Radiation Field for Einstein Vacuum Equations

by

Fang Wang

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Author ....
Department of Mathematics
April 30, 2010

Certified by...
Richard B. Melrose
Simons Professor of Mathematics
Thesis Supervisor

Accepted by ....
Bjorn Poonen
Chairman, Department Committee on Graduate Students

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Abstract

The radiation field introduced by Friedlander provides a direct approach to the asymptotic expansion of solutions to the wave equation near null infinity. We use this concept to study the asymptotic behavior of solutions to the Einstein Vacuum equations, which are close to Minkowski space, at null infinity. By imposing harmonic gauge, the Einstein Vacuum equations reduce to a system of quasilinear wave equations on $\mathbb{R}^{1+n}_{t,x}$. We show that if the space dimension $n \geq 5$ the Møller wave operator is an isomorphism from Cauchy data satisfying the constraint equations to the radiation fields satisfying the corresponding constraint equations on small neighborhoods of suitable weighted b-type Sobolev spaces.

Thesis Supervisor: Richard B. Melrose
Title: Simons Professor of Mathematics

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Contents

Chapter 1

Introduction

The Einstein vacuum equations determine a manifold M^{1+n} with a Lorentzian metric with vanishing Ricci curvature:

$$R_{\mu\nu} = 0. \tag{1.1}$$ eq.1

The set $(\mathbb{R}^{1+n}_{t,x}, m)$: standard Minkowski $m = d^2t - \sum_{i=1}^{n} d^2x^i$ on $\mathbb{R}^{1+n}_{t,x}$ describes the Minkowski space-time solution of the system (1.1). The problem of stability of Minkowski space appears in Cauchy formulation of the Einstein Vacuum equations: for a given n dimensional manifold Σ^n with a Riemannian metric g_0, and a symmetric two tensor k_0, we want to find a $n+1$ dimensional manifold with Lorentzian metirc g of signature $(1, n)$ satisfying the Einstein vacuum equations (1.1), and an embedding $\Sigma^n \subset M^{1+n}$ such that g_0 is the restriction of $-g$ to Σ and k_0 is the second fundamental form of (Σ^n, g_0) in $(M^{1+n}, -g)$. The Cauchy problem is overdetermined which imposes compatibility condition on the Cauchy data: the constraint equations

$$R_0 - [k_0]^i_j[k_0]^j_i + [k_0]^i_i[k_0]^j_j = 0, \quad \nabla^j[k_0]_{ij} - \nabla_i[k_0]^j_j = 0, \forall i = 1, ..., n. \tag{1.2}$$ constraint.1

Here R_0 is the scalar curvature of g_0 and ∇ is covariant differentiation w.r.t. g_0.

The seminal result of Choquet-Bruhat [CB1] followed by the work [CBG] showed existence and uniqueness up to diffeomorphism of a maximal globally hyperbolic smooth space-time arising from any set of smooth initial data. The global stability of Minkowski space for the Einstein vacuum equations was shown in a remarkable work of Christodoulou-Klainerman for strongly asymptotic flat initial data [CK]. The approach taken in that work viewed the Einstein vacuum equations as a system of equations

$$D^{\alpha}W_{\alpha\beta\gamma\delta} = 0, \quad D^{\alpha} * W_{\alpha\beta\gamma\delta} = 0$$

for the Weyl tensor $W_{\alpha\beta\gamma\delta}$ of the metric $g_{\alpha\beta}$ and used generalized energy inequalities associated with the Bel-Robinson energy-momentum tensor, constructed from components of W, and special geometrically constructed vector fields, designed to mimic the rotation and the conformal Morawetz vector fields of the Minkowski space-time, i.e. "almost conformal Killing" vector fields of the unknown metric g. The proof was manifestly invariant.

The Einstein vacuum equations are invariant under diffeomorphism. In the work of Choquet-Bruhat, this allows her to choose a special *harmonic gauge*, also referred as *wave coordinates*, in which the Einstein vacuum equations become a system of quasilinear wave equations on the component of the unknown metric $g_{\mu\nu}$:

$$\widetilde{\Box}_g g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g), \quad \text{where} \quad \widetilde{\Box}_g = g^{\alpha\beta}\partial_{\alpha\beta} \qquad (1.3)$$ eq.2

with $F(u)(v,v)$ depending quadratically on v. Wave coordinates $\{x^{\mu}\}$ are required to be solutions of the wave equations

$$\Box_g x^{\mu} = 0, \quad \text{for} \quad \mu = 0, 1, ..., n,$$

where the geometric wave operator is

$$\Box_g = D_{\alpha}D^{\alpha} = g^{\alpha\beta}\partial_{\alpha}\partial_{\beta} - g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}\partial_{\nu}.$$

The metric $g_{\mu\nu}$ relative to wave coordinates $\{x^{\mu}\}$ satisfies the *harmonic gauge condition*:

$$g_{\mu\nu}\Gamma^{\nu} = g_{\mu\nu}g^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}\partial_{\nu} = g^{\alpha\beta}\partial_{\beta}g_{\alpha\mu} - \tfrac{1}{2}g^{\alpha\beta}\partial_{\mu}g_{\alpha\beta}$$

Under this condition the geometric wave operator $\Box_g$ is equal to the reduced wave operator $\widetilde{\Box}_g$. And the Cauchy problem can reformulate as follows: given a pair of symmetric two tensors (h^0, h^1) on $\mathbb{R}^{1+n}_{t,x}|_{t=0}$, we want to find a Lorentian metric $g = m+h$ of signature $(1, n)$ satisfying the reduced Einstein equations (1.3) such that $((g-m)|_{t=0}, \partial_t g|_{t=0}) = (h^0, h^1)$. Here the Cauchy data (h^0, h^1) satisfy the corresponding constraint equations such that:

$$\Gamma^\mu|_{t=0} = 0, \quad \partial_t \Gamma^\mu|_{t=0} = 0.$$

The constraint equations for (h^0, h^1) is deduced in Section 3.1.3. See (3.6) and (3.7) for details.

The use of harmonic gauge goes back to the work of Einstein on post-Newtonian and post-Minkowski expansions. In the PDE terminology the result of Choquet-Bruhat corresponds to the *local well-posedness* of the Cauchy problem for the Einstein vacuum equations with smooth initial data. The reduced Einstein vacuum equations satisfies the *null condition* for spacial dimension $n \geq 4$, which ensures the global existence theorem for small Cauchy data. The concept of null condition was designed to detect systems for which solutions are asymptotically free and decay like solutions of a linear equation. See [Ch1], [Kl] and [Ho1]. For $n = 3$, it can be shown that the Einstein vacuum equations in harmonic gauge do not satisfy the null condition. Moreover, Choquet-Bruhat [CB2] showed that even without imposing a specific gauge the Einstein equations violate the null condition. However, the Einstein vacuum equations in harmonic gauge satisfy the *weak null condition* and recently Lindblad-Rodnianski reproved the global existence for Einstein vacuum equations in harmonic gauge for general asymptotic flat initial data in harmonic gauge by combining it with the vector field method [LR1] [LR2].

In this paper, we apply the radiation field theory due to Friedlander to study the asymptotic behavior of solutions to Einstein vacuum equations in harmonic gauge. Friedlander's radiation field was used by Hörmander to study the asymptotic behavior of solutions to linear hyperbolic equation in the following coordinates:

$$\rho = \frac{1}{|x|},\ \tau = t - r, \theta = \frac{x}{r}, \quad \text{for } |x| \text{ large.}$$

For instance, consider the Cauchy problem in Minkowski space-time as follows:

$$\Box_m u(t,x) = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad \text{where } u_0, u_1 \in C_c^\infty(\mathbb{R}^n).$$

By writing $u = \rho^{\frac{n-1}{2}} \tilde{u}$ near $\rho = 0$ and studying the equivalent equation

$$(\Box_{\tilde{m}} + \tfrac{(n-1)(n-3)}{4})\tilde{u} = 0$$

with the conformal metric $\tilde{m} = \rho^2 m$ near $\rho = 0$, Friedlander showed that $\tilde{u}$ is smooth up to $\rho = 0$. The radiation field is the image of the map

$$\mathcal{R}_{\mathcal{LP}} : \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ni (u_0, u_1) \longrightarrow \partial_\tau \tilde{u}|_{\rho=0} \in L^2(\mathbb{R}_\tau \times S_\theta^{n-1}),$$

which is an isomorphism. Here $\mathcal{R}_{\mathcal{LP}}$ is essentially the *Møller wave operator* and also the free space translation representation of Lax and Phllips. I generalize this idea by considering the conformal transformation of the reduced Einstein vacuum equations on a suitable compactification of $\mathbb{R}^{1+n}_{t,x}$ as follows. Here $S_1^\pm$ is the compactification

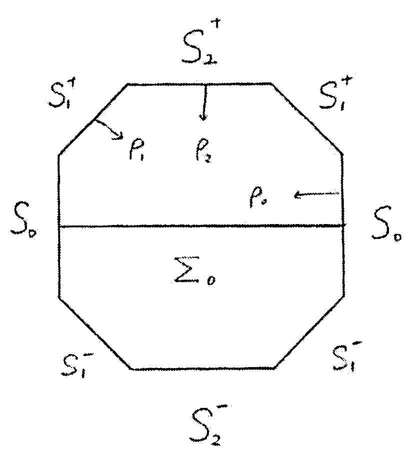

Figure 1-1: the blown-up space X

⟨fig.1⟩

of null infinity and Σ_0 is Cauchy surface, and ρ_0, ρ_1, ρ_2 are defining functions for corresponding boundary hypersurfaces. See Section 2.1 for precise description of this space.

Denote by $\widetilde{\mathcal{U}}_\epsilon^{N,\delta}$ the space of elements (h^0, h^1) satisfies the constraint equations (3.6) and (3.7) and in a small neighborhood of $(0,0)$ in the space $\rho_0^{\frac{n-1}{2}+\delta} H_b^{N+1}(\Sigma_0) \times \rho_0^{\frac{n+1}{2}+\delta} H_b^{N+1}(\Sigma_0)$ for some $N \geq n+6$, $1 > \delta > 0$ and $\epsilon > 0$ small. See Section 2.1.3 for the definition of weighted b-Sobolev spaces. Then by energy estimate method, I show that for $n \geq 5$, if $(h^0, h^1) \in \widetilde{\mathcal{U}}_\epsilon^{N,\delta}$ then the radiation field for $h = g - m$, denoted by $\tilde{h}^{S_1} = (\rho_0\rho_1\rho_2)^{\frac{1-n}{2}} h|_{S_1}$, is well defined, satisfies the corresponding constraint equations (3.18) and lies in a small neighborhood of 0 in the space $\rho_0^{\delta'} \rho_2^{\sigma - \frac{n-1}{2}} H_b^N(S_1)$ for some $0 < \sigma$, $0 < \delta' < \min\{\frac{1}{2}, \delta\}$. Combining this with the linear radiation field theory and the implicit function theorem, I improve the decay order and regularity for the radiation field. Denote by $\widetilde{\mathcal{W}}_\epsilon^{N,\delta}$ the space consisting of elements in a small neighborhood of 0 in $(\rho_0\rho_2)^\delta [H_b^{\frac{1}{2}-\delta}(\overline{\mathbb{R}}; H^{N+\frac{1}{2}+\delta}(\mathbb{S}^{n-1})) \cap L^2(\mathbb{S}^{n-1}; H_b^{N+1}(\overline{\mathbb{R}}))]$ and satisfy the constraint equations (3.18). Then the main theorem of this paper is the following:

⟨mainthm⟩ **Theorem 1.0.1.** *For $n \geq 5$, $N \geq n+6$, $1 > \delta > 0$ and $\epsilon > 0$ small, the nonlinear Møller wave operator*

$$\mathscr{R}_{\mathscr{F}} : (h^0, h^1) \longrightarrow \tilde{h}^{S_1}$$

defines an isomorphism from $\widetilde{\mathcal{U}}_\epsilon^{N,\delta}$ to its image, which is a neighborhood of 0 in $\widetilde{\mathcal{W}}_{C\epsilon}^{N,\delta}$ for some $C > 0$.

For $n = 4$, the perturbation changes the geometry of metric near null infinity a bit more, i.e. the bicharacteristic curve for perturbed metric are not C^1 tangent to the bicharacteristic curve for Minkowski space if they have same limit point on $S_1^\pm$. In this case, we have to combine the harmonic gauge condition with the energy estimates method, which ensures some certain component of the metric is small and hence the geometry of the conformal metric does not changes much actually.

For $n = 3$, the Cauchy data of interest for reduced Einstein vacuum equations has an asymptotic leading term $-M|x|^{-1}\delta_{ij}$ for some constant $M > 0$ small, which has a long range effect at null infinity. Since $-M|x|^{-1}\delta_{ij}$ provides a solution to the linearization of (1.3) in a neighborhood of null infinity in the blow-up space X, we may conclude that the essential change of the geometry of perturbed solution $g = m + h$ only comes from the asymptotic leading term, i.e. the constant M. By a change of coordinates to $\tau = t - |x| - M \log |x|$, which was suggested by Friedlander to study the linear equation $\Box_g u = 0$ with such background metric, I expect to find a corresponding compactification of $\mathbb{R}^{1+n}_{t,x}$ such that the above statements can go through.

Chapter 2

Preliminary Preparation

In this chapter, I set up the basic notation used in this paper and study the conformal transformation of Minkowski metric intensively.

2.1 Geometric Setting

This section is a preparation for describing the problem and dealing with it in a manifold with corners arising from a suitable compactification of $\mathbb{R}^{1+n}_{t,x}$. Refer to [Me] and [Fr] for more details.

2.1.1 Blow-up Space

Let us first introduce the notation we will use throughout this paper. In $\mathbb{R}^{1+n}_{t,x}$, we take $t = x^0 = x_0, x = (x^1, ..., x^n) = (x_1, ..., x_n)$ for simplicity and always use the lower case English alphabet $i, j, k, l, ...$ as indices taking value in $\{1, 2, ..., n\}$, the Greek alphabet $\alpha, \beta, \mu, \nu, ...$ as indices taking value in $\{0, 1, 2, ..., n\}$ and the capital English alphabet $A, B, C, D, ...$ as indices taking value in specific local coordinates. Set

$$r = |x|, \quad \tau = |t| - |x|, \quad \theta = \frac{x}{|x|}, \quad \rho = \frac{1}{|x|}, \quad \phi = \frac{1}{|t|}, \quad y = \frac{x}{|t|}, \quad R = |y|,$$
$$a = 1 - \frac{|t|}{|x|}, \quad b = \frac{1}{|x| - |t|}, \quad \bar{a} = 1 - \frac{|x|}{|t|}, \quad \bar{b} = \frac{1}{|t| - |x|}.$$

Denote by $\overline{\mathbb{R}_{t,x}^{t+n}}$ the radial compactification of $\mathbb{R}_{t,x}^{1+n}$ with boundary defining function $\tilde{\rho}$. Let X be $\overline{\mathbb{R}_{t,x}^{t+n}}$ blown up the embedded submanifold $\partial\overline{\mathbb{R}_{t,x}^{t+n}} \cap \{|t| = |x|\}$:

$$X = [\overline{\mathbb{R}_{t,x}^{t+n}} : \partial\overline{\mathbb{R}_{t,x}^{t+n}} \cap \{|t| = |x|\}],$$

resulting in a manifold with corners up to codimension 2. See Figure 1-1. Here X has 5 boundary hypersurfaces: S_1^+, S_1^- the front faces in $t > 0$ and $t < 0$ separately, S_2^+, S_2^- the top and bottom boundary hypersurfaces and S_0 the middle one. Then

$$X \simeq X_0 := \{(t', x') \in \mathbb{R}_{t',x'}^{t+n} : |t| \leq 1, |x| \leq 1, |t| + |x| \leq \sqrt{2}\},$$
$$S_1^\pm \simeq \overline{\mathbb{R}}_\tau \times \mathbb{S}_\theta^{n-1}, \quad S_2^\pm \simeq \overline{\mathbb{B}}_y^n, \quad S_0 \simeq [-1,1]_s \times \mathbb{S}_\theta^{n-1}.$$

The corresponding defining functions are $\rho_1^+, \rho_1^-, \rho_2^+, \rho_2^-, \rho_0$. Here $\rho_1^\pm, \rho_2^\pm, \rho_0$ and $\tilde{\rho}$ are smooth positive function on X which can be defined in local coordinates as follows:

- In the domain $\Omega_0 = \{|t|^2 + |x|^2 < 1000\}$,

$$\tilde{\rho} = \rho_0 = \rho_1 = \rho_2 = 1$$

- In the domain $\Omega_1 = [-\frac{7}{8}, \frac{7}{8}]_s \times [0,1]_\rho \times \mathbb{S}_\theta^{n-1}$,

$$\tilde{\rho} = \rho_0 = \rho, \quad \rho_2 = \rho_1 = 1.$$

- In the domain $\Omega_2 = [0, \frac{1}{4}]_a \times [0, \frac{1}{4}]_b \times \mathbb{S}_\theta^{n-1}$,

$$\tilde{\rho} = ab, \quad \rho_0 = b, \quad \rho_1 = a, \quad \rho_2 = 1.$$

- In the domain $\Omega_3 = [-\tau_0, \tau_0]_\tau \times [0,1]_\rho \times \mathbb{S}_\theta^{n-1}$ for some constant $\tau_0 > 8$,

$$\tilde{\rho} = \rho_1 = \rho, \quad \rho_0 = \rho_2 = 1.$$

- In the domain $\Omega_4 = [0, \frac{1}{4}]_{\bar{a}} \times [0, \frac{1}{4}]_{\bar{b}} \times \mathbb{S}^n$,

$$\tilde{\rho} = ab, \quad \rho_1 = \bar{a}, \quad \rho_2 = \bar{b}, \quad \rho_0 = 1.$$

- In the domain $\Omega_5 = \{|y| \leq \frac{7}{8}\} \times [0, 1]_\phi$,

$$\tilde{\rho} = \rho_2 = \phi, \quad \rho_1 = \rho_0 = 1.$$

We omit $\pm$ here, which means if the domain is in $\pm t > 0$ then we take $\rho_i = \rho_i^\pm$, or equivalently $\rho_i = \rho_i^+ \rho_i^-$, for $i = 1, 2$. Notice that in the intersection of two domains, the different definitions for defining functions are equivalent:

Definition 2.1.1. We say two defining function $\tilde{\rho}_1$ and $\tilde{\rho}_2$ on a domain $\Omega \subset X$ are equivalent if and only if

$$\frac{\rho_1}{\rho_2} = e^\psi \text{ for some } \psi \in C^\infty(\overline{\Omega}).$$

In many cases, for example the energy estimates, finite choices of equivalent defining functions always give equivalent statements. I change the choice of ρ_i freely from one domain to the other in the rest of this paper if they give equivalent result. Here $\{\Omega_i : 1 \leq i \leq 5\}$ gives a covering of ∂X. However, when solving equations globally on X, we have to consider some of the domains together. See Chapter 3 for details.

Denote by X^2 the double space of X across the front face $S_1 \cup S_1^-$. Then X^2 is a manifold with boundary and ∂X^2 has 3 components $(S_0)^2$, $(S_2^+)^2$, $(S_2^-)^2$, with corresponding defining function $\rho_0, \rho_2^+, \rho_2^-$.

