equivalence relation generated by the shift and $X^{r,s}_n = \bigcup_{d \text{ odd}} X^{r,s}_{n,d}$. The equivalence class of (A, B) is still denoted by (A, B).

Assume $s = 0$. Then there is an obvious bijection $X^{r,0}_{n,d} \to X^{r,0}_{n,-d}, (A, B) \mapsto (B, A)$. This induces an involution on each of the following sets $X^r_{n,\text{even}} = \bigcup_{d \text{ even}} X^{r,0}_{n,d}$, $X^r_{n,\text{odd}} = \bigcup_{d \text{ odd}} X^{r,0}_{n,d}$. Let $Y^r_{n,\text{even}}$ (resp. $Y^r_{n,\text{odd}}$) be the quotient of $X^r_{n,\text{even}}$ (resp. $X^r_{n,\text{odd}}$) by this involution. For $d \geq 0$, the image of $X^{r,0}_{n,\pm d}$ in $Y^r_{n,\text{even}}$ or $Y^r_{n,\text{odd}}$ is denoted $Y^r_{n,d}$ and the image of (A, B) is denoted $\{A, B\}$.

3.7.2 When we consider simultaneously two elements $(A, B) \in X^{r,s}_{n,d}$ and $(A', B') \in X^{r',s'}_{n',d'}$ with $d - d'$ even, with $A = (a_1, \ldots, a_{m+d}), B = (b_1, \ldots, b_m)$ and $A' = (a'_1, \ldots, a'_{m'+d'}), B' = (b'_1, \ldots, b'_{m'})$, we always assume that we have chosen representatives such that $2m + d = 2m' + d'$. We use the same convention for $\{A, B\} \in Y^r_{n,d}$ and $\{A', B'\} \in Y^{r'}_{n',d'}$ with $d, d' \geq 0$ and $d - d'$ even.

There is an obvious addition $X^{r,s}_{n,d} \times X^{r',s'}_{n',d} \to X^{r+r',s+s'}_{n+n',d}, (A, B) + (A', B') = (A'', B''), a''_i = a_i + a'_i, b''_i = b_i + b'_i$. The same formula defines $Y^r_{n,d} \times Y^{r'}_{n',d} \to Y^{r+r'}_{n+n',d}$.

Let $\Lambda^{r,s}_{0,1} \in X^{r,s}_{0,1}$ (resp. $\Lambda^{r,s}_{0,0} \in X^{r,s}_{0,0}$) be the element represented by $(A, B) = (0, \emptyset)$ (resp. $(A, B) = (\emptyset, \emptyset)$). If $s = 0$, let $\Lambda^r_{0,1} \in Y^r_{0,1}$ (resp. $\Lambda^r_{0,0} \in Y^r_{0,0}$) be the image of $\Lambda^{r,0}_{0,1}$ (resp. $\Lambda^r_{0,0}$). We have the following bijective maps: $X^{0,0}_{n,1} \to X^{r,s}_{n,1}$, $\Lambda \mapsto \Lambda + \Lambda^{r,s}_{0,1}$, $\quad Y^0_{n,d} \to Y^r_{n,d}$, $\Lambda \mapsto \Lambda + \Lambda^r_{0,d}, d = 0, 1$.

Since $Y^0_{n,d}, d \geq 1$, and $X^{0,0}_{n,d}$ are obviously in bijection with the set of all pairs of partitions (μ, ν) such that $\sum \mu_i + \sum \nu_i = n$ and thus with $\mathbf{W}^\wedge_n$, $Y^0_{n,0}$ is in bijection with the set of all unordered pairs of partitions $\{\mu, \nu\}$ such that $\sum \mu_i + \sum \nu_i = n$ and thus with $\mathbf{W}_n^{\wedge\prime}$, we get bijections

$$\mathbf{W}_n^{\wedge} \xrightarrow{\sim} X_{n,1}^{r,s}, \ \mathbf{W}_n^{\wedge} \xrightarrow{\sim} Y_{n,1}^{r}, \ \mathbf{W}_n^{\wedge\prime} \xrightarrow{\sim} Y_{n,0}^{r}.$$

3.7.3 An element $(A,B) \in X_{n,d}^{r,s}$ is called distinguished if $d=0$, $a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_m \leq b_m$ or if $d=1$, $a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_m \leq b_m \leq a_{m+1}$. An element $\{A,B\} \in Y_{n,d}^r$ ($d \geq 0$) is called distinguished if (A,B) or (B,A) is distinguished. Let $D_n^{r,s}, D_{n,\text{even}}^r, D_{n,\text{odd}}^r, D_{n,d}^{r,s}, D_{n,d}^r$ be the set of all distinguished elements in $X_n^{r,s}$, $Y_{n,\text{even}}^r, Y_{n,\text{odd}}^r, X_{n,d}^{r,s}, Y_{n,d}^r$ respectively.

Assume $r \geq 1$. For $(A,B) \in \tilde{X}_{n,d}^{r,s}$, we regard A, B as subsets of $\mathbb{N}$. Two elements $(A,B),(C,D) \in X_n^{r,s}$ are said to be similar if $A \cup B = C \cup D$ and $A \cap B = C \cap D$. We define similarity in $Y_{n,\text{even}}^r$ and $Y_{n,\text{odd}}^r$ in the same way.

Let $S = (A \cup B) \backslash (A \cap B)$. A nonempty subset I of S is called an interval of (A,B) or $\{A,B\}$ if it satisfies the following conditions:

(i) if $i < j$ are consecutive elements of I, then $j - i < r + s$;

(ii) if $i \in I$, $j \in S$ and $|i-j| < r+s$, then $j \in I$.

We call I an initial interval if there exists $i \in I$ such that $i < s$ and a proper interval otherwise.

Let $\mathcal{S} \subset X_n^{r,s}$ (resp. $Y_{n,\text{odd}}^r$ or $Y_{n,\text{even}}^r$) be a similarity class and (A,B) (resp. $\{A,B\}$)$\in \mathcal{S}$. Let E be the set of all proper intervals of (A,B) (resp. $\{A,B\}$). The set $\mathcal{A}(E)$ of all subsets of E is a vector space over $\mathbf{F}_2$. If $\mathcal{S} \subset X_n^{r,s}$, it acts simply transitively on $\mathcal{S}$ as follows. The image of (A,B) under $F \subset E$ is the pair (C,D) such that

$$A \cap I = D \cap I, B \cap I = C \cap I \text{ if and only if } I \in F.$$

If $\mathcal{S} \subset Y_{n,\text{odd}}^r$ (or $Y_{n,\text{even}}^r$), as E transforms (A,B) to (B,A), the same formula defines a simply transitive action of $\mathcal{A}(E)/\{\emptyset, E\}$ on $\mathcal{S}$. For $\Lambda \in X_n^{r,s}$ (resp. $Y_{n,\text{odd}}^r$ or $Y_{n,\text{even}}^r$), let $V_\Lambda^{r,s}$ (resp. V_Λ^r) denote the vector space $\mathcal{A}(E)$ (resp. $\mathcal{A}(E)/\{\emptyset, E\}$), where E is the set of all proper intervals of Λ. For $F \in V_\Lambda^{r,s}$ (resp. V_Λ^r), let Λ_F be the image of Λ under the action of F.

3.7.4 Examples (1) $X_{n,1}^{1,0}$ and $Y_{n,0}^1$ are used in [L2] to describe $\mathbf{W}_n^{\wedge}$ and $\mathbf{W}_n^{\wedge\prime}$ respectively.

(2) Assume $\text{char}(\mathbf{k}) \neq 2$. $X_n^{1,1}, Y_{n,\text{even}}^2$ and $Y_{n,\text{odd}}^2$ are used in [L3] to describe the

generalized Springer correspondence for Sp_{2n}, $SO(2n)$ and $SO(2n+1)$ respectively.

(3) $X_n^{2,2}$ and $Y_{n,\text{even}}^4$ are used in [LS2] to describe the generalized Springer correspondence for unipotent classes of $Sp(2n)$ (or $SO(2n+1)$) and $SO(2n)$ respectively.

(4) $Y_{n-1,\text{odd}}^4$, $X_n^{3,1}$ and $X_{n,\text{even}}^{3,1} = \cup_{d \text{ even}} X_{n,d}^{3,1}$ are used in [L5] to describe the generalized Springer correspondence for disconnected groups $O(2n), G_{2n+1}$ with G^0 type A_{2n}, and G_{2n} with G^0 type A_{2n-1} respectively.

(5) We will use $X_n^{2,n+1}, X_n^{n+1,n+1}$ and $Y_{n,\text{even}}^{n+1}$ to describe the Springer correspondence for $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$ and $\mathfrak{o}(2n)$ (or $\mathfrak{o}(2n)^*$). The set $D_n^{2,n+1}$ (resp. $D_n^{n+1,n+1}$, $D_{n,\text{even}}^{n+1}$) is in bijection with the set of $O(2n+1)$ (resp. $Sp(2n)$, $O(2n)$)-nilpotent orbits in $\mathfrak{o}(2n+1)$ (resp. $\mathfrak{sp}_{2n}$, $\mathfrak{o}(2n)$).

(6) We will use $Y_{n,\text{odd}}^{n+1}$ and $X_n^{1,n+1}$ to describe the Springer correspondence for $\mathfrak{o}(2n+1)^*$ and $\mathfrak{sp}(2n)^*$. The set $D_{n,\text{odd}}^{n+1}$ (resp. $D_n^{1,n+1}$) is in bijection with the set of $O(2n+1)$ (resp. $Sp(2n)$)-nilpotent orbits in $\mathfrak{o}(2n+1)^*$ (resp. $\mathfrak{sp}_{2n}^*$).

3.8 Springer correspondence for symplectic Lie algebras

Assume $G = Sp(2n)$. Let $x \in \mathfrak{g}$ be nilpotent.

3.8.1 The orbit c of x is characterized by the following data ([H]):

(d1) The sizes of the Jordan blocks of x give rise to a partition of $2n$. We write it as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2s+1}$, where $\lambda_1 = 0$.

(d2) For each λ_i, there is an integer $\chi(\lambda_i)$ satisfy $0 \leq \chi(\lambda_i) \leq \frac{\lambda_i}{2}$. Moreover, $\chi(\lambda_i) \geq \chi(\lambda_{i-1}), \lambda_i - \chi(\lambda_i) \geq \lambda_{i-1} - \chi(\lambda_{i-1})$, $i = 2, \ldots, 2s+1$.

We can partition the set $\{1, 2, \ldots, 2s+1\}$ in a unique way into blocks of length 1 or 2 such that the following holds:

(b1) If $\chi(\lambda_i) = \lambda_i/2$, then $\{i\}$ is one block;

(b2) All other blocks consist of two consecutive integers.

Note that if $\{i, i+1\}$ is a block, then $\lambda_i = \lambda_{i+1}$ and $\chi(\lambda_i) = \chi(\lambda_{i+1})$.

We attach to the orbit c the sequence $c_1, \ldots, c_{2s+1}$ defined as follows:

(1) If $\{i\}$ is a block, then $c_i = \lambda_i/2 + (n+1)(i-1)$;

(2) If $\{i, i+1\}$ is a block, then $c_i = \lambda_i - \chi(\lambda_i) + (n+1)(i-1)$, $c_{i+1} = \chi(\lambda_{i+1}) + (n+1)i$.

Taking $a_i = c_{2i-1}$, $i = 1, \ldots, s+1$, $b_i = c_{2i}$, $i = 1, \ldots, s$, we get a well defined element $(A, B) \in X_{n,1}^{n+1,n+1}$. We denote it $\rho_G(x)$, $\rho(x)$ or $\rho(\mathrm{c})$.

Lemma. (i) $\mathrm{c} \mapsto \rho(\mathrm{c})$ *defines a bijection from the set of all nilpotent* $Sp(2n)$*-orbits in* $\mathfrak{sp}(2n)$ *to* $D_n^{n+1,n+1}$.

(ii) $A_G(x)^\wedge$ *is isomorphic to* $V_{\rho(x)}^{n+1,n+1}$.

Proof. (i) It is easily checked from the definition that $\rho(\mathrm{c}) \in D_n^{n+1,n+1}$ and the map $\mathrm{c} \mapsto \rho(\mathrm{c})$ is injective. Note that $X_{n,1}^{n+1,n+1} = D_n^{n+1,n+1}$ is in bijection with $\mathbf{W}_n^\wedge$ and the number of nilpotent orbits is equal to $|\mathbf{W}_n^\wedge|$ by Spaltenstein [Spa1]. Hence the bijectivity of the map follows. In fact, given $(A, B) \in D_n^{n+1,n+1}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2s+1}$ be the sequence $a_1 \leq b_1 \leq \cdots \leq a_{s+1}$. If $c_{i+1} < c_i + (n+1)$, then $\{i, i+1\}$ is a block. We can recover $\lambda_i = \lambda_{i+1}$ and $\chi(\lambda_i) = \chi(\lambda_{i+1})$ from (2) of the definition. All blocks of length 2 are obtained in this way. For the other blocks, we can recover λ_i and thus $\chi(\lambda_i) = \lambda_i/2$ from (1) of the definition.

(ii) One easily checks that (A, B) has no proper intervals. It follows that $V_{\rho(x)}^{n+1,n+1} = \{0\}$. On the other hand, $A(x) = 1$ since $Z_G(x)$ is connected by [Spa1]. □

3.8.2 Consider a pair $(x, \phi) \in \mathfrak{A}_\mathfrak{g}$, then $\phi = 1$.

Theorem 3.8.1. *The Springer correspondence* $\gamma_\mathfrak{g} : \mathfrak{A}_\mathfrak{g} \to \mathbf{W}_n^\wedge \cong X_n^{n+1,n+1}$ *is given by*

$$(x, 1) \mapsto \rho(x).$$

Remark. *Theorem rewrites the description of Springer correspondence given by Spaltenstein [Spa1] using pairs of partitions. Note that he works under the assumption that the theory of Springer representations is valid for* $\mathfrak{g}$ *in characteristic 2.*

3.8.3 Let c_{reg} be the nilpotent G-orbit in $\mathfrak{g}$ which is open dense in the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. Let c_0 be the 0 orbit.

Lemma 3.8.2. *The pair* $(\mathrm{c}_{reg}, \bar{\mathbb{Q}}_l)$ *corresponds to the unit representation and the pair* $(\mathrm{c}_{trivial}, \bar{\mathbb{Q}}_l)$ *corresponds to the sign representation.*

Proof. One can show that the Weyl group action on $H^i(\mathcal{B})$ defined in section 1.6 coincides with the classical action. Assume $\chi \in \mathbf{W}_n^\wedge$ correspond to the pair $(\mathrm{c}, \mathcal{F}) \in \mathfrak{A}_\mathfrak{g}$. We write $\chi = \chi_{(\mathrm{c},\mathcal{F})}$. Recall that we have the following decomposition $\varphi_! \bar{\mathbb{Q}}_{lX}|_{\mathcal{N}}[\dim G - n] = \bigoplus_{(\mathrm{c},\mathcal{F}) \in \mathfrak{A}_\mathfrak{g}} \chi_{(\mathrm{c},\mathcal{F})} \otimes IC(\bar{\mathrm{c}}, \mathcal{F})[\dim \mathrm{c}]$. Thus for $x \in \mathcal{N}$, $H^{2i}(\mathcal{B}_x, \bar{\mathbb{Q}}_l) = \bigoplus_{(\mathrm{c},\mathcal{F})} \chi_{(\mathrm{c},\mathcal{F})} \otimes (\mathcal{H}^{2i+\dim \mathrm{c} - \dim G + n} IC(\bar{\mathrm{c}}, \mathcal{F}))_x$. Taking $i = \frac{\dim G - n}{2}$ and $x = 0$, we get $H^{\dim G - n}(\mathcal{B}, \bar{\mathbb{Q}}_l) = \chi_{(\mathrm{c}_0, \bar{\mathbb{Q}}_l)}$ since $A(\mathrm{c}_0) = 1$. It follows that $\chi_{(\mathrm{c}_0, \bar{\mathbb{Q}}_l)}$ is the sign representation. Taking $i = 0$ and $x = 0$, we get $H^0(\mathcal{B}, \bar{\mathbb{Q}}_l) = \chi_{(\mathrm{c}_{reg}, \bar{\mathbb{Q}}_l)} \otimes (\mathcal{H}^0 IC(\overline{\mathrm{c}_{reg}}, \bar{\mathbb{Q}}_l))_0$ since $A(\mathrm{c}_{reg}) = 1$. It follows that $\chi_{(\mathrm{c}_{reg}, \bar{\mathbb{Q}}_l)}$ is the unit representation. □

Proof of Theorem 3.8.1. By the discussion in 3.2.10, it is enough to show that the map $\gamma_\mathfrak{g}$ is compatible with the restriction formula (**R**). When $n = 1$, by Lemma 3.8.2, the pair $(\mathrm{c}_{reg}, 1)$ corresponds to the unit representation and the pair $(\mathrm{c}_0, 1)$ corresponds to the sign representation. When $n = 2$, there are two representations of $\mathbf{W}_2$ restricting to unit representation and two representations of $\mathbf{W}_2$ restricting to sign representation. But again we know the pair $(\mathrm{c}_{reg}, 1)$ corresponds to the unit representation and the pair $(\mathrm{c}_0, 1)$ corresponds to the sign representation. When $n \geq 3$, we show that the map $\gamma_\mathfrak{g}$ is compatible with the restriction formula. Let $x \in \mathfrak{g}$ and $x' \in \mathfrak{l}$ be nilpotent elements. Note that we have $A_G(x) = A_L(x') = 1$. Hence it is enough to show that

$$\langle 1, \varepsilon_{x,x'} \rangle = \langle \mathrm{Res}_{\mathbf{W}_{n-1}}^{\mathbf{W}_n} \rho_G(x), \rho_L(x') \rangle_{\mathbf{W}_{n-1}}. \tag{3.18}$$

Note that $X_{n,d}^{n+1,n+1} = \emptyset$ if $d \neq 1$ is odd. Thus $X_n^{n+1,n+1} = X_{n,1}^{n+1,n+1}$. Let $(A, B) \in X_n^{n+1,n+1}$ correspond to $\chi \in \mathbf{W}_n^\wedge$. The pairs $(A', B') \in X_{n-1}^{n,n}$ which correspond to the components of the restriction of χ to $\mathbf{W}_{n-1}$ are those which can be deduced from (A, B) by decreasing one of the entries c_i by i and decreasing all other entries c_j by $j - 1$. This can be done if and only if $i \geq 3$ and $c_i - c_{i-2} \geq 2n + 3$, $i = 2$, $c_i \geq n + 2$ or $i = 1$, $c_1 \geq 1$. We write $(A, B) \to (A', B')$ if they are related in this way.

Now (3.18) follows since $S_{x,x'} \neq \emptyset$ if and only if x, x' are as in Proposition 3.8.3 (see below) if and only if $\rho_G(x) \to \rho_L(x')$. □

3.8.4 Consider a nilpotent class $\mathrm{c}' \in f_x(\mathcal{P}_x)$ corresponding to $(\lambda'_{2s+1})_{\chi'(\lambda'_{2s+1})} \cdots$

$(\lambda'_1)_{\chi'(\lambda'_1)} := (\lambda', \chi')$. Suppose $Y = f_x^{-1}(c')$ and $X = \varrho_x^{-1}(Y)$ (see 3.2.8).

Proposition 3.8.3 ([Spa1]). *The group $Z_G(x)$ acts transitively on Y. We have $\dim X = \dim \mathcal{B}_x$ if and only if (λ', χ') satisfies a) or b):*

a) Assume $\lambda_i - \lambda_{i-1} \geq 2$, $\chi(\lambda_i) = \lambda_i/2$ and $\chi(\lambda_j) \geq \lambda_j - \lambda_i/2 + 1$ for each $j < i$. $\lambda'_j = \lambda_j$, $j \neq i$, $\lambda'_i = \lambda_i - 2$, $\chi'(\lambda'_j) = \chi(\lambda_j)$ for each $j \neq i$ and $\chi'(\lambda'_i) = \lambda'_i/2$. In this case $\dim Y = 2s - i + 1$.

b) Assume $\lambda_{i+1} = \lambda_i > \lambda_{i-1}$. $\lambda'_j = \lambda_j, j \neq i, i+1$, $\lambda'_{i+1} = \lambda'_i = \lambda_i - 1$, $\chi'(\lambda'_j) = \chi(\lambda_j)$ for each $j \neq i, i+1$ and $\chi'(\lambda'_i) = \chi'(\lambda'_{i+1}) \in \{\chi(\lambda_i), \chi(\lambda_i) - 1\}$ satisfy $0 \leq \chi'(\lambda'_i) \leq \lambda'_i/2$, $\chi(\lambda_{i-1}) \leq \chi'(\lambda'_i) \leq \chi(\lambda_{i-1}) + \lambda_i - \lambda_{i-1} - 1$. We have $\dim Y = 2s - i + 1$ if $\chi'(\lambda'_i) = \chi(\lambda_i)$ and $\dim Y = 2s - i$ if $\chi'(\lambda'_i) = \chi(\lambda_i) - 1$.