2.1.2 Vector Fields.

Let $\mathscr{V}_b(X)$ (resp. $\mathscr{V}_b(X^2)$) be the smooth vector fields on X which are tangent to ∂X (resp. ∂X^2). The relation between $\mathscr{V}_b(X^2)|_X$ and $\mathscr{V}_b(X)$ is

$$\mathscr{V}_b(X^2)|_X = \mathscr{V}_b(X) + C^\infty(X)\partial_{\rho_1}.$$

By [Me], for any p-submanifold $\Sigma \subset X$, $\mathscr{V}_b(X)|_\Sigma = \mathscr{V}_b(\Sigma)$. If $\partial\Sigma = \emptyset$, then $\mathscr{V}_b(\Sigma) = \mathscr{V}(\Sigma)$ in usual sense. Set

$$\partial_0 = \partial_t, \quad \partial_i = \partial_{x^i}, \quad \tilde{\partial}_i = r(\delta_i^j - \theta_i\theta^j)\partial_j,$$
$$Z_{00} = r\partial_r + t\partial_t, \quad Z_{ij} = x_i\partial_j - x_j\partial_i, \quad Z_{0i} = t\partial_i + x_i\partial_t.$$

The vector files $\tilde{\partial}_i$ are the projection of ∂_i onto the unit sphere which satisfies

$$\theta^i\tilde{\partial}_i = 0.$$

Denote by $\tilde{Z}$ any vector field in $\{\partial_\mu, Z_{\mu\nu} : \mu, \nu = 0, ..., n\}$ and $\tilde{\partial}$ any vector field in $\mathscr{V}_b(X)$.

Lemma 2.1.2. *The vector fields above satisfy the following properties:*

$$\partial_\mu \in \rho_0\rho_2\mathscr{V}_b(X),$$
$$\mathscr{V}_b(X) = Span_{C^\infty(X)}\{\partial_\mu, Z_{\mu\nu} : \mu, \nu = 0, 1, ..., n\},$$
$$\mathscr{V}(S^{n-1}) = Span_{C^\infty(S^{n-1})}\{\tilde{\partial}_i : i = 1, ..., n\} = Span_{C^\infty(S^{n-1})}\{Z_{ij} : i, j = 1, ..., n\}.$$

Proof. The third property follows directly by

$$Z_{ij} = \theta_i\tilde{\partial}_j - \theta_j\tilde{\partial}_i, \quad \tilde{\partial}_j = Z_{ij}\theta^i.$$

To prove the first and second one, we only need to check them near ∂X:

- In Ω_1,

$$\partial_t = \rho\partial_s, \quad \partial_i = -\theta_i\rho(\rho\partial_\rho + s\partial_s) + \rho\bar{\partial}_i,$$
$$Z_{00} = -\rho\partial_\rho, \quad Z_{0i} = \theta_i(1 - s^2)\partial_s - \theta_i s\rho\partial_\rho + s\tilde{\partial}_i,$$
$$\Longrightarrow \quad \partial_s = (1 - s^2)^{-1}(Z_{0i}\theta^i - sZ_{00}), \quad \rho\partial_\rho = -Z_{00}.$$

- In Ω_2,

$$\partial_t = b(b\partial_b - a\partial_a), \quad \partial_i = -b\theta_i(b\partial_b - (1-a)a\partial_a) + ab\tilde{\partial}_i,$$
$$Z_{00} = -b\partial_b, \quad Z_{0i} = -\theta_i((2-a)a\partial_a - b\partial_b) + (1-a)\tilde{\partial}_i,$$
$$\implies \quad b\partial_b = -Z_{00}, \quad a\partial_a = -(2-a)^{-1}(Z_{00} - Z_{0i}\theta^i).$$

- In Ω_3,

$$\partial_t = \partial_\tau, \quad \partial_i = -\theta_i(\partial_\tau + \rho^2\partial_\rho) + \rho\tilde{\partial}_i,$$
$$Z_{00} = \tau\partial_\tau - \rho\partial_\rho, \quad Z_{0i} = -\theta_i(\tau\partial_\tau + (1+\rho\tau)\rho\partial_\rho) + (1+\rho\tau)\tilde{\partial}_i,$$
$$\implies \quad \partial_\tau = \partial_t, \quad \rho\partial_\rho = -(2+\rho\tau)^{-1}(Z_0 + Z_{0i}\theta^i).$$

- In Ω_4,

$$\partial_t = \bar{b}((1-\bar{a})\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}}), \quad \partial_i = -\bar{b}\theta_i(\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}}) + \bar{a}\bar{b}(1-\bar{a})^{-1}\tilde{\partial}_i,$$
$$Z_{00} = -\bar{b}\partial_{\bar{b}}, \quad Z_{0i} = -\theta_i((2-\bar{a})\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}}) + (1-\bar{a})^{-1}\tilde{\partial}_i,$$
$$\implies \bar{b}\partial_{\bar{b}} = -Z_{00}, \quad \bar{a}\partial_{\bar{a}} = -(2-\bar{a})^{-1}(Z_{00} + Z_{0i}\theta^i).$$

- In Ω_5,

$$\partial_t = -\phi(\phi\partial_\phi + y^i\partial_{y^i}), \quad \partial_i = \phi\partial_{y^i},$$
$$Z_{00} = -\phi\partial_\phi, \quad Z_{0i} = \partial_{y^i} - y_i(\phi\partial_\phi + y^j\partial_{y^j}),$$
$$\implies \quad \phi\partial_\phi = -Z_{00}, \quad \partial_{y^i} = -(1-R^2)^{-1}y_i(Z_{00} - Z_{0j}y^j) + Z_{0i}.$$

□

2.1.3 Sobolev Spaces and Symbol Spaces.

Suppose $\Sigma \subset X$ is a p-submanifold, then it is a manifold with corners up to codimension ≤ 2. If Σ is a manifold with corners up to codimension 2, then Σ has boundary defining function ρ_Σ equivalent to $\tilde{\rho}|_\Sigma$ and defining functions $\rho_i|_\Sigma$ for boundary hypersurfaces $S_i \cap \Sigma \neq \emptyset$; if Σ is a manifold with boundary, then Σ has a boundary defining function ρ_Σ equivalent to ρ_i for some $i \in \{0,1,2\}$ near $\Sigma \cap S_i \neq \emptyset$, $\Sigma \not\subseteq S_i$; if

$\partial\Sigma = \emptyset$, then ρ_Σ is equivalent to 1. Let m_0 be the Riemannian metric on X induced by the diffeomorphism $\Phi : X \longrightarrow X_0 \subset \mathbb{R}^n$, where we take the metric on X_0 be the restriction of Euclidean metric.

Take $Y = X$ or $Y = \Sigma$, a p-submanifold of X. Then Y is a manifold with corners, with hypersurfaces $\partial_1 Y, ..., \partial_l Y$ and corresponding defining function $\tilde{\rho}_1, ..., \tilde{\rho}_1$. Denote by $\tilde{\rho}_Y$ the total boundary defining function and by $dvol^Y_{m_0}$ the volume form generated by restriction of m_0 to Y.

⟨def.1⟩
Definition 2.1.3. For any domain $\Omega \subset X$, define the weighted b-Sobolev space $\tilde{\rho}_1^{c_1} \cdots \tilde{\rho}_l^{c_l} H_b^N(\Omega \cap Y)$ for any $N \in \mathbb{N}_0$ and $c_1, \cdots, c_l \in \mathbb{R}$ the closure of $C_c^\infty(\Omega \cap Y)$ in the norm:

$$\|v\|_{H_b^N(\Omega\cap Y)} = \Big(\int_{\Omega\cap Y} \sum_{|I|\leq N} |\tilde{Z}|_Y^I v|^2 \frac{dvol^Y_{m_0}}{\tilde{\rho}_Y}\Big)^{\frac{1}{2}},$$

$$\|v\|_{\tilde{\rho}_1^{c_1}\cdots\tilde{\rho}_l^{c_l} H_b^N(\Omega\cap Y)} = \|\tilde{\rho}_1^{-c_1} \cdots \tilde{\rho}_l^{-c_l} v\|_{H_b^N(\Omega\cap Y)}.$$

We also use the symbol space to characterize the asymptotic behavior of functions near boundary of Y with smooth regularity in the interior.

⟨def.2⟩
Definition 2.1.4. For any $c_1, ..., c_l \in \mathbb{C}$, define $\mathscr{A}^{c_1,...,c_l}(Y)$ consisting of elements satisfying

$$\tilde{\rho}_1^{-c_1} \cdots \tilde{\rho}_l^{-c_l} v \in L^\infty(Y), \quad \tilde{\rho}_1^{-c_1} \cdots \tilde{\rho}_l^{-c_l} (\mathcal{V}_b(X)|_Y)^k v \in L^\infty(Y), \quad \forall k \in \mathbb{N}_0.$$

Definition 2.1.3, 2.1.4 only depend on the equivalent class of boundary defining functions. The relation between the weighted b-Sobolev spaces and the Symbol spaces are as follows:

Lemma 2.1.5. *Suppose $c_i \in \mathbb{C}$ and $c_i' \in \mathbb{R}$. If $\Re c_i > c_i'$, then*

$$\mathscr{A}^{c_1,...,c_l}(Y) \subset \tilde{\rho}_1^{c_1'} \cdots \tilde{\rho}_l^{c_l'} H_b^N(Y) \quad \forall N \in \mathbb{N}_0.$$

Lemma 2.1.6. *Suppose* $\dim Y = k$, *then there exists a constant $C_N(Y)$ for any $N \geq \frac{k}{2} + 1$ such that*

$$\|v\|_{L^\infty(Y)} \leq C_N(Y) \|v\|_{H_b^N(Y)}.$$

This implies that for any $c_1, ..., c_l \in \mathbb{R}$

$$\bigcap_{N=0}^{\infty} \tilde{\rho}_1^{c_1} \cdots \tilde{\rho}_l^{c_l} H_b^N(Y) \subset \mathscr{A}^{c_1,...,c_l}(Y).$$

In this paper, we consider the map between Cauchy data and Characteristic data for Einstein vacuum equations, which lie in some weighted b-Sobolev spaces on the p-submanifolds $\Sigma_0 = \overline{\{t=0\}} = \overline{\mathbb{R}^n}$ and S_1 separately.

2.2 Minkowski Space.

Minkowski space-time is a solution to Einstein vacuum equations. In this section, we list the basic geometric properties of its conformal transformation.

2.2.1 Metric.

Denote by m the Minkowski metric on $\mathbb{R}^{1+n}_{t,x}$ and $\tilde{m}$ its conformal transformation on X:

$$m = d^2t - \sum_{i=1}^{n} d^2x^i \quad \text{and} \quad \tilde{m} = \tilde{\rho}^2 m.$$

Here $\tilde{\rho}$ is an arbitrary choice of boundary defining function, which is equivalent to the choice stated in Section 1.1 in each domain. A change of equivalent boundary defining function only results in a smooth factor $e^{2\psi}$ with $\psi \in C^\infty(X)$.

Lemma 2.2.1. *The conformal metric* $\tilde{m}$ *extends to a Lorentz b-metric on* X_0^2 *of signature* $(1,n)$ *with* $S_1^+ \cup S_-$ *being its characteristic surfaces.*

Proof. First notice a smooth factor $e^{2\psi}$ with $\psi \in C^\infty(X)$ preserves this statement. We only need to check the metric near ∂X with specified choice of $\tilde{\rho}$ in each domain.

- In Ω_1,

$$\tilde{m} = d^2s - 2sds\frac{d\rho}{\rho} - (1-s^2)(\frac{d\rho}{\rho})^2 - d^2\theta.$$

- In Ω_2,

$$\tilde{m} = -2da\frac{db}{b} - a(2-a)(\frac{db}{b})^2 - d^2\theta, \quad \tilde{m}|_{\{a=0\}} = -d^2\theta.$$

- In Ω_3,

$$\tilde{m} = -2d\tau d\rho + \rho^2 d^2\tau - d^2\theta, \quad \tilde{m}|_{\{\rho=0\}} = -d^2\theta.$$

- In Ω_4,

$$\tilde{m} = 2d\bar{a}\frac{d\bar{b}}{\bar{b}} + \bar{a}(2-\bar{a})(\frac{d\bar{b}}{d\bar{b}}) - (\frac{d\theta}{1-\bar{a}})^2, \quad \tilde{m}|_{\{\bar{a}=0\}} = -d^2\theta.$$

- In Ω_5,

$$\tilde{m} = (1-|y|^2)(\frac{d\phi}{\phi})^2 - \sum_{i=1}^{n} d^2y^i + 2y^i dy^i(\frac{d\phi}{\phi}).$$

Hence $\tilde{m}$ is a b-Lorentz metric on X and has signature $(0, n-1)$ when restricted on $S_1^+ \cup S_1^-$. □

2.2.2 Connection.

To investigate the geometric property of $\tilde{m}$ near ∂X, we examine the connection components in local coordinates. For simplicity, we change variable for ρ_0, ρ_2 and only write out the nonzero connection components.

- In Ω_1, change variable by $\xi = -\log\rho \in (0, \infty)$. Then in coordinates $\{s, \xi, \theta\}$,

$$\tilde{m} = d^2s + 2sdsd\xi - (1-s^2)d^2\xi - d^2\theta,$$

$$\tilde{m} = \begin{bmatrix} 1 & s & 0 \\ s & -1+s^2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \tilde{m}^{-1} = \begin{bmatrix} 1-s^2 & s & 0 \\ s & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

$$\Gamma^s_{ss}(\tilde{m}) = s, \quad \Gamma^s_{s\xi}(\tilde{m}) = s^2, \quad \Gamma^s_{\xi\xi}(\tilde{m}) = -s(1-s^2), \quad \Gamma^s(\tilde{m}) = 2s,$$

$$\Gamma^\xi_{ss}(\tilde{m}) = -1, \quad \Gamma^\xi_{s\xi}(\tilde{m}) = -s, \quad \Gamma^\xi_{\xi\xi}(\tilde{m}) = -s^2, \quad \Gamma^\xi(\tilde{m}) = -1.$$

- In Ω_2, change variable by $\xi = -\log b \in [\log 4, \infty)$. Then in coordinates $\{a, \xi, \theta\}$,

$$\tilde{m} = 2da d\xi - a(2-a)d^2\xi - d^2\theta,$$

$$\tilde{m} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -a(2-a) & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \tilde{m}^{-1} = \begin{bmatrix} a(2-a) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

$$\Gamma^a_{a\xi}(\tilde{m}) = -(1-a), \quad \Gamma^a_{\xi\xi}(\tilde{m}) = a(1-a)(2-a), \quad \Gamma^a(\tilde{m}) = -2(1-a);$$

$$\Gamma^\xi_{\xi\xi}(\tilde{m}) = 1-a, \quad \Gamma^\xi(\tilde{m}) = 0.$$

- In Ω_3, in coordinates $\{\tau, \rho, \theta\}$,

$$\tilde{m} = -2d\tau d\rho + \rho^2 d^2\tau - d^2\theta,$$

$$\tilde{m} = \begin{bmatrix} \rho^2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \tilde{m}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -\rho^2 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

$$\Gamma^\tau_{\tau\tau}(\tilde{m}) = \rho, \quad \Gamma^\tau(\tilde{m}) = 0,$$

$$\Gamma^\rho_{\rho\tau}(\tilde{m}) = -\rho, \quad \Gamma^\rho_{\tau\tau}(\tilde{m}) = \rho^3, \quad \Gamma^\rho(\tilde{m}) = 2\rho.$$

- In Ω_4, change variable by $\xi = -\log b$, then in coordinates $\{\bar{a}, \xi, \theta\}$,

$$\tilde{m} = -2d\bar{a}d\xi + \bar{a}(2-\bar{a})d^2\xi - (1-\bar{a})^{-2}d^2\theta,$$

$$\tilde{m} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & \bar{a}(2-\bar{a}) & 0 \\ 0 & 0 & -(1-\bar{a})^2 \end{bmatrix}, \quad \tilde{m}^{-1} = \begin{bmatrix} -\bar{a}(2-\bar{a}) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -(1-\bar{a})^{-2} \end{bmatrix};$$

$$\Gamma^{\bar{a}}_{\bar{a}\xi}(\tilde{m}) = -(1-\bar{a}), \quad \Gamma^{\bar{a}}_{\xi\xi}(\tilde{m}) = \Gamma^{\bar{a}}_{\theta\theta}(\tilde{m}) = \bar{a}(1-\bar{a})(2-\bar{a}),$$

$$\Gamma^{\bar{a}}(\tilde{m}) = 2 - (n+1)\bar{a} - (n-1)(1-\bar{a})^{-1}\bar{a},$$

$$\Gamma^\xi_{\xi\xi}(\tilde{m}) = \Gamma^\xi_{\theta\theta}(\tilde{m}) = 1-\bar{a}, \quad \Gamma^\xi(\tilde{m}) = -(n-1)(1-\bar{a})^{-1}.$$

- In Ω_5, change variable by $\zeta = -\log\phi$, in coordinates $\{\zeta, y^1, ..., y^n\}$,

$$\tilde{m} = (1-R^2)d^2\zeta - \sum_{i=1}^{n} d^2y^i - 2y^i dy^i d\zeta,$$

$$\tilde{m} = \begin{bmatrix} 1-R^2 & -y^1 & \cdots & -y^n \\ -y^1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -y^n & 0 & \cdots & -1 \end{bmatrix},$$

$$\tilde{m}^{-1} = \begin{bmatrix} 1 & -y^1 & \cdots & -y^n \\ -y^1 & -1+y^1y^1 & \cdots & y^1y^n \\ \vdots & \vdots & \ddots & \vdots \\ -y^n & y^1y^n & \cdots & -1+y^ny^n \end{bmatrix};$$

$$\Gamma^{\zeta}_{\zeta\zeta}(\tilde{m}) = -R.^2, \quad \Gamma^{\zeta}_{i\zeta}(\tilde{m}) = -y^i, \quad \Gamma^{\zeta}_{ij}(\tilde{m}) = -\delta_{ij}, \quad \Gamma^{\zeta}(\tilde{m}) = n,$$

$$\Gamma^{i}_{\zeta\zeta}(\tilde{m}) = -y^i(1-R^2), \quad \Gamma^{i}_{j\zeta}(\tilde{m}) = y^iy_j, \quad \Gamma^{i}_{jk}(\tilde{m}) = y^i\delta_{jk}, \quad \Gamma^i = -(n+1)y^i.$$

2.2.3 Wave Operator.

The Laplace-Beltrami operator w.r.t. $\tilde{m}$ is defined by

$$\Box_{\tilde{m}} = \tilde{m}^{IJ}\partial_I\partial_J - \Gamma^K\partial_K.$$

in local coordinates. First the commutator of $\Box_m$ with $Z_{\mu\nu}$ gives:

$$[\Box_{\tilde{m}}, Z_{ij}] = 0, \quad [\Box_{\tilde{m}}, Z_{00}] = 0,$$

for $\tilde{\rho} = \rho$ or $\tilde{\rho} = \phi$. The precise formula for $\Box_{\tilde{m}}$ and its commutators with vector fields in $\mathscr{V}_b(X^2)|_X$ are listed in the following.