3.9 Springer correspondence for orthogonal Lie algebras

3.9.1 In this subsection we assume $G = O(2n+1)$.

Let $x = (\lambda_{2s+1})_{\chi(\lambda_{2s+1})} \cdots (\lambda_1)_{\chi(\lambda_1)} \in \mathfrak{g}$ be a nilpotent element (see 3.4.1). Assume $\lambda_1 = 0$. There exists a unique $3 \leq m_0 \leq 2s+1$ such that m_0 is odd and $\lambda_{m_0} > \lambda_{m_0-1}$. We have $\chi(\lambda_j) = \lambda_j$ if $j \leq m_0$; $\lambda_{2j} = \lambda_{2j+1}$, $j \neq \frac{m_0-1}{2}$ and $\lambda_{m_0} = \lambda_{m_0-1} + 1$.

We attach to the orbit c of x the sequence $c_1, \ldots, c_{2s+1}$ defined as follows:

$$(1)\ c_{2j} = \begin{cases} \lambda_{2j} - \chi(\lambda_{2j}) + n + 1 + (j-1)(n+3) & \text{if } 2j < m_0 \\ \lambda_{2j} - \chi(\lambda_{2j}) + 1 + n + 1 + (j-1)(n+3) & \text{if } 2j \geq m_0 \end{cases}$$

$$(2)\ c_{2j-1} = \begin{cases} \chi(\lambda_{2j-1}) + (j-1)(n+3) & \text{if } 2j-1 < m_0 \\ \chi(\lambda_{2j-1}) - 1 + (j-1)(n+3) & \text{if } 2j-1 \geq m_0 \end{cases}.$$

Taking $a_i = c_{2i-1}$, $i = 1, \ldots, s+1$, $b_i = c_{2i}$, $i = 1, \ldots, s$, we get a well-defined element $(A, B) \in X^{2,n+1}_{n,1}$. We denote it $\rho_G(x)$, $\rho(x)$ or $\rho(\mathrm{c})$.

Lemma 3.9.1. (i) $\mathrm{c} \mapsto \rho(\mathrm{c})$ *defines a bijection from the set of all nilpotent $O(2n+1)$-orbits in $\mathfrak{o}(2n+1)$ to $D^{2,n+1}_n$.*

(ii) *$A_G(x)^\wedge$ is isomorphic to $V^{2,n+1}_{\rho(x)}$.*

Proof. (i) It is easily checked from the definition that $\rho(\mathrm{c}) \in D_n^{2,n+1}$ and the map $\mathrm{c} \mapsto \rho(\mathrm{c})$ is injective. Note that $D_n^{2,n+1}$ is in bijection with the set Δ consisting of all pairs of partitions (μ, ν) such that $\sum \mu_i + \sum \nu_i = n, \nu_i \leq \mu_i + 2$. Since the number of nilpotent orbits is equal to $|\Delta|$ by Spaltenstein [Spa1], the bijectivity of the map follows. In fact, given $(A, B) \in D_n^{2,n+1}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2s+1}$ be the sequence $a_1 \leq b_1 \leq \cdots \leq a_{s+1}$. There exists a unique odd integer m_0 such that $c_{2j} > (n+1) + (j-1)(n+3)$ if and only if $2j > m_0$. If $j < \frac{m_0-1}{2}$, then $\lambda_{2j} = \lambda_{2j+1} = \chi(\lambda_{2j}) = \chi(\lambda_{2j+1}) = c_{2j+1} - j(n+3)$. If $j > \frac{m_0-1}{2}$, then $\lambda_{2j} = \lambda_{2j+1} = c_{2j} + c_{2j+1} - (2j-1)(n+3) - (n+1)$ and $\chi(\lambda_{2j}) = \chi(\lambda_{2j+1}) = c_{2j+1} - j(n+3) + 1$. If $j = \frac{m_0-1}{2}$, then $\lambda_{2j} = \chi(\lambda_{2j}) = \lambda_{2j+1} - 1 = \chi(\lambda_{2j+1}) - 1 = c_{2j+1} - j(n+3) + 1$.

(ii) The component group $A_G(x)$ is described in 3.4.1. Let $(A, B) = \rho(x)$ and $c_1, \ldots, c_{2s+1}$ be as above. Let $S = (A \cup B) \backslash (A \cap B)$. Note that $c_1 = 0, c_2, \ldots, c_{m_0}$ all lie in S and they belong to the same interval, which is the initial interval. For $i > m_0$, $\chi(\lambda_i) \neq \lambda_i/2$ if and only if $c_i \in S$. The relations (r2) and (r3) of 3.4.1 say that if c_i, c_j belong to the same interval of (A, B), then ϵ_i, ϵ_j have the same images in $A(x)$. Thus we get an element σ_I of $A(x)$ for each interval I of (A, B) and $\sigma_I^2 = 1$. Moreover (r3) means that $\sigma_I = 1$ if I is the initial interval.

The isomorphism $V_{\rho(x)}^{2,n+1} \to A_G(x)^\wedge$ is given as follows. Let $F \in V_{\rho(x)}^{2,n+1}$. We associate to F the character of $A_G(x)$ which takes value -1 on σ_I if and only if $I \in F$. □

Let $(x, \phi) \in \mathfrak{A}_\mathfrak{g}$. We have defined $\rho(x)$. Let ρ denote also the map $A_G(x)^\wedge \to V_{\rho(x)}^{2,n+1}$.

Theorem 3.9.2. *The Springer correspondence $\gamma_\mathfrak{g} : \mathfrak{A}_\mathfrak{g} \to \mathbf{W}_n^\wedge \cong X_n^{2,n+1}$ is given by*

$$(x, \phi) \mapsto \rho(x)_{\rho(\phi)}.$$

Proof. As in the proof of Theorem 3.8.1, it is enough to prove the map $\gamma_\mathfrak{g}$ is compatible with the restriction formula (**R**). Note that $X_{n,d}^{2,n+1} = \emptyset$ if $d \neq 1$ is odd. Thus $X_n^{2,n+1} = X_{n,1}^{2,n+1}$. Let $(A, B) \in X_n^{2,n+1}$ correspond to $\chi \in \mathbf{W}_n^\wedge$. The pairs $(A', B') \in X_{n-1}^{2,n}$ which correspond to the components of the restriction of χ to $\mathbf{W}_{n-1}$ are those which can be deduced from (A, B) by decreasing one of the entries a_i by i (or b_i by $i+1$) and decreasing all other entries a_j by $j-1$, b_j by j. We can decrease a_i by i (resp. b_i by $i+1$) if and only if $i \geq 2$, $a_i - a_{i-1} \geq n+4$ or $i=1$, $a_i \geq 1$ (resp. $i \geq 2$, $b_i - b_{i-1} \geq n+4$ or $i=1$, $b_i \geq n+2$). We write $(A,B) \to (A',B')$ if they are related in this way. Suppose that $(A,B) \to (A',B')$. One can easily check that if (A,B) and (A',B') are similar to $\Lambda \in D_n^{2,n+1}$ and $\Lambda' \in D_{n-1}^{2,n+1}$ respectively, then $\Lambda \to \Lambda'$.

Let $x \in \mathfrak{g}$ nilpotent and $x' \in \mathfrak{l}$ nilpotent. Then $S_{x,x'} \neq \emptyset$ if and only if x, x' are as in Proposition 3.4.1 if and only if $\Lambda = \rho_G(x) \to \Lambda' = \rho_L(x')$. To verify the map is compatible with the restriction formula, it is enough to show that the set

$$\{(F,F') \in V^{2,n+1}_{\rho_G(x)} \times V^{2,n}_{\rho_L(x')} | \Lambda_F \to \Lambda'_{F'}\} \tag{3.19}$$

is the image of the set

$$\{(\phi,\phi') \in A_G(x)^\wedge \times A_L(x')^\wedge | \langle \phi \otimes \phi', \varepsilon_{x,x'} \rangle \neq 0\} \tag{3.20}$$

under the map ρ.

Let $c_1 \leq \cdots \leq c_{2s+1}$ and $c'_1 \leq \cdots \leq c'_{2s+1}$ correspond to Λ and Λ' respectively. Then $A_G(x)$ is generated by $\{\epsilon_i | c_i \neq c_j, \forall j \neq i\}$, $A_L(x')$ is generated by $\{\epsilon'_i | c'_i \neq c'_j, \forall j \neq i\}$ and A_P is generated by $\{\epsilon_i | c_i \neq c_j, c'_i \neq c'_j, \forall j \neq i\}$. There are various cases to consider. We describe one of the cases in the following and the other cases are similar.

Assume $c_{2k+1} > c_{2k}+1$, $c_{2k+2} = c_{2k+1}+n+2$ and $c'_{2k+1} = c_{2k+1}-(k+1)$, $c'_{2i+1} = c_{2i+1} - i, i \neq k$, $c'_{2i} = c_{2i} - i$, $i = 1, \ldots, s$. Let I (resp. I') be the interval of Λ (resp. Λ') containing c_{2k+1} (resp. c'_{2k+1}) and J' the interval of Λ' containing c'_{2k+2}. Note that $c_{2j+1} - c_{2j} < n+2$, except if $x = (n+1)_{n+1} n_n$. In the latter case $A(x) = 1$. Moreover $c_{2j+2} - c_{2j+1} \geq 2$. Hence all other intervals of Λ and Λ' can be identified naturally. There are two possibilities:

(i) I is a proper interval of Λ. Then $\Lambda_F \to \Lambda_{F'}$ if and only if

(a) $F \backslash \{I\} = F' \backslash \{I', J'\}$;

(b) $F \cap \{I\} = F' \cap \{I', J'\} = \emptyset$ or $\{I\} \subset F, \{I', J'\} \subset F'$.

On the other hand, $A_G(x)$ (resp. $A_L(x')$) is an $\mathbf{F}_2$ vector space with one basis element σ_K (resp. σ'_K) for each proper interval K of Λ (resp. Λ') and $S_{x,x'}$ is the quotient of $A_G(x) \times A_L(x')$ by the subgroup $H_{x,x'}$ generated by elements of the form

$\sigma_I\sigma'_{I'}, \sigma_I\sigma'_{J'}, \sigma_K\sigma'_K$ with K a proper interval of both Λ and Λ'. Now the compatibility between (3.19) and (3.20) is clear.