- In Ω_1,

$$\Box_{\tilde{m}} = (1-s^2)\partial_s^2 - 2s\partial_s\rho\partial_\rho - 2s\partial_s - (\rho\partial_\rho)^2 - \rho\partial_\rho - \triangle_\theta,$$

$$[\Box_{\tilde{m}}, \partial_s] = 2s\partial_s^2, \quad [\Box_{\tilde{m}}, \rho\partial_\rho] = [\Box_{\tilde{m}}, \triangle_\theta] = 0,$$

$$[\Box_{\tilde{m}}, Z_{0i}] = -2s\theta_i(\Box_{\tilde{m}} + \gamma_0), \quad \text{with} \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.$$

- In Ω_2,

$$\begin{aligned}
\Box_{\tilde m} &= -2\partial_a b\partial_b + a(2-a)\partial_a^2 + 2(1-a)\partial_a - \triangle_\theta,\\
[\Box_{\tilde m}, a\partial_a] &= -2\partial_a(b\partial_b - a\partial_a) = \Box_{\tilde m} + (a\partial_a)^2 + a\partial_a + \triangle_\theta,\\
[\Box_{\tilde m}, b\partial_b] &= [\Box_{\tilde m}, \triangle_\theta] = 0,\\
[\Box_{\tilde m}, \partial_a^k] &= -2k(1-a)\partial_a^{k+1} + k(k+1)\partial_a^k,\\
[\Box_{\tilde m}, Z_{0i}] &= -2(1-a)\theta_i(\Box_{\tilde m} + \gamma_0), \quad \text{with} \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.
\end{aligned}$$

- In Ω_3,

$$\begin{aligned}
\Box_{\tilde m} &= -2\partial_\tau\partial_\rho - (\rho\partial_\rho)^2 - \rho\partial_\rho - \triangle_\theta,\\
[\Box_{\tilde m}, \partial_\tau] &= [\Box_{\tilde m}, \triangle_\theta] = 0, \ [\Box_{\tilde m}, \partial_\rho] = 2(\rho\partial_\rho + 1)\partial_\rho.\\
[\Box_{\tilde m}, \rho\partial_\rho] &= -2\partial_\rho\partial_\tau = \Box_{\tilde m} + (\rho\partial_\rho)^2 + \rho\partial_\rho + \triangle_\theta,\\
[\Box_{\tilde m}, Z_{0i}] &= -2(1+\rho\tau)\theta_i(\Box_{\tilde m} + \gamma_0), \quad \text{with} \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.
\end{aligned}$$

- In Ω_4,

$$\begin{aligned}
\Box_{\tilde m} =& 2\partial_{\bar a}\bar b\partial_{\bar b} - \bar a(2-\bar a)\partial_{\bar a}^2 - 2\partial_{\bar a} + (n+1)\bar a\partial_{\bar a} - (1-\bar a)^{-2}\triangle_\theta\\
&+ (n-1)(1-\bar a)^{-1}(\bar a\partial_{\bar a} - \bar b\partial_{\bar b}),\\
[\Box_{\tilde m}, \bar b\partial_{\bar b}] =& [\Box_{\tilde m}, \triangle_\theta] = 0,\\
[\Box_{\tilde m}, \bar a\partial_{\bar a}] =& \Box_{\tilde m} - (\bar a\partial_{\bar a})^2 - n\bar a\partial_{\bar a} + (1-\bar a)^{-3}(1+\bar a)\triangle_\theta\\
&- (n-1)(1-\bar a)^{-2}(\bar a\partial_{\bar a} - \bar b\partial_{\bar b}),\\
[\Box_{\tilde m}, Z_{0i}] =& -2(1-\bar a)\theta_i(\Box_{\tilde m} + \gamma_0), \quad \text{with} \quad \gamma_0 = \tfrac{n^2-1}{4}.
\end{aligned}$$

- In Ω_5,

$$\begin{aligned}
\Box_{\tilde m} &= (\phi\partial_\phi + R\partial_R)^2 + n(\phi\partial_\phi + R\partial_R) - \triangle_y,\\
[\Box_{\tilde m}, \phi\partial_\phi] &= 0, \quad [\Box_{\tilde m}, \partial_{y^i}] = -(2R\partial_R + 1)\partial_{y^i},\\
[\Box_{\tilde m}, Z_{0i}] &= -2y_i(\Box_{\tilde m} + \gamma_0), \quad \text{with} \quad \gamma_0 = \tfrac{n^2-1}{4}.
\end{aligned}$$

2.2.4 Time-like Functions.

Time-like functions is the most important concept for a Lorentzian metric: we use it to define positive quadratic form and the space-like hypersurface, on which the energy norm is defined.

Definition 2.2.2. For a Lorentz metric $\tilde{g}$ on X of signature $(1, n)$, say a function $T \in C^1(\Omega)$ is *time-like* w.r.t. $\tilde{g}$ at $p \in X$ if and only if

$$\langle \nabla T, \nabla T \rangle_{\tilde{g}} > 0 \quad \text{at } p\ ;$$

null w.r.t. $\tilde{g}$ at p if and only if

$$\langle \nabla T, \nabla T \rangle_{\tilde{g}} = 0 \quad \text{at } p\ .$$

A hypersurface $\Sigma \subset X$ is called *space-like* (resp. *null*) w.r.t. $\tilde{g}$ if and only if $\Sigma \cap \Omega$ has a defining function $T \in C^1(X)$ such that T is time-like (resp. null) on Σ.

Definition 2.2.3. A quadratic form field associated to time-like function T w.r.t. $\tilde{g}$ is defined by

$$F(T, v) = \langle \nabla T, \nabla v \rangle_{\tilde{g}} \nabla v - \tfrac{1}{2} \langle \nabla v, \nabla v \rangle_{\tilde{g}} \nabla T + \tfrac{1}{2} \gamma_0 v^2 \nabla T.$$

Here ∇ is the connection w.r.t. $\tilde{g}$ and $\gamma_0 > 0$ is some constant.

Lemma 2.2.4. *Suppose T, T' are two time-like function w.r.t. a Loretzian metric $\tilde{g}$ at $p \in X$ such that $\langle \nabla T, \nabla T' \rangle_{\tilde{g}} > 0$ at p, then $\langle F(T, v), \nabla T' \rangle_{\tilde{g}}$ is a strict positive quadratic form at p.*

⟨cauchy.2⟩

Lemma 2.2.5. *Suppose $\Omega \subset X$ has piecewise smooth boundary $\partial\Omega$ with defining function T', then*

$$\int_{\Omega} \operatorname{div}_{\tilde{g}}(F(T, v))\, dvol_{\tilde{g}} = \int_{\partial\Omega} \langle F(T, v), \nabla T' \rangle_{\tilde{g}}\, d\mu_{\tilde{g}}^{T'}.$$

Here $d\mu^{T'}_{\tilde{g}}$ *is the volume form on* $\partial\Omega$ *such that* $d\mu^{T'}_{\tilde{g}} \wedge dT' = dvol_{\tilde{g}}$ *and*

$$\mathrm{div}_{\tilde{g}}(F(T,v)) = \langle \nabla v, \nabla T\rangle_{\tilde{g}}(\Box_{\tilde{g}} + \gamma_0)v + \nabla^2 T(dv,dv) + \frac{1}{2}(\gamma_0 v^2 - \langle \nabla v, \nabla v\rangle_{\tilde{g}})\Box_{\tilde{g}} T.$$

Lemma 2.2.5 implies the energy estimates if choosing T and Ω properly. Here we list the time-like functions w.r.t. $\tilde{m}$ and the corresponding space-like hypersurfaces in each domain. For the purpose of energy estimates, we modify the covering of X to $\{\overline{\Omega}_i\}_{1\le k\le 4}$ such that each domain is bounded by either space like hypersurfaces or null hypersurface or infinity w.r.t. $\tilde{m}$.

- In $\Omega_1 \cup \Omega_0$, define

$$T_1 = \frac{t}{\psi_1(r)} \quad \text{where}$$

$$\psi(r) = \begin{cases} \frac{3}{2} & \text{for } r < 1 \\ r & \text{for } r > 2 \end{cases}, \quad 0 \le \psi_1'(r) \le 1,\ \psi_1''(r) \ge 0.$$

Set

$$\overline{\Omega}_1 = \{T_1 \in (-\tfrac{7}{8}, \tfrac{7}{8})\}, \quad \Sigma_0 = \{T_1 = 0\}, \quad \Sigma_1 = \{T_1 = \tfrac{7}{8}\}.$$

Then in $\overline{\Omega}_1$,

$$\langle \nabla T_1, \nabla T_1\rangle_{\tilde{m}} = \begin{cases} 1 - s^2 > \frac{15}{64} & \text{if } r > 2 \\ (\tilde{\rho}\psi(r))^{-2}(1 - T^2(\psi'(r))^2) \ge \frac{7}{8}(\tilde{\rho}\psi(r))^{-2} > \frac{7}{32} & \text{if } r \le 2 \end{cases}.$$

Here for $r > 2$,

$$\Box_{\tilde{m}} T_1 = -2s,$$
$$\mathrm{div}_{\tilde{m}} F(T_1, v) = -s(|\nabla_\theta v|^2 + \gamma_0 v^2) + (\Box_{\tilde{m}} + \gamma_0)v,$$
$$\begin{aligned}\langle F(T_1,v), \nabla T_1\rangle_{\tilde{m}} =& \tfrac{1}{2}(1-s^2)^2|\partial_s v|^2 + \tfrac{1}{2}(1+s^2)|s(1-s^2)\partial_s v - \rho\partial_\rho v|^2 \\ &+ \tfrac{1}{2}(1-s^2)(|\nabla_\theta v|^2 + \gamma_0 v^2).\end{aligned}$$

- In Ω_2, define

$$T_2 = -a + \log b, \quad T_2' = -a.$$

Set

$$\overline{\Omega}_2 = \{a \in [0, \tfrac{1}{8}],\ T_2 \in (-\infty, \log \tau_0]\}, \quad \Sigma_2 = \{T_2 = \log \tau_0,\ a \leq \tfrac{1}{8}\}.$$

Then in $\overline{\Omega}_2$,

$$\begin{aligned}
\langle \nabla T_2, \nabla T_2 \rangle_{\tilde{m}} &= 2 + a(2-a) > 2, \\
\langle \nabla T_2', \nabla T_2' \rangle_{\tilde{m}} &= a(2-a) > a, \\
\langle \nabla T_2, \nabla T_2 \rangle_{\tilde{m}} &= 1 + a(2-a) > 1.
\end{aligned}$$

Here

$$\begin{aligned}
\Box_{\tilde{m}} T_2 &= -2(1-a), \\
\mathrm{div}_{\tilde{m}}(F(T_2, v)) &= (1-a)(|\partial_a v|^2 - |\partial_\theta v|^2 - \gamma_0 v^2) + (\Box_{\tilde{m}} + \gamma_0) v, \\
\langle F(T_2, v), \nabla T_2 \rangle_{\tilde{m}} =& \tfrac{1}{2}|b\partial_b v|^2 + \tfrac{1}{2}|(b\partial_b - a(2-a)\partial_a)v|^2 + (1 + a(2-a))|\partial_a v|^2 \\
&+ \tfrac{1}{2}(a(2-a) + 2)(|\partial_\theta v|^2 + \gamma_0 v^2), \\
\langle F(T_2, v), \nabla T_2' \rangle_{\tilde{m}} =& \tfrac{1}{2}|b\partial_b v|^2 + \tfrac{1}{2}|(b\partial_b - a(2-a)\partial_a)v|^2 + \tfrac{1}{2}a(2-a)|\partial_a v|^2 \\
&+ \tfrac{1}{2}(a(2-a) + 1)(|\partial_\theta v|^2 + \gamma_0 v^2), \\
\langle F(T_2, v), \nabla \log b \rangle_{\tilde{m}} =& |\partial_a v|^2 + \tfrac{1}{2}a(2-a)|\partial_a v|^2 + \tfrac{1}{2}|\partial_\theta v|^2 + \tfrac{1}{2}\gamma_0 v^2.
\end{aligned}$$

- In $\Omega_3 \cup \Omega_0$, define for $\tau_0 > 8$,

$$T_3 = t - \psi_2(r), \quad T_3' = -\rho(2\tau_0 - \tau), \quad \text{where}$$

$$\psi_2(r) = \begin{cases} r + \rho & \text{if } r > 2 \\ 2 & \text{if } r < 1 \end{cases}, \quad 0 \leq \psi_2'(r) \leq 1, \quad \psi_2''(r) \geq 0.$$

Set

$$\overline{\Omega}_3 = \overline{\{T_3 \leq \tau_0\} \backslash (\overline{\Omega}_1 \cup \overline{\Omega}_2)},$$

$$\Sigma_3 = (\overline{\Omega}_3 \cap \Sigma_1) \cup (\overline{\Omega}_3 \cap \Sigma_2), \quad \Sigma_4 = \{T_3 = \log \tau_0\}.$$

Here T_3' is only defined near $S_1 \cap \Omega_3$ such that $\rho(2\tau_0 - \tau) \leq 1$. Then in $\overline{\Omega}_3$,

$$\langle \nabla T_3, \nabla T_3 \rangle_{\tilde{m}} = \begin{cases} 2 - \rho^2 > 1 & \text{if } r > 2 \\ \tilde{\rho}^{-2}(1 - (\psi_2'(r))^2) \geq \frac{7}{16} & \text{if } r \leq 2 \end{cases},$$

and for $\rho(2\tau_0 - \tau) \leq 1$,

$$\langle \nabla T_3', \nabla T_3' \rangle_{\tilde{m}} = \rho(2\tau_0 - \tau)(2 - \rho(2\tau_0 - \tau)) > \tau_0 \rho,$$
$$\langle \nabla T_3, \nabla T_3' \rangle_{\tilde{m}} = (2\tau_0 - \tau) + \rho(1 - \rho(2\tau_0 - \tau)) > \tau_0.$$

Moreover,

$$\Box_{\tilde{m}} T_3 = 2\rho,$$
$$\mathrm{div}_{\tilde{m}}(F(T_3, v)) = \rho(|\partial_\theta v|^2 - \rho^2 |\partial_\rho v|^2 + \gamma_0 v^2) + (\Box_{\tilde{m}} + \gamma_0) v,$$
$$\begin{aligned} \langle F(T_3, v), \nabla T_3 \rangle_{\tilde{m}} =& \tfrac{1}{2}|\partial_\tau v|^2 + \tfrac{1}{2}|\partial_\tau v + \rho^2 \partial_\rho v|^2 + (1 - \rho^2)|\partial_\rho v|^2 \\ &+ \tfrac{1}{2}(2 - \rho^2)(|\partial_\theta v|^2 + \gamma_0 |v|^2), \\ \langle F(T_3, v), \nabla T_3' \rangle_{\tilde{m}} =& \tfrac{1}{2}(2\tau_0 - \tau)|\partial_\tau v|^2 + \tfrac{1}{2}(2\tau_0 - \tau)|\partial_\tau v + \rho^2 \partial_\rho v|^2 \\ &+ (1 - \tfrac{1}{2}\rho^2 - \tfrac{1}{2}\rho(2\tau_0 - \tau))\rho |\partial_\rho v|^2 \\ &+ \tfrac{1}{2}(2(\tau_0 - \tau) + \rho(1 - \rho(2\tau_0 - \tau)))(|\partial_\theta v|^2 + \gamma_0 |v|^2). \end{aligned}$$

- In $\Omega_4 \cup \Omega_5$, define for $\alpha > 1$,

$$T_4 = -f_4(a) - \log \bar{b}, \quad T_4' = -f_5(a) \bar{b}^\alpha.$$

Here f_4, f_5 are smooth functions on $\Omega_4 \cup \Omega_5$ such that

$$f_4(a) = \begin{cases} \bar{a} & \text{if } \bar{a} \in [0, \frac{1}{4}) \\ \log \bar{a} + c_4 & \text{if } \bar{a} \in (\frac{3}{4}, 1] \end{cases}, \quad f_4' \in [\tfrac{1}{2}, 1], \quad c_4 \in (0, 1);$$

$$f_5(\bar{a}) = \begin{cases} \bar{a} & \text{if } \bar{a} \in [0, \frac{1}{4}) \\ c_5 \bar{a}^\alpha & \text{if } \bar{a} \in (\frac{3}{4}, 1] \end{cases}, \quad f_5' \in (1, 2), \quad f_5'' \geq 0, \quad c_5 \in (1, \tfrac{3}{2}).$$

Set

$$\overline{\Omega}_4 = \overline{(\Omega_4 \cup \Omega_5)\backslash\Omega_3}.$$

Then in $\overline{\Omega}_4$,

$$\begin{aligned}
\langle \nabla T_4, \nabla T_4\rangle_{\tilde{m}} &= f_4'(\bar{a})\big(2 - \bar{a}(2-\bar{a})f_4'(\bar{a})\big) \geq \tfrac{1}{2},\\
\langle \nabla T_4', \nabla T_4'\rangle_{\tilde{m}} &= \bar{b}^{2\alpha} f_5'(\bar{a})\big(- \bar{a}(2-\bar{a})f_5'(\bar{a}) + 2\alpha f_5(\bar{a})\big) \geq (\alpha - 1)\bar{a}\bar{b}^{2\alpha},\\
\langle \nabla T_4, \nabla T_4'\rangle_{\tilde{m}} &= \bar{b}^{\alpha}\big(f_5'(\bar{a})(1 - \bar{a}(2-\bar{a})f_4'(\bar{a})) + \alpha f_4'(\bar{a})f_5(\bar{a})\big) \geq \tfrac{1}{2}\bar{b}^{\alpha}.
\end{aligned}$$

Here for $\bar{a} \in [0, \frac{1}{4})$,

$$\begin{aligned}
T_4 = -\bar{a} - \log \bar{b}, \quad T_4' &= -\bar{a}\bar{b}^{\alpha},\\
\Box_{\tilde{m}} T_4 &= (n+1)(1-\bar{a}),\\
\mathrm{div}_{\tilde{m}}(F(T_4, v)) =&(1-\bar{a})\big(- |\partial_{\bar{a}} v|^2 + \tfrac{n-1}{2}\bar{a}(2-\bar{a})|\partial_{\bar{a}} v|^2\\
&+ \tfrac{n-1}{2}(1-\bar{a})^{-2}|\nabla_\theta v|^2 + \tfrac{n+1}{2}\gamma_0 v^2\big) + (\Box_{\tilde{m}} + \gamma_0)v,\\
\langle F(T_4, v), \nabla T_4\rangle_{\tilde{m}} =& \tfrac{1}{2}|\partial_\xi v|^2 + \tfrac{1}{2}|\partial_\xi v + \bar{a}(2-\bar{a})\partial_{\bar{a}} v|^2 + (1-\bar{a})^2|\partial_{\bar{a}} v|^2\\
&+ \tfrac{1}{2}(1 + (1-\bar{a})^2)((1-\bar{a})^{-2}|\nabla_\theta v|^2 + \gamma_0 v^2),\\
\langle F(T_4, v), \nabla T_4'\rangle_{\tilde{m}} =& b^{\alpha}\big(\tfrac{1}{2}|\partial_\xi + \bar{a}(2-\bar{a})\partial_a v|^2 + \tfrac{1}{2}\bar{a}(\alpha - 2 + \bar{a} + \alpha(1-\bar{a})^2)|\partial_{\bar{a}} v|^2\\
&+ \tfrac{1}{2}|\partial_\xi v|^2 + \tfrac{1}{2}((1-\bar{a})^2 + \alpha\bar{a})((1-\bar{a})^{-2}|\nabla_\theta v|^2 + \gamma_0 v^2)\big),\\
\langle F(T_4, v), \nabla(-\log \bar{b})\rangle_{\tilde{m}} =& |\partial_{\bar{a}} v|^2 - \tfrac{1}{2}\bar{a}(2-\bar{a})|\partial_{\bar{a}} v|^2 + \tfrac{1}{2}(1-\bar{a})^{-2}|\nabla_\theta v|^2 + \tfrac{1}{2}\gamma_0 v^2;
\end{aligned}$$

and for $\bar{a} \in (\frac{3}{4}, 1]$,

$$\begin{aligned}
T_4 = -\log\phi - c_4, \quad T_4' &= -c_5\phi^{\alpha},\\
\Box_{\tilde{m}} T_4 &= -n,\\
\mathrm{div}_{\tilde{m}}(F(T_4, v)) =& \tfrac{n}{2}|\phi\partial_\phi v + R\partial_R v|^2 - \tfrac{n-2}{2}|\nabla_y v|^2 + \tfrac{n}{2}\gamma_0 v^2 + (\Box_{\tilde{m}} + \gamma_0)v,\\
\langle F(T_4, v), \nabla T_4\rangle_{\tilde{m}} =& \tfrac{1}{2}(|\phi\partial_\phi v + R\partial_R v|^2 + |\nabla_y v|^2 + \gamma_0 v^2),\\
\langle F(T_4, v), \nabla T_4'\rangle_{\tilde{m}} =& \tfrac{1}{2}\alpha c_5\phi^{\alpha}(|\phi\partial_\phi v + R\partial_R v|^2 + |\nabla_y v|^2 + \gamma_0 v^2).
\end{aligned}$$

Here T_i are regular time-like all over $\overline{\Omega}_i$ and T_i' are time-like in the interior of $\overline{\Omega}_i$

but null on $S_1 \cap \overline{\Omega}_i$ w.r.t.$\tilde{m}$ for $1 \leq i \leq 4$. We will show in next section for $n \geq 5$ with a small perturbation of the metric $\tilde{m}$, those properties are preserved w.r.t. the perturbed metric.

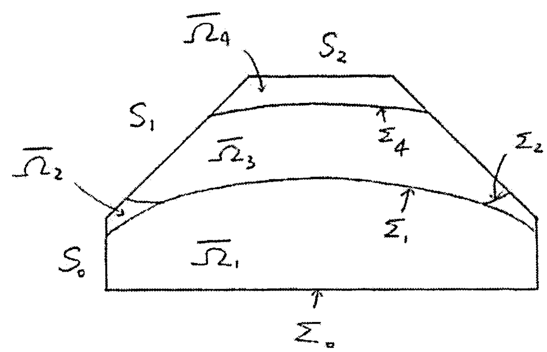

Figure 2-1: domains and hypersurfaces in $X \cap \{t \geq 0\}$.

Chapter 3

Cauchy Problem for Einstein Vacuum Equations with $n \geq 5$.

In this section, I consider the Cauchy problem for the conformal transformation of Einstein Vacuum equations in harmonic gauge on X for $n \geq 5$. We first show that given the Cauchy data small in some weighted b-Sobolev space on Σ_0 the radiation field for the solution is well defined. This is done by energy estimates method. Then applying the linear radiation field theory with a background metric which is close to Minkowski space-time, I show that the map from constraint Cauchy data to constraint characteristic data is an isomorphism on some small neighborhoods of weighted b-Sobolev spaces.

3.1 Problem Description.

In this section, let us set up the Cauchy problem for the conformal transformation of Einstein Vacuum equations in harmonic gauge on the blown-up space X.

3.1.1 Perturbed Metric.

We consider (M, g) as a small perturbation of Minkowski space $(\mathbb{R}^{1+n}_{t,x}, m)$ and take $(M, g) = (\mathbb{R}^{1+n}_{t,x}, m + h)$ with h a symmetric $(0,2)$-tensor on $\mathbb{R}^{1+n}_{t,x}$:

$$h = h_{\mu\nu}dx^\mu dx^\nu = dt^2 + 2h_{0i}dtdx^i + h_{ij}dx^i dx^j.$$

Denote by

$$\tilde{g} = \tilde{\rho}^2 g = \tilde{m} + \tilde{\rho}^2 h \qquad (3.1) \boxed{\texttt{cauchy.3}}$$

the conformal transformation of g on X. In the fixed coordinates (t, x), it is equivalent to say that h a function on $\mathbb{R}^{1+n}_{t,x}$ or X valued in $(n+1) \times (n+1)$ symmetric matrix space $SM(n+1, \mathbb{R}) \simeq \mathbb{R}^{\frac{(n+2)(n+1)}{2}}$. In this sense, set

$$H = g^{-1} - m^{-1} \quad \text{and} \quad \tilde{h} = \tilde{\rho}^{\frac{1-n}{2}} h. \qquad (3.2) \boxed{\texttt{cauchy.4}}$$

To write the perturbation in local coordinates near boundary, let us first denote by

$$h_{\tau\tau} = h_{00}, \quad h_{\tau\rho} = h_{\rho\tau} = -(h_{00} + h_{0i}\theta^i), \quad h_{\rho\rho} = h_{00} + 2h_{0i}\theta^i + h_{ij}\theta^i\theta^j,$$
$$h_{\tau i} = h_{i\tau} = h_{0i}, \quad h_{\rho i} = h_{i\rho} = -(h_{0i} + h_{ij}\theta^j).$$

Then in each domain with the coordinates set in Section 1.1, the perturbation of conformal metric $\tilde{\rho}^2 h$ can be expressed as follows:

- In Ω_1, $s = \frac{t}{r}$, $\rho = \frac{1}{r}$, $\tilde{\rho} = \rho$,

$$\begin{aligned}\tilde{\rho}^2 h =& h_{00}d^2 s + (h_{00}s^2 + 2h_{0i}\theta^i s + h_{ij}\theta^i\theta^j)(\frac{d\rho}{\rho})^2 - 2(h_{00}s + h_{0i})ds\frac{d\rho}{\rho} \\ &+ h_{ij}d\theta^i d\theta^j + 2h_{0i}dsd\theta^i - 2(h_{0i}s + h_{ij}\theta^j)d\theta^j\frac{d\rho}{\rho}.\end{aligned}$$

- In Ω_2, $a = 1 - \frac{t}{r}$, $b = \frac{1}{r-t}$, $\tilde{\rho} = ab$,

$$\begin{aligned}\tilde{\rho}^2 h =& h_{\rho\rho}(\frac{da}{a})^2 + (h_{\rho\rho} + 2ah_{\tau\rho} + a^2 h_{\tau\tau})(\frac{db}{b})^2 + 2(h_{\rho\rho} + ah_{\tau\rho})\frac{da}{a}\frac{db}{b} \\ &+ h_{ij}d\theta^i d\theta^j + 2h_{\rho i}\frac{da}{a}d\theta^i + 2(h_{\rho i} + ah_{\tau i})\frac{db}{b}d\theta^i.\end{aligned}$$

- In Ω_3, $\rho = \frac{1}{r}$, $\tau = t - r$, $\tilde{\rho} = \rho$

$$\tilde{\rho}^2 h = h_{\rho\rho}(\frac{d\rho}{\rho})^2 + \rho^2 h_{\tau\tau} d^2\tau + 2h_{\tau\rho} d\tau d\rho + h_{ij} d\theta^i d\theta^j + 2\rho h_{\tau i} d\theta^i d\tau + 2h_{\rho i} d\theta^i \frac{d\rho}{\rho}.$$

- In Ω_4, $\bar{a} = 1 - \frac{r}{t}$, $\bar{b} = \frac{1}{t-r}$, $\tilde{\rho} = \bar{a}\bar{b}$,

$$\begin{aligned}\tilde{\rho}^2 h =& h_{\rho\rho}(\frac{d\bar{a}}{\bar{a}})^2 + (h_{\rho\rho} + 2\bar{a}h_{i\rho}\theta^i + \bar{a}^2 h_{ij}\theta^i\theta^j)(\frac{d\bar{b}}{\bar{b}})^2 + 2(h_{\rho\rho} + \bar{a}h_{\rho i}\theta^i)\frac{d\bar{a}}{\bar{a}}\frac{d\bar{b}}{\bar{b}} \\ &+ (1-\bar{a})^2 h_{ij} d\theta^i d\theta^j + 2(1-\bar{a})h_{\rho i} d\theta^i \frac{d\bar{a}}{\bar{a}} + 2(1-\bar{a})(h_{\rho i} + \bar{a}h_{ij}\theta^j) d\theta^i \frac{d\bar{b}}{\bar{b}}.\end{aligned}$$

- In Ω_5, $\phi = \frac{1}{t}$, $y = \frac{x}{t}$, $\tilde{\rho} = \phi$,

$$\tilde{\rho}^2 h =(h_{00} + 2h_{0i}y^i + h_{ij}y^i y^j)(\frac{d\phi}{\phi})^2 + h_{ij} dy^i dy^j - 2(h_{0i} + h_{ij}y^j) dy^i \frac{d\phi}{\phi}.$$

Here in Ω_i for $1 \le i \le 4$ we use the polar coordinates. The metric is the same as a restriction to $\mathbb{R}^2 \times \mathbb{S}^{n-1}_\theta$ of the metric on $\mathbb{R}^2 \times \mathbb{R}^n_\theta$ with same components. The restriction is equivalent to imposing a condition

$$|\theta| = 1, \quad \theta_i d\theta^i = 0.$$

To simplify the expression, let us define the following data depending on h which control the geometric perturbation of the conformal metric.

Definition 3.1.1. Define for any $h \in C^0(\Omega; SM(n+1, \mathbb{R}))$ and $\delta > 0$,

$$\begin{aligned}\Lambda_0(h) &= \sum_{\mu,\nu=0,\dots,n} |h_{\mu\nu}|, \\ \Lambda_1(h) &= \sup_{1\le i\le n} |\rho_1^{-1} h_{\rho i}|, \\ \Lambda_{1+\delta}(h) &= |\rho_1^{-1-\delta} h_{\rho\rho}|.\end{aligned}$$

Lemma 3.1.2. *If $h \in C^0(X; SM(n+1, \mathbb{R}))$ satisfying $\Lambda_0(h), \Lambda_1(h), \Lambda_{1+\delta}(h) < \epsilon$ for some $\delta > 0$ and $\epsilon > 0$ small enough, then $\tilde{g}$ extends to a Lorentzian b-metric on X^2 with $S_1^+ \cup S_1^-$ being its characteristic surfaces.*

Proof. The statement is obviously true if $\delta \geq 1$. For $\delta \in (0,1)$, let

$$f(\rho_1) = \tfrac{1}{2} \int_0^{\rho_1} {\rho_1}^{-2} h_{\rho\rho} da = O(\rho_1^{\delta}).$$

and change the tangent coordinates near S_1 as follows:

- In Ω_2, set $\log b = \log b' + f(a)$,
- In Ω_3, set $\tau = \tau' + f(\rho)$,
- In Ω_3, set $\log \bar{b} = \log \bar{b}' - f(\bar{a})$.

Then in the new coordinates we can see the metric components is uniformly bounded and $\{\rho_1 = 0\}$ is its characteristic surface. Since the coordinates changing does not change the surface $S_1^{\pm}$, we prove the statement. Notice that these coordinates changing involves a $C^{0,\delta}$ diffeomorphism of X. □

Definition 3.1.3. For any $k \in \mathbb{N}_0$, denote by $\Theta_k(h)$ a real analytic function of $h \in SM(n+1, \mathbb{R})$ with $C^\infty(X)$ coefficients such that

$$\Theta_k(h) = O(|h|^k) \quad \text{as} \quad \Lambda_0 |h| \to 0.$$

For any $k \in \mathbb{N}_0, l \in \mathbb{N}$, denote by $\Theta_k(h)(h^1, ..., h^l)$ a l form in $(h^1, ..., h^l)$ with coefficients $\Theta_k(h)$.

Lemma 3.1.4. *For any $k, l \in \mathbb{N}_0$*

$$\Theta_k(h) + \Theta_l(h) = \Theta_{\min\{k,l\}}(h), \quad \Theta_k(h)\Theta_l(h) = \Theta_{k+l}(h),$$
$$\Lambda_0(\Theta_k(h)) = \Theta_k(\Lambda_0(h)).$$

Lemma 3.1.5. *With above notation*

$$H^{00} = -h_{00} + \Theta_2(h), \quad H^{0i} = h_{0i} + \Theta_2(h), \quad H^{ij} = -h_{ij} + \Theta_2(h).$$

3.1.2 Einstein Vacuum Equation in Harmonic Gauge.

Say a metric $g = m + h$ satisfies the harmonic gauge condition in coordinates (t, x) if and only if

$$\Box_g x^\mu = 0 \quad \Leftrightarrow \quad \Gamma^\mu(g) = 0 \quad \Leftrightarrow \quad g^{\alpha\beta}\partial_\alpha g_{\mu\beta} = \tfrac{1}{2} g^{\alpha\beta}\partial_\mu g_{\alpha\beta}, \quad \text{for} \quad \mu = 0, 1, ..., n.$$

If $g = m + h$ satisfies the harmonic gauge condition in coordinates (t, x), then

- In Ω_2,

$$\begin{aligned} (b\partial_b - a\partial_a)(2h_{\rho\tau} + \mathrm{tr}_m h) &= \Theta_1(a)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ (b\partial_b - a\partial_a)(2h_{\rho i} - \theta_i \mathrm{tr}_m h) &= \Theta_1(a)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ \Longrightarrow \quad (b\partial_b - a\partial_a) h_{\rho\rho} &= \Theta_1(a)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h). \end{aligned}$$

- In Ω_3,

$$\begin{aligned} \partial_\tau(2h_{\rho\tau} + \mathrm{tr}_m h) &= \Theta_1(\rho)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ \partial_\tau(2h_{\rho i} - \theta_i \mathrm{tr}_m h) &= \Theta_1(\rho)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ \Longrightarrow \quad \partial_\tau h_{\rho\rho} &= \Theta_1(\rho)(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h). \end{aligned}$$

- In Ω_4,

$$\begin{aligned} (\bar{b}\partial_{\bar{b}} - \bar{a}\partial_{\bar{a}})(2h_{\rho\tau} + \mathrm{tr}_m h) &= \Theta_1(\bar{a})(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ (\bar{b}\partial_{\bar{b}} - \bar{a}\partial_{\bar{a}})(2h_{\rho i} - \theta_i \mathrm{tr}_m h) &= \Theta_1(\bar{a})(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h), \\ \Longrightarrow \quad (\bar{b}\partial_{\bar{b}} - \bar{a}\partial_{\bar{a}}) h_{\rho\rho} &= \Theta_1(\bar{a})(\tilde{\partial} h) + \Theta_1(h)(\tilde{\partial} h). \end{aligned}$$

Here $\mathrm{tr}_m h = m^{\alpha\beta} h_{\alpha\beta} = h_{00} - \sum_{i=1}^n h_{ii}$.

In harmonic gauge, the Einstein Vacuum equations reduces to a system of quasi-linear wave equations [LR1].

$$\begin{aligned} &\Box_g h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \quad \mu, \nu = 0, 1, ..., n, \\ &\text{where} \quad \Box_g = \widetilde{\Box}_g = g^{\alpha\beta}\partial_{\alpha\beta} = \Box_m + H^{\alpha\beta}\partial_{\alpha\beta} \end{aligned}$$

and $F_{\mu\nu}(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients that are real analytic functions of h. More precisely,

$$F_{\mu\nu}(h)(\partial h, \partial h) = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$

where $Q_{\mu\nu}(\partial h, \partial h)$ is a null form of constant coefficients and

$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{4}(\partial_\mu \mathrm{tr}_m h)(\partial_\nu \mathrm{tr}_m h) - \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'}(\partial_\mu h_{\alpha\beta})(\partial_\nu h_{\alpha'\beta'}),$$
$$G_{\mu\nu}(h)(\partial h, \partial h) = \Theta_1(h)(\partial h, \partial h).$$

By a conformal transformation (3.1) and (3.2), the reduced Einstein equations (1.3) are equivalent to

$$(\Box_{\tilde g} + \gamma(\tilde h))\tilde h_{\mu\nu} = (\rho_0\rho_2)^{\frac{n-1}{2}} \rho_1^{\frac{n-5}{2}} \tilde F_{\mu\nu}(\tilde h, \tilde\partial \tilde h), \tag{3.3}$$ eq.3

where

$$\gamma(\tilde h) = -\tilde\rho^{\frac{n-1}{2}} \Box_{\tilde g} \tilde\rho^{\frac{1-n}{2}} = \gamma_0 + \tilde\rho^{\frac{n-1}{2}} \Theta_1(\tilde h)) \quad \text{with} \quad \gamma_0 = -\tilde\rho^{\frac{n-1}{2}} \Box_{\tilde m} \tilde\rho^{\frac{1-n}{2}},$$

and

$$\tilde F_{\mu\nu} = \tilde\rho^{-\frac{n+3}{2}} F_{\mu\nu}(\tilde\rho^{\frac{n-1}{2}} \tilde h)(\partial(\tilde\rho^{\frac{n-1}{2}} \tilde h), \partial(\tilde\rho^{\frac{n-1}{2}} \tilde h))$$
$$= \Theta_0(\tilde h)(\tilde\partial \tilde h, \tilde\partial \tilde h) + \Theta_1(\tilde h)(\tilde\partial \tilde h) + \Theta_2(\tilde h).$$

Lemma 3.1.6. *With above notation, γ_0 is a constant in each domain Ω_i for $1 \le i \le 5$ and*

$$\Box_{\tilde g} = \Box_{\tilde m} + (\rho_0\rho_2)^{\frac{n-1}{2}} [\rho_1^{\frac{n-5}{2}} \tilde h_{\rho\rho} P_2 + \rho_1^{\frac{n-3}{2}} (\tilde\partial^2 + \tilde\partial)], \quad \text{where}$$
$$P_2 \in \mathcal{V}_b^2(X), \quad [P_2, \tilde\rho] = 0 \quad \textit{in } \Omega_2, \Omega_3, \Omega_4.$$

Proof. We can compute $\Box_{\tilde g}$ and $\gamma(\tilde h)$ precisely in each domain near ∂X.

- In Ω_1,

$$\begin{aligned}
\Box_{\tilde g} &= \Box_{\tilde m} + [H^{00} - 2sH^{0i}\theta_i + s^2 H^{ij}\theta_i\theta_j]\partial_s^2 + [H^{ij}\theta_i\theta_j](\rho\partial_\rho)^2 + H^{ij}\tilde\partial_i\tilde\partial_j \\
&\quad + 2[-H^{0i}\theta_i + sH^{ij}\theta_i\theta_j]\partial_s(\rho\partial_\rho) + 2[H^{0i} - sH^{ij}\theta_j]\partial_s\tilde\partial_i \\
&\quad + 2[-H^{ij}\theta_j](\rho\partial_\rho)\tilde\partial_i + [-(n+1)H^{0i}\theta_i + sH^{ij}(-\delta_{ij} + (n+2)\theta_i\theta_j)]\partial_s \\
&\quad + [H^{ij}(-\delta_{ij} + (n+1)\theta_i\theta_j)]\rho\partial_\rho + [-nH^{ij}\theta_j]\tilde\partial_i \\
&= \Box_{\tilde m} + \rho^{\frac{n-1}{2}}\Theta_1(\tilde h)(\tilde\partial^2 + \tilde\partial), \\
\gamma &= \gamma_0 + [H^{ij}(-\tfrac{n-1}{2}\delta_{ij} + \tfrac{(n-1)(n+3)}{4}\theta_i\theta_j)], \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.
\end{aligned}$$

- In Ω_2,

$$\begin{aligned}
\Box_{\tilde g} &= \Box_{\tilde m} + a^{-2}[H^{00} - 2H^{i0}\theta_i + H^{ij}\theta_i\theta_j](b\partial_b - a\partial_a)^2 + [H^{ij}\theta_i\theta_j](a\partial_a)^2 \\
&\quad + 2a^{-1}[-H^{0i}\theta_i + H^{ij}\theta_i\theta_j](a\partial_a)(b\partial_b - a\partial_a) + H^{ij}\tilde\partial_i\tilde\partial_j \\
&\quad + 2a^{-1}[H^{0i} - H^{ij}\theta_j]\tilde\partial_i(b\partial_b - a\partial_a) + 2[-H^{ij}\theta_j]\tilde\partial_i(a\partial_a) \\
&\quad + a^{-2}[H^{00} - 2H^{i0}\theta_i + H^{ij}\theta_i\theta_j + aH^{ij}(-\delta_{ij} + n\theta_i\theta_j) - (n-1)aH^{0i}\theta_i] \\
&\quad (b\partial_b - a\partial_a) + [H^{ij}(-\delta_{ij} + (n+1))\theta_i\theta_j]a\partial_a + [-nH^{ij}\theta_j]\tilde\partial_i, \\
&= \Box_{\tilde m} - a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}\tilde h_{\rho\rho}[(b\partial_b - a\partial_a)^2 + (b\partial_b - a\partial_a)] + a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}\Theta_1(\tilde h)(\tilde\partial^2 + \tilde\partial), \\
\gamma &= \gamma_0 + [H^{ij}(-\tfrac{n-1}{2}\delta_{ij} + \tfrac{(n-1)(n+3)}{4}\theta_i\theta_j)], \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.
\end{aligned}$$