(ii) I is an initial interval of Λ. Then $\Lambda_F \to \Lambda_{F'}$ if and only if $F = F'$. On the other hand, $A_G(x)$, $A_L(x')$ and $S_{x,x'}$ are obtained by setting $\sigma_I = \sigma'_{I'} = 1$ in (i). Again the compatibility between (3.19) and (3.20) is clear. □

3.9.2 In this subsection we assume $G = SO(2n)$, $\tilde{G} = O(2n)$ and $\mathfrak{g} = \mathfrak{o}(2n)$. We describe $\tilde{\gamma}_\mathfrak{g} : \tilde{\mathfrak{A}}_\mathfrak{g} \to \mathbf{W}_n^{\wedge\prime}$ instead of $\gamma_\mathfrak{g} : \mathfrak{A}_\mathfrak{g} \to (\mathbf{W}'_n)^\wedge$ (see 3.2.4).

Let $x = (\lambda_{2s})_{\chi(\lambda_{2s})} \cdots (\lambda_1)_{\chi(\lambda_1)} \in \mathfrak{g}$ be a nilpotent element (see 3.4.1). Note that $\lambda_{2i-1} = \lambda_{2i}$. We attach to the orbit c of x the sequence $c_1, \dots, c_{2s}$ defined as follows:

(1) $c_{2j} = \chi(\lambda_{2j}) + (j-1)(n+1)$,

(2) $c_{2j-1} = \lambda_{2j-1} - \chi(\lambda_{2j-1}) + (j-1)(n+1)$.

Taking $a_i = c_{2i-1}$, $b_i = c_{2i}$, $i = 1, \dots, s$, we get a well defined element $\{A, B\} \in Y_{n,0}^{n+1}$. We denote it $\rho_G(x)$, $\rho(x)$ or $\rho(\mathrm{c})$.

Lemma. (i) $\mathrm{c} \mapsto \rho(\mathrm{c})$ *defines a bijection from the set of all nilpotent $O(2n)$-orbits in $\mathfrak{o}(2n)$ to $D_{n,even}^{n+1}$.*

(ii) $A_G(x)^\wedge$ *is isomorphic to* $V_{\rho(x)}^{n+1}$.

Proof. (i) It is easily checked from the definition that $\rho(\mathrm{c}) \in D_{n,\text{even}}^{n+1}$ and the map $\mathrm{c} \mapsto \rho(\mathrm{c})$ is injective. Note that $D_{n,\text{even}}^{n+1}$ is in bijection with the set Δ consisting of all pairs of partitions (μ, ν) such that $\sum \mu_i + \sum \nu_i = n, \nu_i \le \mu_i$. Since the number of nilpotent $O(2n)$-orbits in $\mathfrak{o}(2n)$ is equal to $|\Delta|$ by Spaltenstein [Spa1], the bijectivity of the map follows. In fact, given $\{A, B\} \in D_{n,\text{even}}^{n+1}$ with preimage $(A, B) \in D_{n,0}^{n+1,0}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \le c_2 \le \cdots \le c_{2s}$ be the sequence $a_1 \le b_1 \le \cdots \le a_s \le b_s$. We have $\lambda_{2j} = \lambda_{2j-1} = c_{2j} + c_{2j-1} - (2j-2)(n+1)$ and $\chi(\lambda_{2j}) = \chi(\lambda_{2j-1}) = c_{2j} - (j-1)(n+1)$.

(ii) The component group $A_G(x)$ is described in 3.4.1. Note that in this case, the condition (r3) is void. By similar argument as in the proof of Lemma 3.9.1 (ii), one shows that $A_{\tilde{G}}(x)$ is a vector space over $\mathbf{F}_2$ with basis $(\sigma_I)_{I \in E}$, where E is the set of all intervals of $\rho(x)$. Since $A_G(x)$ consists of the elements in $A_{\tilde{G}}(x)$ which can be written as a product of even number of generators, from the natural identification $A_{\tilde{G}}(x)^\wedge = \mathcal{A}(E)$, we get the isomorphism $A_G(x)^\wedge \cong \mathcal{A}(E)/\{\emptyset, E\} = V^{n+1}_{\rho(x)}$. □

Let $(x, \phi) \in \tilde{\mathfrak{A}}_\mathfrak{g}$. We have defined $\rho(x)$. Let ρ denote also the map $A_G(x)^\wedge \to V^{n+1}_{\rho(x)}$.

Theorem 3.9.3. *The Springer correspondence* $\tilde{\gamma}_\mathfrak{g} : \tilde{\mathfrak{A}}_\mathfrak{g} \to \mathbf{W}_n^{\wedge\prime} \cong Y^{n+1}_{n,even}$ *is given by*

$$(x, \phi) \mapsto \rho(x)_{\rho(\phi)}.$$

Proof. Again it is enough to prove the map $\tilde{\gamma}_\mathfrak{g}$ is compatible with the restriction formula (**R**). Note that $Y^{n+1}_{n,d} = \emptyset$ if $d > 0$ is even. Thus $Y^{n+1}_{n,\text{even}} = Y^{n+1}_{n,0}$. Let $\{A, B\} \in Y^{n+1}_{n,\text{even}}$ correspond to $\chi \in \mathbf{W}_n^{\wedge\prime}$. The pairs $\{A', B'\} \in Y^n_{n-1,\text{even}}$ which correspond to the components of the restriction of χ to $\mathbf{W}'_{n-1}$ are those which can be deduced from $\{A, B\}$ by decreasing one of the entries a_i by i (or b_i by i) and decreasing all other entries a_j by $j-1$, b_j by $j-1$. We can decrease a_i by i (resp. b_i by i) if and only if $i \geq 2$, $a_i - a_{i-1} \geq n+2$ or $i = 1$, $a_i \geq 1$ (resp. $i \geq 2$, $b_i - b_{i-1} \geq n+2$ or $i = 1$, $b_i \geq 1$). We write $\{A, B\} \to \{A', B'\}$ if they are related in this way. Suppose that $\{A, B\} \to \{A', B'\}$. One can easily check that if $\{A, B\}$ and $\{A', B'\}$ are similar to $\Lambda \in D^{n+1}_{n,\text{even}}$ and $\Lambda' \in D^n_{n-1,\text{even}}$ respectively, then $\Lambda \to \Lambda'$.

Let $x \in \mathfrak{g}$ nilpotent and $x' \in \mathfrak{l}$ nilpotent. Then $\tilde{S}_{x,x'} \neq \emptyset$ ($\Leftrightarrow S_{x,x'} \neq \emptyset$) if and only if x, x' are as in Proposition 3.4.1 if and only if $\Lambda = \rho_G(x) \to \Lambda' = \rho_L(x')$. Let $c_1 \leq \cdots \leq c_{2s}$ and $c'_1 \leq \cdots \leq c'_{2s}$ correspond to Λ and Λ' respectively. Then $A_{\tilde{G}}(x)$ is generated by $\{\epsilon_i | c_i \neq c_j, \forall j \neq i\}$, $A_{\tilde{L}}(x')$ is generated by $\{\epsilon'_i | c'_i \neq c'_j, \forall j \neq i\}$ and $\tilde{A}_P$ is generated by $\{\epsilon_i | c_i \neq c_j, c'_i \neq c'_j, \forall j \neq i\}$. The discussion in 3.2.9 allows us to compute $\varepsilon_{x,x'}$ and the set $\{(\phi, \phi') \in A_G(x)^\wedge \times A_L(x')^\wedge | \langle \phi \otimes \phi', \varepsilon_{x,x'} \rangle \neq 0\}$. One verifies the compatibility with the set $\{(F, F') \in V^{n+1}_{\rho_G(x)} \times V^n_{\rho_L(x')} | \Lambda_F \to \Lambda'_{F'}\}$ under the map ρ as in the proof of Theorem 3.9.2. □

3.10 Springer correspondence for duals of symplectic and odd orthogonal Lie algebras

3.10.1 We assume $G = Sp(2n)$ in this subsection. Let $\xi = (\lambda_{2s+1})_{\chi(\lambda_{2s+1})} \cdots (\lambda_1)_{\chi(\lambda_1)} \in \mathfrak{g}^*$ be nilpotent (see 3.5.1), where $\lambda_1 = 0$. We have $\lambda_{2j} = \lambda_{2j+1}$. We attach to the orbit c of ξ the sequence $c_1, \ldots, c_{2s+1}$ defined as follows:

(1) $c_{2j} = \lambda_{2j} - \chi(\lambda_{2j}) + n + 1 + (j-1)(n+2)$,

(2) $c_{2j-1} = \chi(\lambda_{2j-1}) + (j-1)(n+2)$.

Taking $a_i = c_{2i-1}$, $i = 1, \ldots, s+1$, $b_i = c_{2i}$, $i = 1, \ldots, s$, we get a well-defined $(A, B) \in X_{n,1}^{1,n+1}$. We denote it $\rho_G(\xi)$, $\rho(\xi)$ or $\rho(\mathrm{c})$.

Lemma. (i) $\mathrm{c} \mapsto \rho(\mathrm{c})$ *defines a bijection from the set of all nilpotent $Sp(2n)$-orbits in $\mathfrak{sp}(2n)^*$ to $D_n^{1,n+1}$.*

(ii) *$A_G(\xi)^\wedge$ is isomorphic to $V_{\rho(\xi)}^{1,n+1}$.*

Proof. (i) It is easily checked from the definition that $\rho(\mathrm{c}) \in D_n^{1,n+1}$ and the map $\mathrm{c} \mapsto \rho(\mathrm{c})$ is injective. Note that $D_n^{1,n+1}$ is in bijection with the set Δ consisting of all pairs of partitions (μ, ν) such that $\sum \mu_i + \sum \nu_i = n, \nu_i \leq \mu_i + 1$. Since the number of nilpotent orbits is equal to $|\Delta|$ by Corollary 2.5.7, the bijectivity of the map follows. In fact, given $(A, B) \in D_n^{1,n+1}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2s+1}$ be the sequence $a_1 \leq b_1 \leq \cdots \leq a_{s+1}$. Then $\lambda_{2j} = \lambda_{2j+1} = c_{2j} + c_{2j+1} - (2j-1)(n+2) - (n+1)$ and $\chi(\lambda_{2j}) = \chi(\lambda_{2j+1}) = c_{2j+1} - j(n+2)$, $j = 1, \ldots, s$, $\lambda_1 = 0$.