- In Ω_3,

$$\begin{aligned}
\Box_{\tilde g} &= \Box_{\tilde m} + +\rho^{-2}[H^{00} - 2H^{i0}\theta_i + H^{ij}\theta_i\theta_j]\partial_\tau^2 + [H^{ij}\theta_i\theta_j](\rho\partial_\rho)^2 + H^{ij}\tilde\partial_i\tilde\partial_j \\
&\quad + 2[-H^{ij}\theta_j](\rho\partial_\rho)\tilde\partial_i + 2\rho^{-1}[-H^{i0}\theta_i + H^{ij}\theta_i\theta_j]\partial_\tau(\rho\partial_\rho) \\
&\quad + 2\rho^{-1}[H^{i0} - H^{ij}\theta_j]\partial_\tau\tilde\partial_i + [H^{ij}(-\delta_{ij} + (n+1)\theta_i\theta_j)]\rho\partial_\rho \\
&\quad + \rho^{-1}[-(n-1)H^{0i}\theta_i + H^{ij}(-\delta_{ij} + n\theta_i\theta_j)]\partial_\tau + n[-H^{ij}\theta_j]\bar\partial_i \\
&= \Box_{\tilde m} - \rho^{\frac{n-5}{2}}\tilde h_{\rho\rho}\partial_\tau^2 + \rho^{\frac{n-3}{2}}\Theta_1(\tilde h)(\tilde\partial^2 + \tilde\partial), \\
\gamma &= \gamma_0 + [H^{ij}(-\tfrac{n-1}{2}\delta_{ij} + \tfrac{(n-1)(n+3)}{4})\theta_i\theta_j], \quad \gamma_0 = \tfrac{(n-1)(n-3)}{4}.
\end{aligned}$$

- In Ω_4,

$$\begin{aligned}
\Box_{\tilde{g}} =&\Box_{\tilde{m}} + \bar{a}^{-2}[H^{00} - 2H^{i0}\theta_i + H^{ij}\theta_i\theta_j](\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}})^2 + H^{00}(\bar{a}\partial_{\bar{a}})^2 \\
&+ (1-a)^{-2}H^{ij}\tilde{\partial}_i\tilde{\partial}_j - 2\bar{a}^{-1}[H^{00} - H^{0i}\theta_i](\bar{a}\partial_{\bar{a}})(\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}}) \\
&- 2(1-\bar{a})^{-1}H^{0i}(\bar{a}\partial_{\bar{a}})\tilde{\partial}_i + 2(\bar{a}(1-\bar{a}))^{-1}[H^{i0} - H^{ij}\theta_j](\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}})\tilde{\partial}_i \\
&+ (1-\bar{a})^{-2}[-H^{ij}\theta_j - (n-1)H^{0i}]\tilde{\partial}_i + nH^{00}\bar{a}\partial_{\bar{a}} \\
&- \bar{a}^{-2}[H^{00} - 2H^{i0}\theta_i + H^{ij}\theta_i\theta_j + (n-1)\bar{a}(H^{00} - H^{0i}\theta_i) \\
&- \bar{a}(1-\bar{a})^{-1}H^{ij}(-\delta_{ij} + \theta_i\theta_j)](\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}}) \\
=& \Box_{\tilde{m}} - \bar{a}^{\frac{n-5}{2}}\bar{b}^{\frac{n-1}{2}}\tilde{h}_{\rho\rho}[(\bar{a}\partial_{\bar{a}} - \bar{b}\partial_{\bar{b}})^2 + (\bar{b}\partial_{\bar{b}} - \bar{a}\partial_{\bar{a}})] + \bar{a}^{\frac{n-3}{2}}\bar{b}^{\frac{n-1}{2}}\Theta_1(\tilde{h})(\tilde{\partial}^2 + \tilde{\partial}), \\
\gamma =& \gamma_0 + \tfrac{n^2-1}{4}H^{00}, \quad \gamma_0 = \tfrac{n^2-1}{4}.
\end{aligned}$$

- In Ω_5,

$$\begin{aligned}
\Box_{\tilde{g}} &= \Box_{\tilde{m}} + H^{00}(\phi\partial_\phi + y^k\partial_{y^k})^2 + H^{ij}\partial_{y^i}\partial_{y^j} - 2H^{0i}(\phi\partial_\phi + y^k\partial_{y^k})\partial_{y^i} \\
&\quad + nH^{00}(\phi\partial_\phi + y^i\partial_{y^i}) - (n+1)H^{0i}\partial_{y^i}, \\
&= \Box_{\tilde{m}} + \phi^{\frac{n-1}{2}}\Theta_1(\tilde{h})(\tilde{\partial}^2 + \tilde{\partial}), \\
\gamma &= \gamma_0 + \tfrac{n^2-1}{4}H^{00}, \quad \gamma_0 = \tfrac{n^2-1}{4}.
\end{aligned}$$

□

We also deduce the equation for $\tilde{\partial}^k\tilde{h}$ from (3.3) for all $k \in \mathbb{N}_0$ and $\tilde{\partial} \in \mathscr{V}_b(X)$.

Lemma 3.1.7. *The equations (3.3) implies that*

$$(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h}_{\mu\nu} = [\Box_{\tilde{m}}, \tilde{\partial}^k]\tilde{h}_{\mu\nu} + \tilde{f}^k_{\mu\nu}(\tilde{h}) \qquad (3.4)\ \boxed{\texttt{eq.5}}$$

where $\tilde{f}^k_{\mu\nu}(h)$ can be expressed as

$$\rho_1^{\frac{n-5}{2}}(\rho_0\rho_2)^{\frac{n-1}{2}} \sum_{\alpha_1+\cdots\alpha_l\le k+2,\ 2\le l\le k+2,\ 0\le\alpha_i\le k+1} \Theta_0(\tilde{h})(\tilde{\partial}^{\alpha_1}\tilde{h}, \cdots, \tilde{\partial}^{\alpha_l}\tilde{h}). \qquad (3.5)\ \boxed{\texttt{cauchy.6}}$$

Proof. For $k = 0$, obviously

$$\tilde{f}^0_{\mu\nu}(\tilde{h}) = \tilde{\rho}^{\frac{n-1}{2}}\gamma_1(\tilde{h}) + (\rho_0\rho_2)^{\frac{n-1}{2}}\rho_1^{\frac{n-5}{2}}\tilde{F}_{\mu\nu}.$$

is of form (3.5). For $k \geq 1$,

$$(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h}_{\mu\nu} = \tilde{\partial}^k\tilde{f}_{\mu\nu} + [\Box_{\tilde{g}}, \tilde{\partial}^k]\tilde{h}_{\mu\nu},$$

By direct computation, both $[\Box_{\tilde{g}} - \Box_{\tilde{m}}, \tilde{\partial}^k]\tilde{h}_{\mu\nu}$ and $\tilde{\partial}^k\tilde{f}_{\mu\nu}$ can be expressed as (3.5). □

Here $\tilde{\partial}^k$ means an element in $\mathscr{V}_b(X)^k$. In particular coordinates, we can choose a basis of $\mathscr{V}_b(X)$ and view k as a multi-index. In (3.4), $\tilde{f}^k$ in the form of (3.5) is quadratic with variable coefficients and the commutator $[\Box_{\tilde{m}}, \tilde{\partial}^k]$ gives linear terms of derivative order $\leq k+1$ or a nonlinear term in the form of (3.5).

3.1.3 Cauchy Data in Harmonic Gauge.

Given Cauchy data (g_0, k_0) on $\mathbb{R}^n$ which satisfies the constraint equations (1.2) and is close to $(-\sum_{i=1}^n d^2x^i, 0)$, we can construct $(g|_{t=0}, \partial_t g|_{t=0})$ satisfying the harmonic gauge condition at $t = 0$, [LR1].

Theorem 3.1.8 (Lindblad, Rodnianski). *The solution to reduced Einstein equations (1.3) with the constructed Cauchy data $(g|_{t=0}, \partial_t g|_{t=0})$ satisfies the harmonic gauge condition globally and hence provides a true solution to Einstein vacuum equations (1.1).*

This is basically a uniqueness theorem for wave equation on the connection components $\Gamma^\mu(g)$ for $i = 0, ..., n$. In this paper, we consider the Cauchy problem for reduced Einstein equation (1.3) with initial data

$$(g|_{t=0}, \partial_t g|_{t=0}) = (-\sum_{i=1}^n d^2x^i + h^0, h^1)$$

such that

$$\Gamma^\mu(g)|_{t=0} = 0, \quad \partial_t\Gamma^\mu g|_{t=0}.$$

Denote by

$$G_0 = (m + h^0)^{-1} = m + \Theta_1(h^0)$$
$$\Longrightarrow \quad \partial_t (m+h)^{\alpha\beta}|_{t=0} = -G_0^{\alpha\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'}.$$

Hence $\Gamma^\mu(g)|_{t=0} = 0$ implies that for $j = 1, ..., n,$

$$\begin{aligned} G_0^{0\beta} h^1_{0\beta} + G_0^{i\beta}\partial_i h^0_{0\beta} - \tfrac{1}{2} G_0^{\alpha\beta} h^1_{\alpha\beta} &= 0, \\ G_0^{0\beta} h^1_{j\beta} + G_0^{i\beta}\partial_i h^0_{j\beta} - \tfrac{1}{2} G_0^{\alpha\beta} \partial_j h^0_{\alpha\beta} &= 0; \end{aligned} \qquad (3.6) \; \boxed{\texttt{constraint.2}}$$

and $\partial_t \Gamma^\mu(g)|_{t=0} = 0$ implies that for $j = 1, ..., n,$

$$\begin{aligned} \left(G_0^{0\beta}\partial_t^2 h_{0\beta} - \tfrac{1}{2} G_0^{\alpha\beta}\partial_t^2 h_{\alpha\beta}\right)|_{t=0} &= -G_0^{i\beta}\partial_i h^1_{0\beta} + E_0(h^0, h^1), \\ \left(G_0^{0\beta}\partial_t^2 h_{j\beta}\right)|_{t=0} &= -G_0^{i\beta}\partial_i h^1_{j\beta} + \tfrac{1}{2} G_0^{\alpha\beta}\partial_j h^1_{\alpha\beta} + E_j(h^0, h^1); \end{aligned}$$

where

$$\begin{aligned} E_0(h^0, h^1) = &\; -\tfrac{1}{2} G_0^{\alpha\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} h^1_{\alpha\beta} + G_0^{0\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} h^1_{0\beta} + G_0^{i\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} \partial_i h^0_{0\beta}, \\ E_j(h^0, h^1) = &\; -\tfrac{1}{2} G_0^{\alpha\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} \partial_j h^0_{\alpha\beta} + G_0^{0\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} h^1_{j\beta} + G_0^{i\alpha'} G_0^{\beta\beta'} h^1_{\alpha'\beta'} \partial_i h^0_{j\beta}. \end{aligned}$$

However, the reduced Einstein vacuum equations at $t = 0$ gives

$$\partial_t^2 h_{\mu\nu} = (G_0^{00})^{-1} F'_{\mu\nu}(h^0, h^1).$$

Here $F'_{\mu\nu}(h^0, h^1)$ are real analytic functions of $(h^0, h^1, \partial_i h^0, \partial_i h^1, \partial_i \partial_j h^0)$. Hence this problem is still overdetermined, which impose a constraint condition on h^0, h^1 such that for $j = 1, ..., n,$

$$\begin{aligned} G_0^{0\beta} F'_{0\beta}(h^0, h^1) - \tfrac{1}{2} G_0^{\alpha\beta} F'_{\alpha\beta}(h^0, h^1) &= G_0^{00}(-G_0^{i\beta}\partial_i h^1_{0\beta} + E_0(h^0, h^1)), \\ G^{0\beta} F'_{j\beta}(h^0, h^1) &= G_0^{00}(-G_0^{i\beta}\partial_i h^1_{j\beta} + \tfrac{1}{2} G_0^{\alpha\beta}\partial_j h^1_{\alpha\beta} + E_j(h^0, h^1)). \end{aligned} \qquad (3.7) \; \boxed{\texttt{constraint.3}}$$

Now (3.6) and (3.7) give the constraint condition for Cauchy data for reduced Einstein vacuum equations.

Corallary 3.1.9. *Given Cauchy data* $(h^0, h^1) \in C^\infty(\mathbb{R}^n; SM(n+1, \mathbb{R}))$ *small and* *satisfying the constraint condition (3.6) and (3.7), then a solution to (1.3) provides a true solution to (1.1) in harmonic gauge.*

Definition 3.1.10. Define $\widetilde{\mathcal{U}}_\epsilon^{N,\delta}$ the space consisting of the elements (h^0, h^1) which satisfies the constrain condition (3.6) and (3.7) such that

$$\|h^0\|_{\rho_0^{\frac{n-1}{2}+\delta} H_b^{N+1}(\Sigma_0)} + \|h^1\|_{\rho_0^{\frac{n+1}{2}+\delta} H_b^{N}(\Sigma_0)} < \epsilon.$$

Here $\Sigma_0 = \overline{\mathbb{R}^n}$ is the radial compactification of $\mathbb{R}^n$ and ρ_0 is the boundary defining function for $S_0 \cap \Sigma_0$ as set in Section 2.1.

3.2 Energy Estimates for $n \geq 5$.

In this section, we show by energy estimates that for $n \geq 5$ if

$$(h^0, h^1) \in \widetilde{\mathcal{U}}_\epsilon^{N,\delta}, \quad \text{for } N > n+6,\ \delta \in (0, \tfrac{1}{2}) \text{ and } \epsilon > 0 \text{ small},$$

then there exists a global solution $\tilde{h}$ to (3.4) which is $C^{0,\delta'}$ up to S_1 for some $\delta' \in (0, \delta]$ and hence the radiation field in Friedlander's sense is well defined:

$$\mathcal{R}_{\mathcal{F}}(h^0, h^1) = \tilde{h}|_{S_1} \in \rho_0^{\delta} \rho_2^{\sigma - \frac{n-1}{2}} H_b^N(S_1) \quad \text{for some } \sigma > 0.$$

3.2.1 Preparation.

In Section 2.1, we give a covering $\{\overline{\Omega}_i : 1 \leq i \leq 4\}$ of X. Each of these domains is bounded by space-like hypersurfaces defined by time-like functions or characteristic hypersurface S_1 or infinity w.r.t. $\tilde{m}$. For $n \geq 5$ and with small perturbation, we can choose the same time-like functions and hence same space-like hypersurfaces to proceed the energy estimates. Refer to Section 2.1, 2.2 for notations. Let us first deduce an inverse formula for symmetric matrix close to $\tilde{m}$ in chosen coordinates.

Lemma 3.2.1. *Suppose in chosen coordinates,*

$$\tilde{g} = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix},$$

where $A^T = A$, $B^T = B$ and A, B are invertible. Then

$$\tilde{g}^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad \text{where}$$

$$\begin{aligned} G_{11} =& A - CB^{-1}C^T)^{-1}, \\ G_{12} =& G_{21}^T = -(A - CB^{-1}C^T)^{-1}CB^{-1}, \\ G_{22} =& B^{-1} + B^{-1}C^T(A - CB^{-1}C^T)^{-1}CB^{-1}. \end{aligned}$$

In this section for h small, if not specified, B is always set as follows:

$$B = -Id_{n\times n} + [h_{ij}]_{n\times n}, \quad \Rightarrow \quad B^{-1} = -Id_{n\times n} - [h_{ij}]_{n\times n} + \Theta_2(h);$$

and A is a 2×2 symmetric matrix and C is a $2 \times n$ matrix. We view the metric $\tilde{g}$ as a restriction to $\mathbb{R}^2 \times \mathbb{S}_\theta^{n-1}$ of the metric on $\mathbb{R}^2 \times \mathbb{R}_\theta^n$ with same components.

In the following, I omit the constant C if it is independent of ϵ_i for ϵ_i small enough.

3.2.2 In $\overline{\Omega}_1$

We solve the equation (3.3) from Σ_0 to Σ_1 with Cauchy data on Σ_0 such that

$$\sum_{|k|\le N+1} ||\tilde{\partial}^k \tilde{h}||_{\rho_0^\delta H_b^0(\Sigma_0)} < \epsilon_0, \quad 0 < \delta < \tfrac{1}{2}, \quad \epsilon_0 > 0, \quad \text{small.}$$

Notice that, in $\overline{\Omega}_1$, everything stated in the following also works for $n = 4$.

- Perturbed metric: by changing variable $\xi = -\log \rho \in (0, \infty)$ near the boundary

S_0, the metric $\tilde{g}$ can be expressed in coordinate (s, ξ, θ) as follows,

$$A = \begin{bmatrix} 1 + h_{00} & s + h_{00}s + h_{0i} \\ s + h_{00}s + h_{0i}\theta^i & -1 + s^2 + h_{00}s^2 + 2sh_{0i}\theta^i + h_{ij}\theta^i\theta^j \end{bmatrix},$$

$$C = \begin{bmatrix} h_{01} & \cdots & h_{0n} \\ sh_{01} + h_{1j}\theta^j & \cdots & sh_{0n} + h_{nj}\theta^j \end{bmatrix}.$$

Hence the m

$$\tilde{g} = \tilde{m} + \Theta_1(h) = \tilde{m} + \rho_0^{\frac{n-1}{2}}\Theta_1(\tilde{h}), \quad \tilde{g}^{-1} = \tilde{m}^{-1} + \Theta_1(h) = \tilde{m}^{-1} + \rho_0^{\frac{n-1}{2}}\Theta_1(\tilde{h}).$$

- Connection components: for any $I, K, J \in \{s, \xi, \theta\}$,

$$\Gamma_{IJ}^K(\tilde{g}) = \Gamma_{IJ}^K(\tilde{m}) + \Theta_1(h) + \Theta_0(h)(\tilde{\partial}h) = \Gamma_{IJ}^K(\tilde{m}) + \rho_0^{\frac{n-1}{2}}[\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})].$$

- Partial derivatives:

$$\tilde{\partial} \in \{\partial_s, \rho\partial_\rho, \tilde{\partial}_i\}.$$

- Commutator:

$$[\Box_{\tilde{m}}, \tilde{\partial}^k] = \sum_{1 \le i \le k+1} c_i(s)\tilde{\partial}^i,$$

where $c_i(s)$ are smooth bounded function on $\overline{\Omega}_1$.

- Equations:

$$(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h} = \sum_{0 \le i \le k+1} c_i(s)\tilde{\partial}^i\tilde{h} + \tilde{f}^k.$$

where $\tilde{f}^k$ can be expressed as (3.5).

- Time-like function:

$$\langle \nabla T_1, \nabla T_1 \rangle_{\tilde{g}} = \langle \nabla T_1, \nabla T_1 \rangle_{\tilde{m}} + \Theta_1(h).$$

- Space-like hypersurface: define for $T_1 \in [0, \frac{7}{8}]$,

$$\Sigma_{T_1} = \{T_1 = \text{constant}\},$$

and consider the domain $\Omega_1^{T_1}$ bounded by Σ_0, Σ_{T_1} and S_0 for energy estimates.

- Quadratic form:

$$1 - |\rho_0^{\frac{n-1}{2}} \Theta_1(\tilde{h})| \leq \frac{\langle F(T_1, v), \nabla T_1 \rangle_{\tilde{g}}}{\langle F(T_1, v), \nabla T_1 \rangle_{\tilde{m}}} \geq 1 + |\rho_0^{\frac{n-1}{2}} \Theta_1(\tilde{h})|.$$

- Sobolev norm: define for $\|\rho_0^{\frac{n-1}{2}} \tilde{h}\|_\infty < \epsilon_1$ with $\epsilon_1 > 0$ small,

$$M_1^N(T_1, v) = \left(\sum_{k=0}^{N} \int_{\Sigma_{T_1}} e^{-2\delta \log \rho_0} \langle F(T_1, \tilde{\partial}^k v), \nabla T_1 \rangle_{\tilde{g}} d\mu_{\tilde{g}}^{T_1} \right)^{\frac{1}{2}} \simeq \|v\|_{\rho_0^\delta H_b^N(\Sigma_{T_1})}.$$

Here for ϵ_1 small enough and near the boundary S_0,

$$d\mu_{\tilde{g}}^{T_1} \simeq \frac{d\rho}{\rho} d\theta.$$

Moreover, we have

$$\sum_{k \leq \frac{N}{2}+1} \|\tilde{\partial}^k v\|_\infty \leq C \rho_0^\delta M_1^N(T_1, v).$$

where C is a constant independent of ϵ_1 if it is small enough.

- Divergence of quadratic form field: if $\|\rho_0^{\frac{n-1}{2}} \tilde{h}\|_\infty + \|\rho_0^{\frac{n-1}{2}} \tilde{\partial} \tilde{h}\|_\infty < \epsilon_1'$ with $\epsilon_1' > 0$ small, then

$$\begin{aligned} &\operatorname{div}_{\tilde{g}}(e^{-2\delta \log \rho_0} F(T_1, \tilde{\partial}^k \tilde{h})) \\ &= e^{-2\delta \log \rho_0}(-2\delta \langle F(T_1, \tilde{\partial}^k \tilde{h}), \nabla \log \rho_0 \rangle_{\tilde{g}} + \operatorname{div}_{\tilde{g}}(F(T_1, \tilde{\partial}^k \tilde{h}))) \\ &\leq \rho_0^{-2\delta}(\langle \nabla \tilde{\partial}^k \tilde{h}, \nabla T_1 \rangle_{\tilde{g}} \tilde{f}^k(\tilde{h}) + C \langle F(T_1, \tilde{\partial}^k \tilde{h}), \nabla T_1 \rangle_{\tilde{m}}), \end{aligned}$$

where C is a constant independent of ϵ_1' if ϵ_1' small enough. Here ∇ is w.r.t. $\tilde{g}$

and (3.5) implies that for ϵ_1' small,

$$\partial_{T_1}\int_{\Omega_1^{T_1}}\sum_{k=0}^{N}\mathrm{div}_{\tilde{g}}(e^{-2\delta\log\rho_1}F(T_1,\tilde{\partial}^k\tilde{h}))d\mu_{\tilde{g}}^{T_1} \leq C(1+M_1^N(T_1,\tilde{h}))(M_1^N(T_1,\tilde{h}))^2,$$

where C is a constant independent of ϵ_1' if ϵ_1' is small enough.

- Energy estimates: for $T_1\in[0,\frac{7}{8}]$

$$(M_1^N(T_1,\tilde{h}))^2-(M_1^N(0,\tilde{h}))^2=\int_{\Omega_1^{T_1}}\mathrm{div}_{\tilde{g}}(e^{-2\delta\log\rho_1}F(T_1,\tilde{\partial}^k\tilde{h}))dvol_{\tilde{g}},$$

$$\Longrightarrow\quad \partial_{T_1}(M_1^N(T_1,\tilde{h}))^2\leq C(1+M_1^N(T1,\tilde{h}))(M_1^N(T_1,\tilde{h}))^2,$$

which implies

$$M_1^N(T_1,\tilde{h})\leq CM_1^N(0,\tilde{h})<C\epsilon_0. \qquad (3.8)\ \boxed{\texttt{cauchy.7}}$$

Here C is a constant independent of ϵ_1' if it is small enough. Choose ϵ_0 small enough, we have $\|\rho_0^{\frac{n-1}{2}}\tilde{h}\|_\infty+\|\rho_0^{\frac{n-1}{2}}\tilde{\partial}\tilde{h}\|_\infty\leq\epsilon_1'$ satisfied. Hence the (3.8) is valid until Σ_1.

3.2.3 In $\overline{\Omega_2}$.

We solve the equation (3.3) from Σ_1 to Σ_2 with Cauchy data on Σ_1 such that

$$\sum_{|k|\leq N+1}\|\tilde{\partial}^k\tilde{h}\|_{\rho_0^\delta H_b^0(\Sigma_1)}<\epsilon_1,\quad 0<\delta<\tfrac{1}{2},\quad \epsilon_1>0,\ \text{small.}$$

- Perturbed metric: by changing variable $\xi=-\log b$, the metric $\tilde{g}$ can be expressed in coordinates (a, ξ, θ) as follows,

$$A = \begin{bmatrix} a^{-2}h_{\rho\rho} & 1 - h_{\tau\rho} - a^{-1}h_{\rho\rho} \\ 1 - h_{\tau\rho} - a^{-1}h_{\rho\rho} & -a(2-a) + h_{\rho\rho} + 2ah_{\tau\rho} + a^2 h_{\tau\tau} \end{bmatrix},$$

$$C = \begin{bmatrix} a^{-1}h_{\rho 1} & \cdots & a^{-1}h_{\rho n} \\ -(h_{\rho 1} + ah_{\tau 1}) & \cdots & -(h_{\rho n} + ah_{\tau n}) \end{bmatrix},$$

$$G_{11} = \begin{bmatrix} a(2-a) + \Theta_1(h) & 1 - a^{-1}(h_{\rho\rho} + \Theta_2(h)) + \Theta_1(h) \\ 1 - a^{-1}(h_{\rho\rho} + \Theta_2(h)) + \Theta_1(h) & -a^{-2}(h_{\rho\rho} + \Theta_2(h)) \end{bmatrix},$$

$$G_{12} = \begin{bmatrix} h_{\rho 1} + 2ah_{1j}\theta^j + \Theta_2(h) & \cdots & h_{\rho n} + 2ah_{nj}\theta^j + \Theta_2(h) \\ -a^{-1}(h_{\rho 1} + \Theta_2(h)) & \cdots & -a^{-1}(h_{\rho n} + \Theta_2(h)) \end{bmatrix},$$

$$G_{22} = -Id_{n\times n} - [h_{ij}]_{n\times n} + \Theta_2(h).$$

- Connection components:

$$\begin{aligned}
&\Gamma^a_{aa}(\tilde{g}) = a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
&\Gamma^a_{\xi a}(\tilde{g}) = -(1-a) + a^{\frac{n-3}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
&\Gamma^a_{\xi\xi}(\tilde{g}) = a(2-a)(1-a) + a^{\frac{n-1}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
&\Gamma^a_{ai}(\tilde{g}) = a^{\frac{n-3}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
&\Gamma^a_{\xi i}(\tilde{g}) = a^{\frac{n-1}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
&\Gamma^a_{ij}(\tilde{g}) = a^{\frac{n-1}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))\\
\Longrightarrow\ &\Gamma^a(\tilde{g}) = -2(1-a) + a^{\frac{n-3}{2}} b^{\frac{n-1}{2}} (\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))
\end{aligned}$$

$$
\begin{aligned}
&\Gamma^{\xi}_{aa}(\tilde g) = \tfrac{1}{2}a^{\frac{n-7}{2}}b^{\frac{n-1}{2}}\big(\tfrac{n-5}{2}\tilde h_{\rho\rho} + a\partial_a \tilde h_{\rho\rho}\big) + a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
&\Gamma^{\xi}_{\xi a}(\tilde g) = a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
&\Gamma^{\xi}_{\xi\xi}(\tilde g) = 1 - a + a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
&\Gamma^{\xi}_{ai}(\tilde g) = a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
&\Gamma^{\xi}_{\xi i}(\tilde g) = a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
&\Gamma^{\xi}_{ij}(\tilde g) = a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))\\
\Longrightarrow\quad &\Gamma^{\xi}(\tilde g) = a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}\tilde h_{\rho\rho} + a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))
\end{aligned}
$$

- Partial derivatives:

$$\tilde\partial \in \{a\partial_a, b\partial_b, \tilde\partial_i\}.$$

- Commutator: for $k_1, k_2, k_3 \in \mathbb{N}_0$,

$$
\begin{aligned}
&[\Box_{\tilde m}, (a\partial_a)^{k_1}(b\partial_b)^{k_2}\triangle_\theta^{k_3}] = [\Box_{\tilde m}, (a\partial_a)^{k_1}](b\partial_b)^{k_2}\triangle_\theta^{k_3}\\
&= \big(c_{k_1}(a\partial_a)^{k_1-1}\Box_{\tilde m} + \sum_{i+2j\le k_1+1} c_{ij}(a\partial_a)^i\triangle_\theta^j\big)(b\partial_b)^{k_2}\triangle_\theta^{k_3},
\end{aligned}
\tag{3.9}
$$ cauchy.8

where c_{k_1}, c_{ij} are constants.

- Equations: inserting (3.9) into (3.4) gives:

$$
\begin{aligned}
(\Box_{\tilde g} + \gamma_0)\tilde\partial^k h_{\mu\nu} &= c_k\tilde\partial^{k-1}\Box_{\tilde m}h_{\mu\nu} + \sum_{i\le k+1} c_{ki}\tilde\partial^i h_{\mu\nu} + \tilde f^k\\
&= \sum_{i\le k+1} c_i'\tilde\partial^i\tilde h_{\mu\nu} + \tilde f_k',
\end{aligned}
$$

where c_i' are constants and $\tilde f_k'$ can be expressed as (3.5).

- Time-like functions:

$$
\begin{aligned}
&\langle\nabla T_2, \nabla T_2\rangle_{\tilde g} = 2 - a^{\frac{n-5}{2}}b^{\frac{n-1}{2}}\tilde h_{\rho\rho} + a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}\Theta_1(\tilde h),\\
&\langle\nabla T_2', \nabla T_2'\rangle_{\tilde g} = a(2-a) + a^{\frac{n-1}{2}}b^{\frac{n-1}{2}}\Theta_1(\tilde h),\\
&\langle\nabla T_2, \nabla T_2'\rangle_{\tilde g} = 1 + a(2-a) - a^{\frac{n-3}{2}}b^{\frac{n-1}{2}}\tilde h_{\rho\rho} + a^{\frac{n-1}{2}}b^{\frac{n-1}{2}}\Theta_1(\tilde h).
\end{aligned}
$$

For $n \geq 5$, if $\|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{h}\|_\infty < \epsilon_2$ for some $\epsilon_2 > 0$ small then T_2 is regularly time-like w.r.t. $\tilde{g}$ and T_2' is time-like in the interior and null on S_1; this is same as w.r.t. $\tilde{m}$.

- Space-like hypersurfaces: define for $T_2' \in [-\frac{1}{8}, 0]$,

$$\Sigma_{T_2'} = \{T_2' = \text{constant}\} \cap \Omega_2, \quad \Sigma_2^{\rho_1} = \Sigma_2 \cap \{T_2' \leq -\rho_1\},$$

and consider the domain $\Omega_2^{T_2'}$ bounded by S_0, Σ_1, $\Sigma_2^{-T_2'}$ and $\Sigma_{T_2'}$ for energy estimates.

- Quadratic forms:

$$1 - |a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \Theta_1(\tilde{h})| \leq \frac{\langle F(T_2, v), \nabla T_2\rangle_{\tilde{g}}}{\langle F(T_2, v), \nabla T_2\rangle_{\tilde{m}}} \leq 1 + |a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \Theta_1(\tilde{h})|,$$
$$1 - |a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \Theta_1(\tilde{h})| \leq \frac{\langle F(T_2, v), \nabla T_2'\rangle_{\tilde{g}}}{\langle F(T_2, v), \nabla T_2'\rangle_{\tilde{m}}} \leq 1 + |a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \Theta_1(\tilde{h})|;$$

- Sobolev norm: define for $\|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{h}\|_\infty \leq \epsilon_2$ with $\epsilon_2 > 0$ small,

$$\begin{aligned} M_2^N(T_2', v) =& (\int_{\Sigma_{T_2'}} \sum_{k=0}^N e^{-2\delta \log b} \langle F(T_2, \tilde{\partial}^k v), \nabla T_2'\rangle_{\tilde{g}} d\mu_{\tilde{g}}^{T_2'})^{\frac{1}{2}} \\ \simeq& (\int_{\Sigma_{T_2'}} \sum_{k=0}^N b^{-2\delta} (|b\partial_b \tilde{\partial}^k v|^2 + a|\partial_a \tilde{\partial}^k v|^2 + |\nabla_\theta \tilde{\partial}^k v|^2 + |\tilde{\partial}^k v|^2) d\mu_{\tilde{g}}^{T_2'})^{\frac{1}{2}} \\ L_2^N(\rho_1, v) =& (\int_{\Sigma_2^{\rho_1}} \sum_{k=0}^N e^{-2\delta \log b} \langle F(T_2, \tilde{\partial}^k v), \nabla T_2\rangle_{\tilde{g}} d\mu_{\tilde{g}}^{T_2})^{\frac{1}{2}} \\ \simeq& (\int_{\Sigma_2^{\rho_1}} \sum_{k=0}^N (|b\partial_b \tilde{\partial}^k v|^2 + |\partial_a \tilde{\partial}^k v|^2 + |\nabla_\theta \tilde{\partial}^k v|^2 + |\tilde{\partial}^k v|^2) d\mu_{\tilde{g}}^{T_2})^{\frac{1}{2}}. \end{aligned}$$

Here for ϵ_2 small enough,

$$d\mu_{\tilde{g}}^{T_2} \simeq da d\theta, \quad d\mu_{\tilde{g}}^{T_2'} \simeq \frac{db}{db} d\theta.$$

Moreover,

$$\sum_{k\le \frac{N}{2}+1} |\tilde{\partial}^k v| \le C b^\delta M_2^N(T_2', v),$$

$$\sum_{k\le \frac{N}{2}+1} |\partial_a \tilde{\partial}^k v| \le C b^\delta a^{-\frac{1}{2}} M_2^N(T_2', v),$$

where C is a constant independent of ϵ_2 if it is small enough.

- Divergence of quadratic form field: if $\|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{h}\|_\infty + \|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{\partial}\tilde{h}\|_\infty \le \epsilon_2'$ with $\epsilon_2' > 0$ small, then

$$\begin{aligned}
&\operatorname{div}_{\tilde{g}}(e^{-2\delta \log b} F(T_2, v)) \\
&= b^{-2\delta}(-2\delta\langle F(T_2, v), \nabla \log b\rangle_{\tilde{g}} + \operatorname{div}_{\tilde{g}}(F(T_2, v))) \\
&\le b^{-2\delta}\big(a^{-1}(1 - 2\delta + \epsilon_2')(\tilde{\partial}\tilde{h}))|)\langle F(T_2, v), \nabla T_2'\rangle_{\tilde{g}} + \langle \nabla v, \nabla T_2\rangle_{\tilde{g}}(\Box_{\tilde{g}} + \gamma_0)v\big).
\end{aligned}$$

Replace v by $\tilde{\partial}^k \tilde{h}$ for $|k| \le N$ and then we have

$$\begin{aligned}
\partial_{T_2'} \int_{\Omega_2^{T_2'}} \sum_{|k|\le N} (\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k \tilde{h} \langle \nabla \tilde{\partial}^k \tilde{h}, \nabla T_2\rangle_{\tilde{g}} dvol_{\tilde{g}} \\
\le C a^{-\frac{1}{2}}(1 + M_2^N(T_2', \tilde{h}))(M_2^N(T_2', \tilde{h}))^2,
\end{aligned}$$

where C is a constant independent of ϵ_2' if it is small enough.

- Energy estimates: notice here $\rho_1 = a = -T_2'$. Then for $T_2' \in [-\frac{1}{8}, 0]$,

$$\begin{aligned}
(M_2^N(T_2', \tilde{h}))^2 - (L_2^N(\rho_1, \tilde{h}))^2 - M_2^N(-\tfrac{1}{8}, \tilde{h})^2 \\
= \sum_{|k|\le N} \int_{\Omega_2^\rho} \operatorname{div}_{\tilde{g}}(e^{-2\delta \log b} F(T_2, \tilde{\partial}^k \tilde{h})) dvol_{\tilde{g}},
\end{aligned}$$

which implies that for ϵ_2' small

$$\begin{aligned}
\partial_{T_2'}(M_2^N(T_2', \tilde{h}))^2 + \partial_{T_2'}(L_2^N(\rho_1, \tilde{h}))^2 \\
\le \big((1 - 2\delta + \epsilon_2')a^{-1} + C a^{-\frac{1}{2}}(1 + M_2^N(T_2', \tilde{h}))\big)(M_2^N(T_2', \tilde{h}))^2.
\end{aligned} \tag{3.10}$$

cauchy.9

Since $\delta > 0$, we can choose $\epsilon_2' < 2\delta$ and small. Then (3.10) implies:

$$
\begin{aligned}
\text{either} \quad & M_2^N(T_2', \tilde{h}) < M_2^N(-\tfrac{1}{8}, \tilde{h}) < 1 \quad \forall\, T_2' \in [-\frac{1}{8}, 0], \\
\text{or} \quad & -\partial_a M_2^N(-a, \tilde{h}) \leq \tfrac{1-2\delta+\epsilon_2'}{2} M_2^N(-a, \tilde{h}) + C a^{-\frac{1}{2}} (M_2^N(-a, \tilde{h}))^2, \\
\Longrightarrow \quad & -\partial_a (M_2^N(-a, \tilde{h}) a^{\frac{1+\epsilon_2'}{2} - \delta}) \leq C (M_2^N(-a, \tilde{h}) a^{\frac{1+\epsilon_2}{2} - \delta})^2 a^{-1+\delta-\frac{\epsilon_2'}{2}}, \\
\Longrightarrow \quad & M_2(-a, \tilde{h}) \leq \frac{M_2^N(-\frac{1}{8}, \tilde{h}) a^{\delta - \frac{1+\epsilon_2'}{2}}}{8^{\frac{1+\epsilon_2'}{2} - \delta} - M_2^N(-\frac{1}{8}, \tilde{h}) C \int_0^{\frac{1}{8}} a^{-1+\delta-\frac{\epsilon_2'}{2}} da}.
\end{aligned}
$$

If ϵ_1 is small enough, then

$$
M_2^N(T_2', \tilde{h}) + L_2^N(\rho_1, \tilde{h}) \leq (C_1 + C_2 a^{\delta - \frac{\epsilon_2'+1}{2}}) \epsilon_1, \tag{3.11}
$$
cauchy.10

where C_1, C_2 are constant independent of ϵ_1 if it is small enough. Now (3.11) gives

$$
\begin{aligned}
& \sum_{|k| \leq \frac{N}{2}+1} |\partial_a \tilde{\partial}^k \tilde{h}| \leq C b^\delta (C_1 a^{-\frac{1}{2}} + C_2 a^{\delta - \frac{\epsilon_2'}{2} - 1}) \epsilon_1 \\
\Longrightarrow \quad & \sum_{|k| \leq \frac{N}{2}+1} |\tilde{\partial}^k \tilde{h}| \leq C b^\delta (1 + \int_0^{\frac{1}{8}} C_1 a^{-\frac{1}{2}} + C_2 a^{\delta - \frac{\epsilon_2'}{2} - 1} da) \epsilon_1 \leq C' b^\delta \epsilon_1.
\end{aligned}
$$

Here C' is a constant independent of ϵ_1 if it is small enough. For ϵ_1 small, $\|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{h}\|_\infty + \|a^{\frac{n-5}{2}} b^{\frac{n-1}{2}} \tilde{\partial}\tilde{h}\|_\infty \leq \epsilon_2'$ holds and hence (3.11) is valid for all a up to 0.