(ii) The component group $A_G(\xi)$ is described in 3.5.1. Let $(A, B) = \rho(\xi)$ and $c_1, \ldots, c_{2s+1}$ be as above. Let $S = (A \cup B) \backslash (A \cap B)$. Then $\chi(\lambda_i) \neq (\lambda_i - 1)/2$ if and only if $c_i \in S$. The relation (r2) of 3.5.1 says that if c_i, c_j belong to the same interval of (A, B), then ϵ_i, ϵ_j have the same images in $A_G(\xi)$. Thus we get an element σ_I of $A_G(\xi)$ for each interval I of (A, B) and $\sigma_I^2 = 1$. Moreover (r3) means that $\sigma_I = 1$ if I is the initial interval.

The isomorphism $V_{\rho(\xi)}^{1,n+1} \rightarrow A_G(\xi)^\wedge$ is given as follows. We associate to F the character of $A_G(\xi)$ which takes value -1 on σ_I if and only if $I \in F$. □

Let $(\xi, \phi) \in \mathfrak{A}_{\mathfrak{g}^*}$. We have defined $\rho(\xi)$. Let ρ denote also the map $A_G(\xi)^\wedge \rightarrow V_{\rho(\xi)}^{1,n+1}$.

Theorem 3.10.1. *The Springer correspondence $\gamma_{\mathfrak{g}^*} : \mathfrak{A}_{\mathfrak{g}^*} \rightarrow \mathbf{W}_n^\wedge \cong X_n^{1,n+1}$ is given by*

$$(\xi, \phi) \mapsto \rho(\xi)_{\rho(\phi)}.$$

Proof. By similar argument as in the proof of Theorem 3.8.1, it is enough to prove the map $\gamma_{\mathfrak{g}^*}$ is compatible with the restriction formula $(\mathbf{R}')$. Note that $X_{n,d}^{1,n+1} = \emptyset$ if

$d \neq 1$ is odd. Thus $X_n^{1,n+1} = X_{n,1}^{1,n+1}$. Let $(A,B) \in X_n^{1,n+1}$ correspond to $\chi \in \mathbf{W}_n^\wedge$. The pairs $(A',B') \in X_{n-1}^{1,n}$ which correspond to the components of the restriction of χ to $\mathbf{W}_{n-1}$ are those which can be deduced from (A,B) by decreasing one of the entries a_i by i (or b_i by $i+1$) and decreasing all other entries a_j by $j-1$, b_j by j. We can decrease a_i by i (resp. b_i by $i+1$) if and only if $i \geq 2$, $a_i - a_{i-1} \geq n+3$ or $i=1$, $a_i \geq 1$ (resp. $i \geq 2$, $b_i - b_{i-1} \geq n+3$ or $i=1$, $b_i \geq n+2$). We write $(A,B) \to (A',B')$ if they are related in this way. Suppose that $(A,B) \to (A',B')$. One can easily check that if (A,B) and (A',B') are similar to $\Lambda \in D_n^{1,n+1}$ and $\Lambda' \in D_{n-1}^{1,n}$ respectively, then $\Lambda \to \Lambda'$.

Let $\xi \in \mathfrak{g}^*$ and $\xi' \in \mathfrak{l}^*$ be nilpotent. Then $S_{\xi,\xi'} \neq \emptyset$ if and only if ξ, ξ' are as in Proposition 3.5.2 if and only if $\Lambda = \rho_G(\xi) \to \Lambda' = \rho_L(\xi')$. The verification of compatibility with restriction formula is entirely similar to that in Theorem 3.9.2. □

3.10.2 We assume $G = O(2n+1)$ in this subsection.

Let $\xi = (m; (\lambda_{2s})_{\chi(\lambda_{2s})} \cdots (\lambda_1)_{\chi(\lambda_1)}) \in \mathfrak{g}^*$ be nilpotent (see 3.6.1). We have $\lambda_{2j-1} = \lambda_{2j}$. We attach to the orbit c of ξ the sequence $c_1, \ldots, c_{2s+1}$ defined as follows:

(1) $c_{2j} = \chi(\lambda_{2j}) + (j-1)(n+1)$, $j = 1, \ldots, s$

(2) $c_{2j-1} = \lambda_{2j-1} - \chi(\lambda_{2j-1}) + (j-1)(n+1)$, $j = 1, \ldots, s$,

(3) $c_{2s+1} = m + s(n+1)$.

Taking $a_i = c_{2i-1}$, $i = 1, \ldots, s+1$, $b_i = c_{2i}$, $i = 1, \ldots, s$, we get a well defined element $\{A,B\} \in Y_{n,1}^{n+1}$. We denote it $\rho_G(\xi)$, $\rho(\xi)$ or $\rho(\mathrm{c})$.

Lemma 3.10.2. (i) $\mathrm{c} \mapsto \rho(\mathrm{c})$ *defines a bijection from the set of all nilpotent* $O(2n+1)$*-orbits in* $\mathfrak{o}(2n+1)^*$ *to* $D_{n,odd}^{n+1}$.

(ii) $A_G(\xi)^\wedge$ *is isomorphic to* $V_{\rho(\xi)}^{n+1}$.

Proof. (i) It is easily checked from the definition that $\rho(\mathrm{c}) \in D_{n,\mathrm{odd}}^{n+1}$ and the map $\mathrm{c} \mapsto \rho(\mathrm{c})$ is injective. Note that $D_{n,\mathrm{odd}}^{n+1}$ is in bijection with the set Δ consisting of all pairs of partitions (μ,ν) such that $\sum \mu_i + \sum \nu_i = n$, $\mu_{i+1} \leq \nu_i$. Since the number of nilpotent orbits is equal to $|\Delta|$ by Corollary 2.5.7, the bijectivity of the map follows. In fact, given $\{A,B\} \in D_{n,\mathrm{odd}}^{n+1}$ with inverse image $(A,B) \in D_{n,1}^{n+1,0}$, the corresponding nilpotent orbit can be obtained as follows. Let $c_1 \leq c_2 \leq \cdots \leq c_{2s+1}$

be the sequence $a_1 \leq b_1 \leq \cdots \leq a_{s+1}$. Then $\lambda_{2j} = \lambda_{2j-1} = c_{2j} + c_{2j-1} - (2j-2)(n+1)$, $\chi(\lambda_{2j}) = \chi(\lambda_{2j-1}) = c_{2j} - (j-1)(n+1)$, $j = 1, \ldots, s$ and $m = c_{2s+1} - s(n+1)$. The corresponding orbit is $(m; (\lambda_{2s})_{\chi(\lambda_{2s})} \cdots (\lambda_1)_{\chi(\lambda_1)})$.

(ii) The component group $A_G(\xi)$ is described in 3.6.1. Let $\{A, B\} = \rho(\xi)$, (A, B) and $c_1, \ldots, c_{2s+1}$ be as above. Let $S = (A \cup B) \backslash (A \cap B)$. Note that $c_{2s} < c_{2s+1}$, thus $c_{2s+1} \in S$. For $i = 1, \ldots, 2s$, $\chi(\lambda_i) \neq \lambda_i/2$ if and only if $c_i \in S$. The relation (r2) of 3.6.1 says that for $1 \leq i < j \leq 2s$, if c_i, c_j belong to the same interval of $\{A, B\}$, then ϵ_i, ϵ_j have the same images in $A_G(\xi)$. Let I_0 be the interval containing c_{2s+1}. The relation (r3) says that $\epsilon_i = 1$ if $c_i \in I_0$. Thus we get an element σ_I of $A_G(\xi)$ for each interval $I \neq I_0$ of $\{A, B\}$ and $\sigma_I^2 = 1$.

The isomorphism $V_{\rho(\xi)}^{n+1} \to A_G(\xi)^\wedge$ is given as follows. Let $F \in V_{\rho(\xi)}^{n+1} = \mathcal{A}(E)/\{\emptyset, E\}$ and $\tilde{F}$ the inverse image of F in $\mathcal{A}(E)$ that does not contain I_0. We associate to F the character of $A_G(\xi)$ which takes value -1 on σ_I if and only if $I \in \tilde{F}$. □

Let $(\xi, \phi) \in \mathfrak{A}_{\mathfrak{g}^*}$. We have defined $\rho(\xi)$. Let ρ denote also the map $A_G(\xi)^\wedge \to V_{\rho(\xi)}^{n+1}$.

Theorem 3.10.3. *The Springer correspondence $\gamma_{\mathfrak{g}^*} : \mathfrak{A}_{\mathfrak{g}^*} \to \mathbf{W}_n^\wedge \cong Y_{n,odd}^{n+1}$ is given by*

$$(\xi, \phi) \mapsto \rho(\xi)_{\rho(\phi)}.$$

Proof. Again it is enough to prove the map $\gamma_{\mathfrak{g}^*}$ is compatible with the restriction formula ($\mathbf{R}'$). Note that $Y_{n,d}^{n+1} = \emptyset$ if $d \neq 1$ is odd. Thus $Y_{n,\text{odd}}^{n+1} = Y_{n,1}^{n+1}$. Let $\{A, B\} \in Y_{n,\text{odd}}^{n+1}$ with inverse image $(A, B) \in X_{n,1}^{n+1,0}$ correspond to $\chi \in \mathbf{W}_n^\wedge$. The pairs $\{A', B'\} \in Y_{n-1,\text{odd}}^{n}$ with inverse images $(A', B') \in X_{n-1,1}^{n,0}$ which correspond to the components of the restriction of χ to $\mathbf{W}_{n-1}$ are those which can be deduced from (A, B) by decreasing one of the entries a_i by i (or b_i by i) and decreasing all other entries a_j by $j-1$, b_j by $j-1$. We can decrease a_i by i (resp. b_i by i) if and only if $i \geq 2$, $a_i - a_{i-1} \geq n+2$ or $i = 1$, $a_i \geq 1$ (resp. $i \geq 2$, $b_i - b_{i-1} \geq n+2$ or $i = 1$, $b_i \geq 1$). We write $\{A, B\} \to \{A', B'\}$ if they are related in this way. Suppose that $\{A, B\} \to \{A', B'\}$. One can easily check that if $\{A, B\}$ and $\{A', B'\}$ are similar to $\Lambda \in D_{n,\text{odd}}^{n+1}$ and $\Lambda' \in D_{n-1,\text{odd}}^{n}$ respectively, then $\Lambda \to \Lambda'$.