- Radiation field: For ϵ_1 small, $\tilde{h}$ is $C^{\delta - \frac{\epsilon_2'}{2}}$ up to S_1 in Ω_2 and $\tilde{h}|_{S_1 \cap \Omega_2}$ is well defined. Define

$$
Q_2^N(T_2', \tilde{h}) = (\int_{\Sigma_{T_2'}} \sum_{k+|\alpha| \leq N} |b \partial_b \nabla_\theta^\alpha v| \frac{db}{b} d\theta)^2
$$

Then by (3.11)

$$\partial_{T_2'}(Q_2^N(T_2',\tilde{h}))^2 \leq Ca^{-\frac{1}{2}}M_2^N(T_2',\tilde{\rho})Q_2^N(T_2',\tilde{h})$$
$$\leq C\epsilon_1(C_1a^{-\frac{1}{2}} + C_2a^{\delta-\frac{\epsilon_2}{2}-1})Q_2^N(T_2',\tilde{h})$$

which implies

$$Q_2^N(T_2',\tilde{h}) < C_2'\epsilon_1 \quad \forall\, T_2' \in [-\tfrac{1}{8}, 0].$$

Here C_2' is a constant independent of ϵ_1 if it is small. Hence

$$\|\tilde{h}\|_{H_b^N(S_1\cap\Omega_2)} = Q_2^N(0,\tilde{h}) \leq C_2'\epsilon_1.$$

3.2.4 In $\overline{\Omega}_3$

We solve the equation (3.3) from Σ_3 to Σ_4 with Cauchy data on Σ_3 such that

$$L_3^N(\rho_1,\tilde{h}) \leq (1+\rho_1^{\delta'-\frac{1}{2}})\epsilon_2, \quad 0<\delta'\leq\delta, \quad \epsilon_2>0, \quad \text{small.}$$

where

$$L_3^N(\rho_1,\tilde{h})^2 = L_2^N(\rho_1,\tilde{h}) + \|\tilde{h}\|^2_{H^{N+1}(\Sigma_3\cap\Sigma_1)},$$

Notice here $\epsilon_2 \lesssim \epsilon_1 + \epsilon_0 \lesssim \epsilon_0$ and δ' can be chosen as $\delta - \frac{1}{2}\epsilon_2'$ as set in Ω_2. The space-like hypersurface Σ_3 consists of two parts: Σ_2 if it is above Σ_1 and Σ_1 otherwise. Hence $\Sigma_3 \cap \Sigma_1$ is in a finite region.

• Perturbed metric: in the coordinates (τ, ρ, θ) near $S_1 \cap \Omega_3$,

$$A = \begin{bmatrix} \rho^2(1+h_{\tau\tau}) & -1+h_{\tau\rho} \\ -1+h_{\tau\rho} & \rho^{-2}h_{\rho\rho} \end{bmatrix}, \quad C = \begin{bmatrix} \rho h_{\tau 1} & \cdots & \rho h_{\tau n} \\ \rho^{-1}h_{\rho 1} & \cdots & \rho^{-1}h_{\rho n} \end{bmatrix},$$

$$G_{11} = \begin{bmatrix} -\rho^{-2}(h_{\rho\rho}+\Theta_2(h)) & -(1+h_{\tau\rho}+h_{\rho\rho}+\Theta_2(h)) \\ -(1+h_{\tau\rho}+h_{\rho\rho}+\Theta_2(h)) & -\rho^2(1+h_{\tau\tau}+h_{\rho\rho}+2h_{\tau\rho}+\Theta_2(h)) \end{bmatrix},$$

$$G_{12} = \begin{bmatrix} -\rho^{-1}(h_{\rho 1}+\Theta_2(h)) & \cdots & -\rho^{-1}(h_{\rho n}+\Theta_2(h)) \\ -\rho(h_{\tau 1}+h_{\rho 1}+\Theta_2(h)) & \cdots & -\rho(h_{\tau n}+h_{\rho n}+\Theta_2(h)) \end{bmatrix},$$

$$G_{22} = -Id_{n\times n} - [h_{ij}]_{n\times n} + \Theta_2(h).$$

• Connection components: apply the gauge condition

$$\begin{aligned}
&\Gamma^\tau_{\tau\tau}(\tilde g) = \rho + \rho^{\frac{n-1}{2}}(\Theta_1(\tilde h)) + \Theta_0(\tilde h)(\tilde\partial\tilde h) \\
&\Gamma^\tau_{\rho\rho}(\tilde g) = -\tfrac{1}{2}\rho^{\frac{n-7}{2}}(\tfrac{n-5}{2}\tilde h_{\rho\rho} + \rho\partial_\rho \tilde h_{\rho\rho}) + \rho^{\frac{n-5}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\tau_{\rho\tau}(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\tau_{\rho i}(\tilde g) = \rho^{\frac{n-5}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\tau_{\tau i}(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\tau_{ij}(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
\Longrightarrow\quad &\Gamma^\tau(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))
\end{aligned}$$

$$\begin{aligned}
&\Gamma^\rho_{\tau\tau}(\tilde g) = \rho^3 + \rho^{\frac{n+3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\rho_{\rho\rho}(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\rho_{\rho\tau}(\tilde g) = -\rho + \rho^{\frac{n+1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\rho_{\rho i}(\tilde g) = \rho^{\frac{n-3}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\rho_{\tau i}(\tilde g) = \rho^{\frac{n+1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
&\Gamma^\rho_{ij}(\tilde g) = \rho^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h)) \\
\Longrightarrow\quad &\Gamma^\rho(\tilde g) = 2\rho + \rho^{\frac{n-1}{2}}(\Theta_1(\tilde h) + \Theta_0(\tilde h)(\tilde\partial\tilde h))
\end{aligned}$$

- Partial derivatives: near $S_1 \cap \Omega_3$,

$$\tilde{\partial} \in \{\partial_\tau, \rho\partial_\rho, \tilde{\partial}_i\}$$

and extend them smoothly and linear independently to finite region.

- Commutator: near $S_1 \cap \Omega_3$,

$$\begin{aligned}
&[\Box_{\tilde{m}}, (\rho\partial_\rho)^{k_1}\partial_\tau^{k_2}\triangle_\theta^{k_3}] = [\Box_{\tilde{m}}, (\rho\partial_\rho)^{k_1}](\partial_\tau)^{k_2}\triangle_\theta^{k_3} \\
&= [c_{k_1}(\rho\partial_\rho)^{k_1-1}\Box_{\tilde{m}} + \sum_{i+2j\le k_1+1} c_{ij}(\rho\partial_\rho)^i\triangle_\theta^j](\partial_\tau)^{k_2}\triangle_\theta^{k_3},
\end{aligned} \tag{3.12}$$ cauchy.16

where c_{k_1} and c_{ij} are constants.

- Equations: inserting (3.12) into (3.4) gives:

$$(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h} = \sum_{i\le k+1} c_i\tilde{\partial}^i\tilde{h} + \tilde{f}'_k$$

where $\tilde{f}'_k$ can be expressed as (3.5).

- Time-like functions: near $S_1 \cap \Omega_3$,

$$\begin{aligned}
\langle\nabla T_3, \nabla T_3\rangle_{\tilde{g}} &= 2 - \rho^2 - \rho_1^{\frac{n-5}{2}}\Theta_1(\tilde{h}), \\
\langle\nabla T'_3, \nabla T'_3\rangle_{\tilde{g}} &= \rho(2\tau_0 - \tau)[2 - \rho(2\tau_0 - \tau)] + \rho_1^{\frac{n-1}{2}}\Theta_1(\tilde{h}), \\
\langle\nabla T_3, \nabla T'_3\rangle_{\tilde{g}} &= 2\tau_0 - \tau + \rho[1 - \rho(2\tau_0 - \tau)] - \rho_1^{\frac{n-3}{2}}\Theta_1(\tilde{h}).
\end{aligned}$$

- Space-like hypersurfaces: define

$$\begin{aligned}
\Sigma_{T'_3} &= \{T'_3 = \text{constant}\} \cap \Omega_3, \\
\Sigma_3^{\rho_1} &= \Sigma_3 \cap \{T'_3 \le 3\tau_0\rho_1\} = (\Sigma_1 \cap \overline{\Omega}_3) \cup \Sigma_2^{\rho_1}, \\
\Sigma_4^{\rho_1} &= \Sigma_4 \cap \{T'_3 \le \tau_0\rho_1\}.
\end{aligned}$$

We only consider T'_3 near S_1. Here $\Sigma_3^{\rho_1}$ and $\Sigma_3^{\rho_1}$ works as $\Sigma_2^{\rho_1}$ in Ω_2; $\Sigma_{T'_3}$ works as $\Sigma_{T'_2}$ in Ω_2. Consider the domain $\Omega_3^{T'_3}$ bounded by Σ_3, Σ_4 and $\Sigma_{T'_3}$ for energy estimates.

- Quadratic form:

$$1-|\rho_1^{\frac{n-5}{2}}\Theta_1(\tilde h)| \le \frac{\langle F(T_3,v),\nabla T_3\rangle_{\tilde g}}{\langle F(T_3,v),\nabla T_3\rangle_{\tilde m}} \le 1+|\rho_1^{\frac{n-5}{2}}\Theta_1(\tilde h)|,$$
$$1-|\rho_1^{\frac{n-5}{2}}\Theta_1(\tilde h)| \le \frac{\langle F(T_3,v),\nabla T_3'\rangle_{\tilde g}}{\langle F(T_3,v),\nabla T_3'\rangle_{\tilde m}} \le 1+|\rho_1^{\frac{n-5}{2}}\Theta_1(\tilde h)|;$$

- Sobolev norm: define for $\|\rho_1^{\frac{n-5}{2}}\tilde h\|_\infty \le \epsilon_3$ with $\epsilon_3>0$ small enough,

$$\begin{aligned}
M_3^N(T_3',v) =& \Big(\int_{\Sigma_{T_3'}}\sum_{k=0}^N e^{-\lambda T_3}\langle F(T_3,\tilde\partial^k v),\nabla T_3'\rangle_{\tilde g} d\mu_{\tilde g}^{T_3'}\Big)^{\frac12}\\
\simeq& \Big(\int_{\Sigma_{T_3'}}\sum_{k=0}^N (|\partial_\tau\tilde\partial^k v|^2+\rho|\partial_\rho\tilde\partial^k v|^2+|\nabla_\theta\tilde\partial^k v|^2+|\tilde\partial^k v|^2) d\mu_{\tilde g}^{T_3'}\Big)^{\frac12},\\
L_4^N(\rho_1,v) =& \Big(\int_{\Sigma_4^{\rho_1}}\sum_{k=0}^N e^{-\lambda T_3}\langle F(T_3,\tilde\partial^k v),\nabla T_3\rangle_{\tilde g} d\mu_{\tilde g}^{T_3}\Big)^{\frac12}\\
\simeq& \Big(\int_{\Sigma_4^{\rho_1}}\sum_{k=0}^N (|\partial_\tau\tilde\partial^k v|^2+|\partial_\rho\tilde\partial^k v|^2+|\nabla_\theta\tilde\partial^k v|^2+|\tilde\partial^k v|^2) d\mu_{\tilde g}^{T_3}\Big)^{\frac12}
\end{aligned}$$

where $\lambda>0$ is some large constant depending on N only if ϵ_3 is small enough. Here for ϵ_3 small and near $S_1\cap\Omega_3$,

$$d\mu_{\tilde g}^{T_3}\simeq d\rho d\theta,\quad d\mu_{\tilde g}^{T_3'}\sim d\tau d\theta.$$

Moreover,

$$\sum_{|k|\le\frac N2+1}|\tilde\partial^k v|\le C M_3^N(T_3',v),$$
$$\sum_{|k|\le\frac N2+1}|\partial_\rho\tilde\partial^k v|\le C\rho_1^{-\frac12}M_3^N(T_3',v),$$

where C is a constant only depend on λ, hence N, if ϵ_3 is small enough.

- Divergence of quadratic form field: if $\|\rho_1^{\frac{n-5}{2}}\tilde h\|_\infty+\|\rho_1^{\frac{n-5}{2}}\tilde\partial\tilde h\|_\infty\le\epsilon_3'$ with $\epsilon_3'>0$

small and λ large, then

$$\begin{aligned}
&\operatorname{div}_{\tilde g}(e^{-\lambda T_3}F(T_3,v)) \\
&= e^{-\lambda T_3}[-\lambda\langle F(T_3,v),\nabla T_3\rangle_{\tilde g} + \operatorname{div}_{\tilde g}(F(T_3,v))] \\
&\le e^{-\lambda T_3}\big(\epsilon_3'\rho_1^{\frac{n-7}{2}}\langle F(T_3,v),\nabla T_3\rangle_{\tilde g} + \langle\nabla v,\nabla T_3\rangle_{\tilde g}(\Box_{\tilde g}+\gamma_0)v\big),
\end{aligned}$$

Replace v by $\tilde\partial^k h$ for $|k|\le N$ and then we have

$$\begin{aligned}
\partial_{T_3'}\int_{\Omega_3^{T_3'}}\sum_{|k|\le N}(\Box_{\tilde g}+\gamma_0)\tilde\partial^k\tilde h\langle\nabla\tilde\partial^k\tilde h,\nabla T_3\rangle_{\tilde g}dvol_{\tilde g}& \\
\le C\rho_1^{-\frac12}(1+M_3^N(T_3',\tilde h))(M_3^N(T_3',\tilde h))^2.&
\end{aligned}$$

- Energy estimates: here $T_3' = -\rho(2\tau_0-\tau)\simeq -\rho_1$ is only defined near $S_1\cap\Omega_3$. First, the domain bounded by Σ_3, Σ_4, $\Sigma_{T_3'=-\frac12}$ is compact finite region. It is obvious that if ϵ_2 is small enough, we can solve the equation up to $\Sigma_{T_3'=-\frac12}$ and $\Sigma_4^{\frac{1}{2\tau_0}}\}$ such that

$$(M_3^N(-\tfrac12,\tilde h))^2 + (L_4^N(\tfrac{1}{2\tau_0},\tilde h))^2 \le C\epsilon_2.$$

Then for $T_3'\in[-\frac12,0]$,

$$(M_3^N(T_3',\tilde h))^2 + (L_4^N(\rho_1,\tilde h))^2 - (L_3^N(\rho_1,\tilde h))^2 = \int_{\Omega_3^{\rho_1}}\operatorname{div}(e^{-\lambda T_3}F(T_3))dvol_{\tilde g}$$

implies for ϵ_3' small enough,

$$\begin{aligned}
&\partial_{T_3'}(M_3^N(T_3',\tilde h))^2 + \partial_{T_3'}(L_4^N(\rho_1,\tilde h))^2 - \partial_{T_3'}(L_3^N(\rho_1,\tilde h))^2 \\
&\le \epsilon_3'\rho_1^{\frac{n-7}{2}}(M_3^N(T_3',\tilde h))^2 + C\rho_1^{-\frac12}(1+M_3^N(T_3',\tilde h))(M_3^N(T_3',\tilde h))^2.
\end{aligned} \tag{3.13}$$ cauchy.17

Then for $\epsilon_3' < 1-2\delta'$ and small enough, (3.13) gives

$$M_3^N(T_3',\tilde h) + L_4^N(\rho_1,\tilde h) \le (C_1 + C_2\rho_1^{\delta'-\frac12})\epsilon_2, \tag{3.14}$$ cauchy.18

where C_1, C_2 are constants independent of ϵ_3' if it is small enough. Then (3.14) implies

$$\sum_{|k|\leq\frac{N}{2}+1}|\partial_\rho\tilde{\partial}^k\tilde{h}|\leq C(C_1\rho_1^{-\frac{1}{2}}+C_2\rho_1^{\delta'-1})\epsilon_2,$$

$$\Longrightarrow\sum_{|k|\leq\frac{N}{2}+1}\|\tilde{\partial}^k\tilde{h}\|_\infty<C\epsilon_2(1+\int_0^{\frac{1}{2}}C_1'\rho_1^{-\frac{1}{2}}+C_2'\rho_1^{\delta'-1}d\rho_1).$$

If ϵ_2 is small enough, then $\|\rho_1^{\frac{n-5}{2}}\tilde{h}\|_\infty+\|\rho_1^{\frac{n-5}{2}}\tilde{\partial}\tilde{h}\|_\infty<\epsilon_3'$ is valid for all T_3' up to 0. So is (3.14).

- Radiation field: here $\tilde{h}$ is $C^{\delta'}$ up to S_1. Hence the radiation field $\tilde{h}|_{S_1}$ is well defined in $\Omega_3\cap S_1$. Define

$$Q_3^N(T_3',\tilde{h})=(\int_{\Sigma_{T_3'}}\sum_{k+|\alpha|\leq N}|\partial_\tau\nabla_\theta^\alpha v|\frac{db}{b}d\theta)^2$$

Then by (3.14)

$$\partial_{T_3'}(Q_3^N(T_3',\tilde{h}))^2\leq C\rho_1^{-\frac{1}{2}}M_3^N(T_3',\tilde{\rho})Q_3^N(T_3',\tilde{h})\leq C(C_1\rho_1^{-\frac{1}{2}}+C_2\rho_1^{\delta'-1})Q_2^N(T_2',\tilde{h})$$

which implies

$$Q_3^N(T_3',\tilde{h})<C_3'\epsilon_2,\quad\forall\,T_3'\in[-\frac{1}{2},0],$$

where C_3' is a constant independent of ϵ_2 if it is small.

$$\|\tilde{h}\|_{H_b^N(S_1\cap\Omega_3)}=Q_3^N(0,\tilde{h})<C_3'\epsilon_2.$$

3.2.5 In $\overline{\Omega}_4$

.We solve the equation (3.4) from Σ_4 up to $S_2\cup S_1$ with Cauchy data on Σ_4 such that

$$L_4^N(\rho_1,\tilde{h})\leq(1+\rho_1^{\delta'-\frac{1}{2}})\epsilon_3,\quad\delta'\in(0,\delta),\quad\epsilon_3>0,\quad\text{small.}$$

- Perturbed metric: for $\bar{a}\in[0,\frac{1}{4})$, by changing variable $\xi=-\log\bar{b}$, the metric can be expressed in the coordinates $(\bar{a}, \xi, \theta)$ as:

$$A = \begin{bmatrix} \bar{a}^{-2}h_{\rho\rho} & -1 - \bar{a}^{-1}h_{\rho\rho} - h_{\rho i}\theta^i \\ -1 - \bar{a}^{-1}h_{\rho\rho} - h_{\rho i}\theta^i & a(2-a) + h_{\rho\rho} + 2\bar{a}h_{\rho i}\theta^i + \bar{a}^2 h_{ij}\theta^i\theta^j \end{bmatrix},$$

$$C = (1-\bar{a}) \begin{bmatrix} \bar{a}^{-1}h_{\rho 1} & \cdots & \bar{a}^{-1}h_{\rho n} \\ -(h_{\rho 1} + \bar{a}h_{1j}\theta^j) & \cdots & -(h_{\rho n} + \bar{a}h_{nj}\theta^j) \end{bmatrix},$$

$$B = (1-\bar{a})^2(-Id_{n\times n} + [h_{ij}]_{n\times n}),$$

$$G_{11} = \begin{bmatrix} -\bar{a}(2-\bar{a}) + \Theta_1(h) & -1 - \bar{a}^{-1}(h_{\rho\rho} + \Theta_2(h)) + \Theta_1(h) \\ -1 - \bar{a}^{-1}(h_{\rho\rho} + \Theta_2(h)) + \Theta_1(h) & -\bar{a}^{-2}(h_{\rho\rho} + \Theta_2(h)) \end{bmatrix},$$

$$G_{12} = (1-\bar{a})^{-1} \begin{bmatrix} -h_{\rho 1} - \bar{a}h_{01} + \Theta_2(h) & \cdots & -h_{\rho n} - \bar{a}h_{0n} + \Theta_2(h) \\ -\bar{a}^{-1}(h_{\rho 1} + \Theta_1(h)) & \cdots & -\bar{a}^{-1}(h_{\rho n} + \Theta_1(h)) \end{bmatrix},$$

$$G_{22} = (1-\bar{a})^{-2}(-Id_{n\times n} - [h_{ij}]_{n\times n} + \Theta_2(h));$$

for $\bar{a} \in (\frac{1}{8}, 1]$, by changing variable $\zeta = -\log\phi$ and in the coordinates (ζ, y),

$$\tilde{g} = \tilde{m} + \rho_2^{\frac{n-1}{2}}\Theta_1(\tilde{h}), \quad \tilde{g}^{-1} = \tilde{m}^{-1} + \rho_2^{\frac{n-1}{2}}\Theta_1(\tilde{h}).$$