Let $\xi \in \mathfrak{g}^*$ and $\xi' \in \mathfrak{l}^*$ be nilpotent. Then $S_{\xi,\xi'} \neq \emptyset$ if and only if ξ, ξ' are as in Proposition 3.6.2 if and only if $\Lambda = \rho_G(\xi) \to \Lambda' = \rho_L(\xi')$. To verify the map is compatible with the restriction formula, it is enough to show that the set

$$\{(F,F') \in V^{n+1}_{\rho_G(\xi)} \times V^n_{\rho_L(\xi')} | \Lambda_F \to \Lambda'_{F'}\} \tag{3.21}$$

is the image of the set

$$\{(\phi,\phi') \in A_G(\xi)^\wedge \times A_L(\xi')^\wedge | \langle \phi \otimes \phi', \varepsilon_{\xi,\xi'} \rangle \neq 0\} \tag{3.22}$$

under the map ρ.

Let $c_1 \leq \cdots \leq c_{2s+1}$ and $c'_1 \leq \cdots \leq c'_{2s+1}$ correspond to the pre-image (A,B) and (A',B') of Λ and Λ' in $D^{n+1,0}_{n,1}$ and $D^{n,0}_{n-1,1}$ respectively. Then $A_G(\xi)$ is generated by $\{\epsilon_i | c_i \neq c_j, \forall j \neq i\}$, $A_L(\xi')$ is generated by $\{\epsilon'_i | c'_i \neq c'_j, \forall j \neq i\}$ and A_P is generated by $\{\epsilon_i | c_i \neq c_j, c'_i \neq c'_j, \forall j \neq i\}$. There are various cases to consider. We describe one of the cases in the following and the other cases are similar.

Assume $k \geq 1$, $c_{2k} > c_{2k-1}+1$, $c_{2k+1} = c_{2k}+n$ and $c'_{2k} = c_{2k}-k$, $c'_{2i} = c_{2i}-(i-1)$, $i \neq k$, $c'_{2i+1} = c_{2i+1} - i$, $i = 1,\ldots,s$. Let I (resp. I') be the interval of Λ (resp. Λ') containing c_{2k+1} (resp. c'_{2k+1}) and J' the interval of Λ' containing c'_{2k}. Note that $c_{2j} - c_{2j-1} = 2\chi(\lambda_{2j}) - \lambda_{2j} < n$ and $c_{2j+1} - c_{2j} = n+1+\lambda_{2j+1} - \chi(\lambda_{2j+1}) - \chi(\lambda_{2j}) \geq 2$, except if $m = 0$, $\chi(\lambda_{2s}) = \lambda_{2s} = n$. In the latter case, ξ' correspond to $m' = 0$, $\chi'(\lambda'_{2s}) = \lambda'_{2s} = n-1$ and $A_G(\xi) = A_L(\xi') = 1$. Hence all other intervals of Λ and Λ' can be identified naturally. Let I_0 (resp. I'_0) be the interval of Λ (resp. Λ') containing c_{2s+1} (resp. c'_{2s+1}) There are two possibilities:

(i) $I \neq I_0$. Let $\tilde{F}$ (resp. $\tilde{F}'$) be the pre-image of F (resp. F') in $\mathcal{A}(E)$ (resp. $\mathcal{A}(E')$) that does not contain I_0 (resp. I'_0). Then $\Lambda_F \to \Lambda_{F'}$ if and only if

(a) $\tilde{F} \backslash \{I\} = \tilde{F}' \backslash \{I', J'\}$;

(b) $\tilde{F} \cap \{I\} = \tilde{F}' \cap \{I', J'\} = \emptyset$ or $\{I\} \subset \tilde{F}, \{I', J'\} \subset \tilde{F}'$.

On the other hand, $A_G(\xi)$ (resp. $A_L(\xi')$) is an $\mathbf{F}_2$ vector space with one basis element σ_K (resp. σ'_K) for each interval $K \neq I_0$ (resp. $K \neq I'_0$) of Λ (resp. Λ') and $S_{\xi,\xi'}$ is the quotient of $A_G(\xi) \times A_L(\xi')$ by the subgroup $H_{\xi,\xi'}$ generated by elements of the form $\sigma_I \sigma'_{I'}, \sigma_I \sigma'_{J'}, \sigma_K \sigma'_K$ with $K \neq I_0$ an interval of both Λ and Λ'. Now the compatibility between (3.21) and (3.22) is clear.

(ii) $I = I_0$. Then $\Lambda_F \to \Lambda_{F'}$ if and only if $\tilde{F} = \tilde{F}'$. On the other hand, $A_G(\xi)$,

$A_L(\xi')$ and $S_{\xi,\xi'}$ are obtained by setting $\sigma_I = \sigma'_{I'} = 1$ in (i). Again the compatibility between (3.21) and (3.22) is clear. $\square$

3.11 Complement

3.11.1 In [L7], Lusztig gives an a priori description of the Weyl group representations that correspond to the pairs (c, 1) under Springer correspondence, where c is a unipotent class in G or a nilpotent orbit in $\mathfrak{g}$. We list the results of [L7] here.

Let R be a root system of type B_n, C_n or D_n with Π a set of simple roots and $\mathbf{W}$ the weyl group. There exists a unique $\alpha_0 \in R\backslash\Pi$ such that $\alpha - \alpha_i \notin R, \forall\ \alpha_i \in \Pi$. Let $J \subset \Pi \cup \{\alpha_0\}$ be such that $J = |\Pi|$. Let $\mathbf{W}_J$ be the subgroup of $\mathbf{W}$ generated by $s_\alpha, \alpha \in J$.

(i) Denote $\mathcal{S}_\mathbf{W}$ the set of special representations of $\mathbf{W}$. The set of unipotent classes when char($\mathbf{k}$) $\neq 2$ is in bijection with the set $\mathcal{S}^1_\mathbf{W} = \{j^\mathbf{W}_{\mathbf{W}^*_J} E,\ E \in \mathcal{S}_{\mathbf{W}^*_J}\}$ (see [L2, L3]), where $\mathbf{W}^*_J$ is defined as $\mathbf{W}_J$ by taking α_0 such that $\check{\alpha}_0 - \check{\alpha}_i \notin \check{R}, \forall\ \alpha_i \in \Pi$ ($\check{R}$ is the coroot lattice and $\check{\alpha}_0, \check{\alpha}_i$ are coroots).

(ii) The set of unipotent classes when char($\mathbf{k}$) $= 2$ is in bijection with the set $\mathcal{S}^2_\mathbf{W} = \{j^\mathbf{W}_{\mathbf{W}_J} E,\ E \in \mathcal{S}^1_{\mathbf{W}_J}\}$ (see [L6]).

(iii) The set of nilpotent classes when char($\mathbf{k}$) $= 2$ is in bijection with the set $\mathcal{T}^2_\mathbf{W}$ defined by induction on $|\mathbf{W}|$ as follows (see [L7]). If $\mathbf{W} = \{1\}$, $\mathcal{T}^2_\mathbf{W} = \mathbf{W}^\wedge$. If $\mathbf{W} \neq \{1\}$, then $\mathcal{T}^2_\mathbf{W}$ is the set of all $E \in \mathbf{W}^\wedge$ such that either $E \in \mathcal{S}^1_\mathbf{W}$ or $E = j^\mathbf{W}_{\mathbf{W}_J} E_1$ for some $\mathbf{W}_J \neq \mathbf{W}$ and some $E_1 \in \mathcal{T}^2_{\mathbf{W}_J}$.

3.11.2 One can show that the set of nilpotent orbits in $\mathfrak{g}^*$ when char($\mathbf{k}$) $= 2$ is in bijection with the set $\mathcal{T}^{2*}_\mathbf{W}$ defined by induction on $|\mathbf{W}|$ as follows. If $\mathbf{W} = \{1\}$, $\mathcal{T}^{2*}_\mathbf{W} = \mathbf{W}^\wedge$. If $\mathbf{W} \neq \{1\}$, then $\mathcal{T}^{2*}_\mathbf{W}$ is the set of all $E \in \mathbf{W}^\wedge$ such that either $E \in \mathcal{S}^1_\mathbf{W}$ or $E = j^\mathbf{W}_{\mathbf{W}^*_J} E_1$ for some $\mathbf{W}^*_J \neq \mathbf{W}$ and some $E_1 \in \mathcal{T}^{2*}_{\mathbf{W}^*_J}$, where $\mathbf{W}^*_J$ is defined as in 3.11.1 (i).

Chapter 4

Dual of Exceptional Lie Algebras

4.1 Introduction

Let G be a connected semisimple algebraic group defined over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \geq 1$ and $\mathfrak{g}$ the Lie algebra of G. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. We show that the number of nilpotent G-orbits in $\mathfrak{g}^*$ is finite. It suffices to assume that G is simple and simply connected. The nilpotent orbits for type B, C or D have been studied in chapter 2. In this chapter we study the nilpotent G-orbits in $\mathfrak{g}^*$ when G is an exceptional group of type G_2, F_4 or E_n $(n = 6, 7, 8)$.

Denote $\mathcal{O}_{\mathfrak{g}^*}$ the set of nilpotent G-orbits in $\mathfrak{g}^*$ and $\mathfrak{A}_{\mathfrak{g}^*}$ the set of all pairs $(\mathrm{c}, \mathcal{F})$ where $\mathrm{c} \in \mathcal{O}_{\mathfrak{g}^*}$ and $\mathcal{F}$ is an irreducible G-equivariant local system on c (up to isomorphism). Denote W the Weyl group of G and $W^\wedge$ the set of irreducible representations of W (up to isomorphism). In chapter 2, we have constructed a Springer correspondence $\gamma_{\mathfrak{g}^*} : W^\wedge \to \mathfrak{A}_{\mathfrak{g}^*}$ when G is of type B, C or D and $p = 2$. The same construction gives an injective map $\gamma_{\mathfrak{g}^*} : W^\wedge \to \mathfrak{A}_{\mathfrak{g}^*}$ for G an exceptional group. Note that the pairs $(\mathrm{c}, \bar{\mathbb{Q}}_l)$ lie in the image of the map $\gamma_{\mathfrak{g}^*}$. Let $\mathfrak{R}(\mathfrak{g}^*)$ be the inverse image of the set $\{(\mathrm{c}, \bar{\mathbb{Q}}_l) | \mathrm{c} \in \mathcal{O}_{\mathfrak{g}^*}\}$ under the map $\gamma_{\mathfrak{g}^*}$. We describe the set $\mathfrak{R}(\mathfrak{g}^*)$ explicitly and give an a priori definition of this set following [L7].

In [K, Theorem 2.2], Kac constructs an invariant non-degenerate bilinear form on a symmetrizable Kac-Moody Lie algebra over $\mathbb{C}$. Assume G is of type G_2 and $p \neq 3$, of type F_4, E_6 or E_7 and $p \neq 2$, or of type E_8 (resp. G is of type E_6 or

E_7 and $p = 2$). Then the Cartan matrix is symmetrizable and one can apply the method used in [*loc.cit*] to construct an invariant non-degenerate bilinear form (resp. pairing) on $\mathfrak{g}$ (resp. between $\mathfrak{g}_s$ (the Lie algebra of the simply connected group) and $\mathfrak{g}_{ad}$ (the Lie algebra of the adjoint group)). Thus in the above cases $\mathfrak{g}^*$ and $\mathfrak{g}$ (resp. $\mathfrak{g}_s^*$ and $\mathfrak{g}_{ad}$) can be identified via the bilinear form (resp. pairing). It follows that the nilpotent orbits in $\mathfrak{g}^*$ and $\mathfrak{g}$ (resp. in $\mathfrak{g}_s^*$ and $\mathfrak{g}_{ad}$) are identified and the set $\mathfrak{N}(\mathfrak{g}^*)$ (resp. $\mathfrak{N}(\mathfrak{g}_s^*)$) is identified with the similarly defined set $\mathfrak{N}(\mathfrak{g})$ (resp. $\mathfrak{N}(\mathfrak{g}_{ad})$) for $\mathfrak{g}$ (resp. $\mathfrak{g}_{ad}$), which has been described in [Sp1, Sh2, AL, Spa3, HS, L7]. Hence it remains to study the nilpotent orbits in $\mathfrak{g}^*$ and the set $\mathfrak{N}(\mathfrak{g}^*)$ for G_2 in characteristic 3 and F_4 in characteristic 2.

Let $\mathbf{F_q}$ be a finite field of characteristic 3 (resp. 2). From now on we assume G is a connected group of type G_2 (resp. F_4) defined over $\mathbf{F_q}$.

4.2 Recollections and outline

4.2.1 Let T be a maximal torus of G, R the root system of (G,T) and Π a set of simple roots in R. We have a Chevalley basis $\{h_\alpha,\ \alpha \in \Pi;\ e_\alpha,\ \alpha \in R\}$ of $\mathfrak{g}$ satisfying $[h_\alpha, h_\beta] = 0$; $[h_\alpha, e_\beta] = A_{\alpha\beta}e_\beta; [e_\alpha, e_{-\alpha}] = h_\alpha$; $[e_\alpha, e_\beta] = 0$, if $\alpha + \beta \notin R$; $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$, if $\alpha + \beta \in R$, where the constants $A_{\alpha,\beta}$ and $N_{\alpha,\beta}$ are integers.

For each $\alpha \in R$, there is a unique 1-dimensional connected closed unipotent subgroup $U_\alpha \subset G$ and an isomorphism x_α of the additive group $\mathbb{G}_a$ onto U_α such that $sx_\alpha(t)s^{-1} = x_\alpha(\alpha(s)t)$ for all $s \in T$ and $t \in \mathbb{G}_a$. We assume that $dx_\alpha(1) = e_\alpha$ and $n_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ normalizes T. Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^* = \{\xi \in \mathfrak{g}^* | \xi(e_\alpha) = 0,\ \forall\ \alpha \in R\}$. We define $e'_\alpha \in \mathfrak{g}^*$ by $e'_\alpha(\mathfrak{t}) = 0$; $e'_\alpha(e_\beta) = \delta_{-\alpha,\beta}, \forall\ \beta \in R$ and define $h'_\alpha \in \mathfrak{t}^*$ by $h'_\alpha(h) = \alpha(h),\ \forall\ h \in \mathfrak{t}$. We have

$$x_\alpha(t).\xi = \xi - t\xi(h_\alpha)e'_\alpha,\ \xi \in \mathfrak{t}^*;\ x_\alpha(t).e'_{-\alpha} = e'_{-\alpha} + th'_\alpha - t^2e'_\alpha;$$
$$x_\alpha(t).e'_\beta = \sum_i (-1)^i t^i M_{\alpha,-i\alpha-\beta,i} e'_{i\alpha+\beta}, \alpha \neq -\beta, \tag{4.1}$$

where $M_{\alpha,\beta,i} = \frac{1}{i!}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}\cdots N_{\alpha,(i-1)\alpha+\beta}$ (for determination of structural constants $N_{\alpha,\beta}$ see [Ch]).

4.2.2 Let B be the Borel subgroup UT of G, where $U = \{\prod_{\alpha \in R^+} x_\alpha(t_\alpha), t_\alpha \in \mathbb{G}_a\}$. By Bruhat decomposition, every element in G can be written uniquely in the form $bn_w u_w$ for some $w \in W = NT/T$, some $b \in B$ and some $u_w \in U_w = \{\prod_{\alpha>0, w(\alpha)<0} x_\alpha(t_\alpha) | t_\alpha \in \mathbb{G}_a\}$, where n_w is a representative of w in NT. We can choose $n_\alpha = n_\alpha(1)$ to be the representative of the simple reflections s_α, $\alpha \in \Pi$.

4.2.3 We study the nilpotent $G(\mathbf{F_q})$-orbits in $\mathfrak{g}(\mathbf{F_q})^*$. The strategy is as follows. We find representatives for the nilpotent orbits and calculate the number of rational points in the centralizers and thus count the number of rational points in each orbit. Then we show that the numbers of rational points in all orbits add up to q^{2N}, where N is the number of positive roots in R. As the number of nilpotent elements in $\mathfrak{g}(\mathbf{F_q})^*$ is q^{2N} (shown by G. Lusztig), we get all the orbits.

To compute the centralizer, we use Bruhat decomposition (see 4.2.2) and the formulas (4.1). In particular, we need knowledge on the set $\{w \in W | Z_G(\xi) \cap (BwB) \neq \emptyset\}$. We can assume that $\xi \in \mathfrak{n}^*$ and thus $\xi = \sum_{\alpha \in R^+} a_\alpha e'_\alpha$. Denote $\Delta^\xi = \{\alpha \in R^+ | a_\alpha \neq 0\}$. For $\alpha, \beta \in R^+$, recall that we have order relation $\alpha > \beta$ meaning that $\alpha - \beta$ can be written as a sum of positive roots. Let Δ^ξ_{min} be the set of minimal elements in Δ^ξ under this order relation. If $Z_G(\xi) \cap BwB \neq \emptyset$, then for any $\alpha \in \Delta^\xi_{\text{min}}$, there exists $\beta \in \Delta^\xi_{\text{min}}$ such that $w(\alpha) \geq \beta$.

4.2.4 For a parabolic subgroup P of G, let U_P, $\mathfrak{n}_P$, L, $\mathfrak{l}$, $\mathfrak{p}^*, \mathfrak{n}_P^*, \mathfrak{l}^*$ be as in 3.2.5. Let c' be a nilpotent L-orbit in $\mathfrak{l}^*$. Since $\mathfrak{g}^*$ has finitely many nilpotent G-orbits (see sections 4.3 and 4.4), there exists a unique nilpotent G-orbit c in $\mathfrak{g}^*$ such that $\mathrm{c} \cap (\mathrm{c}' + \mathfrak{n}_P^*)$ is dense in $\mathrm{c}' + \mathfrak{n}_P^*$. Following [LS1], we say that c is obtained by inducing c' from $\mathfrak{l}^*$ to $\mathfrak{g}^*$. One can adapt the proof in [LS1] to show that the orbit c is independent of the choice of P. We denote $\mathrm{c} = \mathrm{Ind}_{\mathfrak{l}^*}^{\mathfrak{g}^*} \mathrm{c}'$.

4.2.5 Let $\xi \in \mathfrak{g}^*$. For a Borel subgroup B of G with Lie algebra $\mathfrak{b}$ and nilpotent radical $\mathfrak{n}$, we define $\mathfrak{b}^* = \{\xi \in \mathfrak{g}^* | \xi(\mathfrak{n}) = 0\}$. Let $\mathcal{B}_G$ be the variety of all Borel subgroups of G. We define $\mathcal{B}_\xi^G = \{B \in \mathcal{B}_G | \xi \in \mathfrak{b}^*\}$. We have

Propsition. $\dim \mathcal{B}_\xi^G = (\dim Z_G(\xi) - \dim T)/2$.

The proposition is proved using induction on $\dim G$ and the following arguments.

(a) Let L be a Levi subgroup of a proper parabolic subgroup of G. Assume $\xi \in \mathfrak{l}^*$ is a nilpotent element such that $\dim Z_L(\xi) = 2\dim \mathcal{B}_\xi^L + \dim T$. Then $\dim Z_G(\xi) = 2\dim \mathcal{B}_\xi^G + \dim T$. The argument is the same as [Spa2, II 3.14].

(b) If ξ lies in the orbit obtained by inducing the orbit of ξ' in $\mathfrak{l}^*$ (see 4.2.5), where L is as in (a), and $\dim Z_L(\xi') = 2\dim \mathcal{B}_{\xi'}^L + \dim T$. Then $\dim Z_G(\xi) = 2\dim \mathcal{B}_\xi^G + \dim T$. The argument is the same as [LS1, Theorem 1.3].

4.2.6 We describe the set $\mathfrak{R}(\mathfrak{g}^*)$ following the methods used in [AL]. Let $\mathrm{c} \in \mathcal{O}_{\mathfrak{g}^*}$ and $\xi \in \mathrm{c}$. We write $\rho_{\xi,G} = \gamma_{\mathfrak{g}^*}(\mathrm{c}, \bar{\mathbb{Q}}_l)$. The map $\xi \to \rho_{\xi,G}$ is determined by the following properties for which the proofs are essentially the same as in [AL] and are omitted.

(a) If ξ lies in the orbit induced from the nilpotent orbit of $\xi' \in \mathfrak{l}^*$, where L is a Levi subgroup of a proper parabolic subgroup of G, then $\rho_{\xi,G} = j_{W'}^W(\rho_{\xi',L})$, where W' is the corresponding parabolic subgroup of W and j is the truncated induction (see [AL] and the references there).

(b) $b_{\rho_{\xi,G}} = \dim \mathcal{B}_\xi^G$, where for $\rho \in \mathrm{Irr}(W)$, b_ρ is the minimal integer d such that ρ occurs in the W-module $\mathfrak{P}_d(V)$ of all homogeneous polynomials of degree d on the reflection space V.

(c) If $\xi \in \mathfrak{l}^*$, where L is a Levi subgroup of a proper parabolic subgroup of G, then $\rho_{\xi,G}$ has non-zero multiplicity in $\mathrm{Ind}_{W'}^W \hat{\rho}_{\xi,L}$, where $\hat{\rho}_{\xi,L} = \sum(-1)^i H^i(\mathcal{B}_\xi^L)$ and W' is the corresponding parabolic subgroup of W.

4.2.7 Let $R^\vee$ be the set of coroots. For $\alpha \in R$, we denote $\alpha^\vee$ the corresponding coroot. Define $\tilde{\Theta} = \{\beta \in R | \beta^\vee - \alpha^\vee \notin R^\vee, \forall\, \alpha \in \Pi\}$, and for a positive integer r, $\tilde{\mathcal{A}}_r' = \{J \subset \tilde{\Theta} | J$ is linearly independent and $\sum_{\alpha\in\Pi} \mathbb{Z}\alpha^\vee / \sum_{\beta\in J} \mathbb{Z}\beta^\vee$ is finite of order r^k for some $k \in \mathbb{Z}_{>0}\}$. For $J \in \tilde{\mathcal{A}}_r'$, let W_J be the subgroup of W generated by the reflections $s_\alpha, \alpha \in J$. Following [L7], we define a set $\mathcal{T}_W^{r*}$ defined by induction on $|W|$ as follows. If $W = \{1\}$, $\mathcal{T}_W^{r*} = W^\wedge$. If $W \neq \{1\}$, then $\mathcal{T}_W^{r*}$ is the set of all $E \in W^\wedge$ such that either $E \in \mathcal{S}_W^1$ or $E = j_{W_J}^W E_1$ for some $J \in \tilde{A}_r'$ and some $E_1 \in \mathcal{T}_{W_J}^{r*}$, where $\mathcal{S}_W^1$ is defined as in [L2, 1.3]. The j-induction can be computed using the tables in [A]. One can then verify easily that the set $\mathfrak{R}(\mathfrak{g}^*)$ coincides with the set $\mathcal{T}_W^{p*}$ (where p is the characteristic exponent of the base field).

4.2.8 For irreducible characters of Weyl groups of type G_2 and F_4 (in table 4.2 and table 4.4), we use the same notation as in [A].

4.3 Type G_2

We assume that $\Pi = \{\alpha, \beta\}$ with α short and β long. Fix $\zeta \in \mathbf{F_q} \backslash \{x^2 | x \in \mathbf{F_q}\}$ and $\varpi \in \mathbf{F_q} \backslash \{x^3 + x | x \in \mathbf{F_q}\}$. The representatives ξ for nilpotent $G(\mathbf{F_q})$-orbits in $\mathfrak{g}(\mathbf{F_q})^*$ and the number of rational points in the corresponding centralizers are listed in Table 4.1. The map $\xi \to \rho_{\xi,G}$ (see 4.2.6) is described in Table 4.2.

Orbit	Representative ξ	$\lvert Z_G(\xi)(\mathbf{F_q})\rvert$
G_2	$e'_\alpha + e'_\beta$	$\mathbf{q}^2$
$G_2(a_1)$	$e'_\beta + e'_{2\alpha+\beta}$	$6\mathbf{q}^4$
$G_2(a_1)$	$e'_\beta + e'_{2\alpha+\beta} - \varpi e'_{3\alpha+\beta}$	$3\mathbf{q}^4$
$G_2(a_1)$	$e'_\beta - \zeta e'_{2\alpha+\beta}$	$2\mathbf{q}^4$
$\widetilde{A_1}$	e'_α	$\mathbf{q}^4(\mathbf{q}^2 - 1)$
A_1	e'_β	$\mathbf{q}^6(\mathbf{q}^2 - 1)$
$\emptyset$	0	$\mathbf{q}^6(\mathbf{q}^2 - 1)(\mathbf{q}^6 - 1)$

Table 4.1:

Orbit of ξ	$\rho_{\xi,G}$	Orbit of ξ	$\rho_{\xi,G}$
G_2	$\chi_{1,1}$	A_1	$\chi_{1,4}$
$G_2(a_1)$	$\chi_{2,1}$	$\emptyset$	$\chi_{1,2}$
$\widetilde{A_1}$	$\chi_{2,2}$		

Table 4.2:

4.4 Type F_4

We assume that $\Pi = \{p, q, r, s\}$ with p, q long, r, s short and $(q, r) \neq 0$. We denote $apbqcrds$ the root $ap+bq+cr+ds$. Fix $\eta \in \mathbf{F_q} \backslash \{x^2+x | x \in \mathbf{F_q}\}$ and $\varpi \in \mathbf{F_q} \backslash \{x^3+x | x \in \mathbf{F_q}\}$. The representatives ξ for nilpotent $G(\mathbf{F_q})$-orbits in $\mathfrak{g}(\mathbf{F_q})^*$ and the number of rational points in the corresponding centralizers are listed in Table 4.3. The map $\xi \to \rho_{\xi,G}$ (see 4.2.6) is described in Table 4.4.

Orbit	Representative ξ	$\lvert Z_G(\xi)(\mathbf{F_q})\rvert$
F_4	$e'_p + e'_q + e'_r + e'_s$	$\mathbf{q}^4$
$F_4(a_1)$	$e'_p + e'_{qr} + e'_{q2r} + e'_s$	$2\mathbf{q}^6$
$F_4(a_1)$	$e'_p + e'_q + e'_{qr} + e'_s + \eta e'_{q2r}$	$2\mathbf{q}^6$
$F_4(a_2)$	$e'_p + e'_{qr} + e'_{rs} + e'_{q2r2s}$	$\mathbf{q}^8$
B_3	$e'_p + e'_{qrs} + e'_{q2r} + e'_{pq2rs}$	$\mathbf{q}^{10}$
C_3	$e'_s + e'_{q2r} + e'_{pqr}$	$\mathbf{q}^8(\mathbf{q}^2-1)$
$F_4(a_3)$	$e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$	$24\mathbf{q}^{12}$
$F_4(a_3)$	$e'_{pq} + e'_{pqr} + e'_{q2rs} + e'_{q2r2s} + \eta e'_{pq2r}$	$8\mathbf{q}^{12}$
$F_4(a_3)$	$e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s} + \eta e'_{pq2r2s}$	$4\mathbf{q}^{12}$
$F_4(a_3)$	$e'_{pq} + e'_{pqr} + e'_{q2rs} + e'_{q2r2s} + \eta e'_q$	$4\mathbf{q}^{12}$
$F_4(a_3)$	$e'_{pqr} + e'_{qrs} + e'_{q2r} + e'_{q2r2s} + \varpi e'_{pq2r2s}$	$3\mathbf{q}^{12}$
$(B_3)_2$	$e'_p + e'_{qr} + e'_{q2r2s}$	$\mathbf{q}^{10}(\mathbf{q}^2-1)$
$C_3(a_1)$	$e'_{pqr} + e'_{q2rs} + e'_{q2r2s}$	$2\mathbf{q}^{12}(\mathbf{q}^2-1)$
$C_3(a_1)$	$e'_{pq} + e'_{pqr} + e'_{q2rs} + \eta e'_{pq2r}$	$2\mathbf{q}^{12}(\mathbf{q}^2-1)$
B_2	$e'_{pqr} + e'_{q2r2s}$	$2\mathbf{q}^{12}(\mathbf{q}^2-1)^2$
B_2	$e'_{pq} + e'_{pqr} + e'_{q2r2s} + \eta e'_{pq2r}$	$2\mathbf{q}^{12}(\mathbf{q}^4-1)$
$\widetilde{A_2} + A_1$	$e'_{pqr} + e'_{q2rs} + e'_{p2q2r2s}$	$\mathbf{q}^{14}(\mathbf{q}^2-1)$
$A_2 + \widetilde{A_1}$	$e'_{p2q2r} + e'_{q2r2s} + e'_{pq2rs}$	$\mathbf{q}^{16}(\mathbf{q}^2-1)$
$\widetilde{A_2}$	$e'_{pqrs} + e'_{q2rs}$	$\mathbf{q}^{14}(\mathbf{q}^2-1)(\mathbf{q}^6-1)$
A_2	$e'_{p2q2r} + e'_{pq2r2s} + e'_{p2q3r2s}$	$\mathbf{q}^{20}(\mathbf{q}^2-1)$
$A_1 + \widetilde{A_1}$	$e'_{p2q2r2s} + e'_{p2q3rs}$	$\mathbf{q}^{20}(\mathbf{q}^2-1)^2$
$(A_2)_2$	$e'_{p2q2r} + e'_{pq2r2s}$	$\mathbf{q}^{20}(\mathbf{q}^2-1)(\mathbf{q}^6-1)$
$\widetilde{A_1}$	$e'_{p2q3r2s}$	$2\mathbf{q}^{21}(\mathbf{q}^2-1)(\mathbf{q}^3-1)(\mathbf{q}^4-1)$
$\widetilde{A_1}$	$e'_{p2q2r2s} + e'_{p2q3r2s} + \eta e'_{p2q4r2s}$	$2\mathbf{q}^{21}(\mathbf{q}^2-1)(\mathbf{q}^3+1)(\mathbf{q}^4-1)$
A_1	$e'_{2p3q4r2s}$	$\mathbf{q}^{24}(\mathbf{q}^2-1)(\mathbf{q}^4-1)(\mathbf{q}^6-1)$
$\emptyset$	0	$\mathbf{q}^{24}(\mathbf{q}^2-1)(\mathbf{q}^6-1)(\mathbf{q}^8-1)(\mathbf{q}^{12}-1)$

Table 4.3:

Orbit of ξ	$\rho_{\xi,G}$	Orbit of ξ	$\rho_{\xi,G}$	Orbit of ξ	$\rho_{\xi,G}$
F_4	$\chi_{1,1}$	$(B_3)_2$	$\chi_{2,1}$	A_2	$\chi_{8,4}$
$F_4(a_1)$	$\chi_{4,2}$	$C_3(a_1)$	$\chi_{16,1}$	$A_1 + \widetilde{A_1}$	$\chi_{9,4}$
$F_4(a_2)$	$\chi_{9,1}$	B_2	$\chi_{9,2}$	$(A_2)_2$	$\chi_{1,2}$
B_3	$\chi_{8,1}$	$\widetilde{A_2} + A_1$	$\chi_{6,1}$	$\widetilde{A_1}$	$\chi_{4,5}$
C_3	$\chi_{8,3}$	$A_2 + \widetilde{A_1}$	$\chi_{4,3}$	A_1	$\chi_{2,4}$
$F_4(a_3)$	$\chi_{12,1}$	$\widetilde{A_2}$	$\chi_{8,2}$	$\emptyset$	$\chi_{1,4}$

Table 4.4:

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