- Connection components: for $\bar{a} \in [0, \frac{1}{4})$

$$\begin{aligned}
\Gamma^{\bar{a}}_{\bar{a}\bar{a}} =& \bar{b}^{\frac{n-1}{2}}\bar{a}^{\frac{n-5}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Gamma^{\bar{a}}_{\bar{a}\xi} =& -(1-\bar{a}) + \bar{b}^{\frac{n-1}{2}}\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Gamma^{\bar{a}}_{\xi\xi} =& +\bar{a}(2-\bar{a})(1-\bar{a}) + \bar{b}^{\frac{n-1}{2}}[\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Gamma^{\bar{a}}_{\bar{a}i} =& \bar{b}^{\frac{n-3}{2}}\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Gamma^{\bar{a}}_{\xi i} =& \bar{b}^{\frac{n-1}{2}}\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Gamma^{\bar{a}}_{ij} =& \bar{a}(2-\bar{a})(1-\bar{a})\delta_{ij} + \bar{b}^{\frac{n-1}{2}}[\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})) \\
\Longrightarrow \quad \Gamma^{\bar{a}} =& 2(1-\bar{a}) - (n-1)\bar{a}(2-\bar{a})(1-\bar{a})^{-1} + \bar{b}^{\frac{n-3}{2}}[\bar{a}^{\frac{n-3}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h}))
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{\xi}_{\bar{a}\bar{a}} =& -\tfrac{1}{2}\bar{a}^{\frac{n-7}{2}}\bar{b}^{\frac{n-1}{2}}(\tfrac{n-5}{2}\tilde{h}_{\rho\rho} + \bar{a}\partial_{\bar{a}}\tilde{h}_{\rho\rho}) + \bar{a}^{\frac{n-7}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Gamma^{\xi}_{\xi\bar{a}} =& \bar{a}^{\frac{n-5}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Gamma^{\xi}_{\xi\xi} =& (1-a) + \bar{a}^{\frac{n-3}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Gamma^{\xi}_{\bar{a}i} =& \bar{a}^{\frac{n-5}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Gamma^{\xi}_{\xi i} =& \bar{a}^{\frac{n-3}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Gamma^{\xi}_{ij} =& (1-a)\delta_{ij} + \bar{a}^{\frac{n-3}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h})\\
\Longrightarrow \quad \Gamma^{\xi} =& -(n-1)(1-\bar{a})^{-1} + \bar{a}^{\frac{n-5}{2}}\bar{b}^{\frac{n-1}{2}}\tilde{h}_{\rho\rho} + \bar{a}^{\frac{n-3}{2}}\bar{b}^{\frac{n-1}{2}}(\Theta_1(\tilde{h}) + \Theta_0(\tilde{h})(\tilde{\partial}\tilde{h});
\end{aligned}
$$

for $\bar{a} \in (\frac{1}{8}, 1]$, for any $I, J, K \in \{\zeta, y^i : i = 1, ..., n\}$,

$$\Gamma^K_{IJ}(\tilde{g}) = \Gamma^K_{IJ}(\tilde{m}) + \rho_2^{\frac{n-1}{2}}\Theta_1(\tilde{h}).$$

- Partial derivatives:

$$\tilde{\partial} \in \{Z_{00}, Z_{ij}, Z_{0i} : i = 1, ..., n\}.$$

- Commutator:

$$
\begin{aligned}
[\Box_{\tilde{m}}, Z_{00}] = [\Box_{\tilde{m}}, Z_{ij}] = 0, \quad [\Box_{\tilde{m}}, Z_{0i}] = -2y^i(\Box_{\tilde{m}} + \gamma_0),\\
[\Box_{\tilde{m}}, \tilde{\partial}^k] = \sum_{0\le i\le k-1} c_i\tilde{\partial}^i(\Box_{\tilde{m}} + \gamma_0).
\end{aligned}
\tag{3.15}
$$

cauchy.21

Here c_i are smooth and bounded function on $\overline{\Omega}_4$.

- Equations: inserting (3.15) into (3.4) gives

$$(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h} = \tilde{f}'_k.$$

Here $\tilde{f}'_k$ can be expressed as (3.5).

- Time-like functions: to simplify the computation, we use

$$T_4 = -\bar{a} - \log \bar{b} \quad \text{and} \quad T'_4 = -\bar{a}\bar{b}^{\alpha}$$

for some $\alpha > 1$ to be determined later in the whole domain $\overline{\Omega}_4$ without modifying them to be smooth near $\bar{a} = 1$. This does not change the norm type.

$$\langle \nabla T_4, \nabla T_4 \rangle_{\tilde{g}} = \langle \nabla T_4, \nabla T_4 \rangle_{\tilde{m}} - \rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \tilde{h}_{\rho\rho} + \rho_1^{\frac{n-3}{2}} \rho_2^{\frac{n-1}{2}} \Theta_1(\tilde{h}),$$
$$\langle \nabla T_4, \nabla T_4' \rangle_{\tilde{g}} = \langle \nabla T_4, \nabla T_4' \rangle_{\tilde{m}} + \rho_2^{\alpha + \frac{n-1}{2}} \rho_1^{\frac{n-3}{2}} \Theta_1(\tilde{h}).$$

- Space-like hypersurfaces: define

$$\Sigma_{T_4'} = \{T_4' = \text{constant}\}$$

and consider the domain $\Omega_4^{T_4'}$ bounded by Σ_4 and $\Sigma_{T_4'}$ for energy estimates.

- Quadratic form:

$$1 + |\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \Theta_1(\tilde{h})| \leq \frac{\langle F(T_4, v), \nabla T_4 \rangle_{\tilde{g}}}{\langle F(T_4, v), \nabla T_4 \rangle_{\tilde{m}}} \leq 1 + |\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \Theta_1(\tilde{h})|,$$
$$1 + |\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \Theta_1(\tilde{h})| \leq \frac{\langle F(T_4, v), \nabla T_4' \rangle_{\tilde{g}}}{\langle F(T_4, v), \nabla T_4' \rangle_{\tilde{m}}} \leq 1 + |\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \Theta_1(\tilde{h})|.$$

- Sobolev norm: define for $\|\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \tilde{h}\|_\infty \leq \epsilon_4$ with $\epsilon_4 > 0$ small,

$$M_4^N(T_4', v) = \Big(\int_{\Sigma_{T_4'}} \sum_{k=0}^{N} e^{-\log \bar{b}} \langle F(T_4, \tilde{\partial}^k v), \nabla T_4' \rangle_{\tilde{g}} d\mu_{\tilde{g}}^{T_4'} \Big)^{\frac{1}{2}}$$
$$\simeq \Big(\int_{\Sigma_{T_4'}} \sum_{k=0}^{N} \rho_2^{-1+\alpha} (\alpha \bar{a} |\partial_{\bar{a}} v|^2 + |\bar{b} \partial_{\bar{b}} v|^2 + (1 + \bar{a}\alpha)((1-\bar{a})^{-2} |\nabla_\theta v|^2 + \gamma_0 v^2)) d\mu_{\tilde{g}}^{T_4'} \Big)^{\frac{1}{2}},$$

where $\alpha > 1$ is a constant to be determined later. Here for ϵ_4 small,

$$d\mu_{\tilde{g}}^{T_4'} \simeq \begin{cases} \bar{b}^{-\alpha} \frac{d\bar{b}}{\bar{b}} d\theta & \text{if } \bar{a} \sim 0 \\ \alpha \phi^{-\alpha} dy & \text{if } \bar{a} \sim 1 \end{cases}.$$

Moreover,

$$\sum_{|k|\leq \frac{N}{2}+1} |\tilde{\partial}^k \tilde{h}| \leq C\rho_2^{\frac{1}{2}} M_4^N(T_3', v),$$

$$\sum_{|k|\leq \frac{N}{2}+1} |\partial_a \tilde{\partial}^k \tilde{h}| \leq C\rho_1^{-\frac{1}{2}} \rho_2^{\frac{1}{2}} M_4^N(T_3', v)$$

- Divergence of quadratic form field: for $\|\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \tilde{h}\|_\infty + \|\rho_1^{\frac{n-5}{2}} \rho_2^{\frac{n-1}{2}} \tilde{\partial}\tilde{h}\|_\infty < \epsilon_4'$ with $\epsilon_4' > 0$ small,

$$\begin{aligned}
\mathrm{div}_{\tilde{g}}(e^{-\log \bar{b}} F(T_4, v)) &= \bar{b}^{-1}\big(\langle F(T_4, v), \nabla(-\log \bar{b})\rangle_{\tilde{g}} + \mathrm{div}_{\tilde{g}}(F(T_4, v))\big) \\
&\leq \bar{b}^{-1}\big(\langle \nabla v, \nabla T_4\rangle_{\tilde{g}}(\Box_{\tilde{g}} + \gamma_0)v \\
&\quad + (\epsilon_4' + C(n, \alpha))(-T_4')^{-1}\langle F(T_4, v), \nabla T_4'\rangle_{\tilde{g}}\big).
\end{aligned}$$

Here $C(n, \alpha)$ is a constant only depending on n and α which comes from $\mathrm{div}_{\tilde{m}}(F(T_4, v))$ and its comparative with $\langle F(T_4, v), \nabla T_4'\rangle_{\tilde{m}}$. Previous computation in Section 2.4 gives

$$\begin{aligned}
C(n, \alpha) = \max\{ &\sup_{0\leq \bar{a}\leq 1} \frac{\bar{a}(1 + \frac{n-1}{2}(2-a)(1-a) - \frac{1}{2}(2-\bar{a}))}{\frac{1}{4}\bar{a}(2-\bar{a})^2 + \frac{1}{2}(\alpha - 2 + a + \alpha(1-\bar{a})^2)}, \\
&\sup_{0\leq \bar{a}\leq 1} \frac{\bar{a}((n+1)(1-a)+1)}{(1-a)^2 + \alpha a}\}.
\end{aligned}$$

Choose α large enough, such that $\epsilon_4' + C(n, \alpha) < 1 - 2\delta'$. Replace v by $\tilde{\partial}^k \tilde{h}$, which gives

$$\partial_{T_4'} \int_{\Omega_4^{T_4'}} \bar{b}^{-1}(\Box_{\tilde{g}} + \gamma_0)\tilde{\partial}^k\tilde{h}\langle \nabla\tilde{\partial}^k\tilde{h}, \nabla T_4\rangle_{\tilde{g}} d\mu_{\tilde{g}}^{T_4'} \leq C(-T_4')^{-\frac{1}{2}}(M_4^N(T_4', \tilde{h}))^3.$$

Here C is independent of ϵ_4' if it is small enough.

- Energy estimates: notice here $T_4' \simeq -\rho_1 \tau_0^{-\alpha}$ on Σ_4 near S_1. Denote by $\min_{\overline{\Omega}_4}\{T_4'\} = t_0 < 0$. Then for $T_4' \in [t_0, 0]$,

$$(M_4^N(T_4', \tilde{h}))^2 - (L_4^N(\rho_1, \tilde{h}))^2 = \int_{\Omega_4^{T_4'}} \mathrm{div}_{\tilde{g}}(e^{-\log \bar{b}} F(T_4, v)) dvol_{\tilde{g}},$$

which implies for ϵ_4' small,

$$\partial_{T_4'}(M_4^N(T_4',\tilde{h}))^2 \leq \partial_{T_4'}(L_4^N(\rho_1,\tilde{h}))^2 + (1-2\delta')(-T_4')^{-1}(M_4^N(T_4',\tilde{h}))^2 + C(-T_4')^{-\frac{1}{2}}(M_4^N(T_4',\tilde{h}))^3.$$

Hence

$$M_4^N(T_4',\tilde{h}) \leq (C_1 + C_2(-T_4')^{\delta'-\frac{1}{2}})\epsilon_3. \tag{3.16}$$ cauchy.23

Here (3.16) gives

$$\sum_{|k|\leq \frac{N}{2}+1} |\partial_a \tilde{\partial}^k \tilde{h}| \leq C_1\epsilon_3(C_1\bar{a}^{-\frac{1}{2}}\bar{b}^{\frac{1}{2}} + C_2\bar{a}^{\delta'-1}\bar{b}^{\alpha(\delta'-\frac{1}{2})+\frac{1}{2}}),$$
$$\Longrightarrow \sum_{|k|\leq \frac{N}{2}+1} |\tilde{\partial}^k \tilde{h}| \leq C\epsilon_3 \bar{b}^{\alpha(\delta'-\frac{1}{2})+\frac{1}{2}}. \tag{3.17}$$ cauchy.24

Choose α such that $\alpha(\delta'-\frac{1}{2})+\frac{1}{2} > -\frac{n-1}{2}$, this can be done since we can choose α large and ϵ_4' small enough such that

$$\alpha(\epsilon_4' + C(n,\alpha)) \leq n.$$

Here $\alpha C(n,\alpha) \sim \frac{n+3}{2}$ for $\alpha \to \infty$. Denote by

$$\sigma = \alpha(\delta'-\tfrac{1}{2}) + \tfrac{n}{2} > 0$$

Choose ϵ_3 small enough, (3.17) gives $\|\rho_1^{\frac{n-5}{2}}\rho_2^{\frac{n-1}{2}}\tilde{h}\|_\infty + \|\rho_1^{\frac{n-5}{2}}\rho_2^{\frac{n-1}{2}}\tilde{\partial}\tilde{h}\|_\infty \leq \epsilon_4'$ is valid. Hence (3.16) works until $T_4' = 0$. Notice that $\sigma > 0$ only depends on n if the Cauchy data small enough.

- Radiation field: here $\tilde{h}$ is $C^{0,\delta'}$ up to $S_1 \cap \Omega_4$. Hence the radiation field $\tilde{h}|_{S_1\cap\Omega_4}$ is well defined. Define for $\rho_1 = -T_4'\tau_0^\alpha$

$$Q_4^N(\rho_1,\tilde{h}) = \Big(\int_{\Sigma_{T_4'}} \sum_{k+|\beta|\leq N} |\bar{b}\partial_b^k \nabla_\theta^\beta \tilde{h}|\bar{b}^{-2\sigma+n-1}\frac{d\bar{b}}{\bar{b}}d\theta\Big)^{\frac{1}{2}}$$

Then

$$\partial_{\rho_1}(Q_4^N(\rho_1,\tilde{h}))^2 \le CM_4^N(T_4',\tilde{h})Q_4^N(\rho_1,\tilde{h}) \le C(C_1'+C_2'\rho_1^{\delta'-1})Q_4^N(\rho_1,\tilde{h})$$

which implies

$$Q_4^N(\rho_1,\tilde{h}) < C_4'\epsilon_3.$$

Here C_4' only depend on the Cauchy data only on Σ_4 if ϵ_3 is small enough.

$$\|\tilde{h}\|_{\rho_2^{\sigma-\frac{n-1}{2}}H_b^N(S_1\cap\Omega_4)} \le Q_4^N(0,\tilde{h}) < C_4'\epsilon_3.$$

3.3 Nonlinear Møller Wave Operator.

In Section 3.2, we showed for $n \ge 5$ by energy estimates that given Cauchy data $(h^0,h^1) \in \widetilde{\mathcal{U}}_\epsilon^{N,\delta}$ for $N > n+6$, $\delta \in (0,\frac{1}{2})$ and $\epsilon > 0$ small, then the Einstein vacuum equations have a solution $g = m + h$ such that $\tilde{h} = \tilde{\rho}^{\frac{1-n}{2}}h$ is $C^{0,\delta'}$ continuous to S_1 and

$$\|\tilde{h}\|_{\rho_0^\delta\rho_2^{\sigma-\frac{n-1}{2}}H_b^N(S_1)} < C(\|h^0\|_{\rho_0^{\frac{n-1}{2}+\delta}H_b^{N+1}(\Sigma_0)} + \|h^1\|_{\rho_0^{\frac{n-1}{2}+\delta}H_b^N(\Sigma_0)})$$

for some $0 < \delta' \le \delta$ and $\sigma > 0$. These result can be extended to $\delta \ge \frac{1}{2}$ easily by same estimates but with $0 < \delta' \le \frac{1}{2}$.

Definition 3.3.1. Define the nonlinear Møller wave operator for $N > 0$, $0 < \delta < 1$, $\sigma > 0$ and $\epsilon > 0$ small:

$$\mathscr{R}_{\mathscr{F}} : \mathcal{U}_\epsilon^{N,\delta} \ni (h_0,h_1) \longrightarrow \tilde{h}|_{S_1} \in \rho_0^\delta\rho_2^{\sigma-\frac{n-1}{2}}H_b^N(\overline{\mathbb{R}}\times\mathbb{S}^{n-1}).$$

We study this map intensively in this section.

3.3.1 Refinement of Image Space.

First let us consider the Cauchy data with only conormal singularity at $\partial\Sigma_0$.

Proposition 3.3.2. *If* $(h_0, h_1) \in \widetilde{\mathcal{U}}_\epsilon^{N,\delta} \cap \mathscr{A}^{\frac{n-1}{2}+\delta}(\Sigma_0)$ *for* $\delta > 0$ *then for some* $\sigma > 0$

$$h \in \mathscr{A}^{\frac{n-1}{2}+\delta, \frac{n-1}{2}, \sigma}(X)$$

Proof. Fist it is easy to see in $\overline{\Omega}_1$, we have a sequence of constant $C_1^{N'}$ for all N' such that

$$M_1^{N'}(T_1, \tilde{h}) \leq C_1^{N'} M_1^{N'}(0, \tilde{h}) \quad \forall\, T_1 \in [0, \tfrac{1}{8}].$$

Then from energy estimates, for ϵ small enough and $i = 2, 3, 4$,

$$\|\rho_1^{\frac{n-5}{2}}(\rho_0\rho_2)^{\frac{n-1}{2}}\tilde{h}\|_\infty + \|\rho_1^{\frac{n-5}{2}}(\rho_0\rho_2)^{\frac{n-1}{2}}\tilde{\partial}\tilde{h}\|_\infty < C\epsilon,$$
$$M_i^N(T_i', \tilde{h}) < C\epsilon(1 + (-T_i)^{\delta'-\frac{1}{2}}).$$

where C is a constant. Hence $\epsilon_2', \epsilon_3', \epsilon_4'$ are fixed and small. For any $N' > N$, (3.5) and interpolation methods imply that in each domain $\overline{\Omega}_i$, $i = 2, 3, 4$, the cubic term in the estimates can be replaced by

$$C(-T_i')^{-\frac{1}{2}} M_i^N(T_i', \tilde{h})(M_i^{N'}(T_i', \tilde{h}))^2$$

Since $(-T_i')^{-\frac{1}{2}} M_i^N(T_i', \tilde{h})$ is integrable and fixed w.r.t. T_i' near 0, we have

$$M_i^{N'}(T_i', \tilde{h}) \leq C_1 + C_2(-T_i')^{\delta'-\frac{1}{2}}$$

where $\delta' \in (0, \delta)$ only depend on δ and $\epsilon_2', \epsilon_3', \epsilon_4'$, and

$$|C_1| + |C_2| \leq C\big(\|h^0\|_{\rho_0^{\frac{n-1}{2}+\delta} H_b^{N'+1}(\Sigma_0)} + \|h^1\|_{\rho_0^{\frac{n-1}{2}+\delta} H_b^{N'}(\Sigma_0)}\big).$$

Hence $\tilde{h}$ only has conormal singularity at ∂X. ∎

Fix some $(h_0, h_1) \in \widetilde{\mathcal{U}}_\epsilon^{N,\delta} \cap \mathscr{A}^{\frac{n-1}{2}+\delta}(\Sigma_0)$ for $N > n + 6$, $\delta \in (0, 1)$ and $\epsilon > 0$ small.

Then applying the linear wave operator $\Box_{\tilde{m}}$ to the solution h gives

$$\begin{aligned}\Box_{\tilde{m}} h_{\mu\nu} &= (\Box_{\tilde{g}})h_{\mu\nu} + (\Box_{\tilde{m}} - \Box_{\tilde{g}})h_{\mu\nu} \\ &= F_{\mu\nu}(\partial h, \partial h) - H^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} \\ &\in \mathscr{A}^{n+1+2\delta, n-1, 2+2\sigma}(X).\end{aligned}$$

Repeating k times until $2k + 2\sigma > \frac{n-1}{2} + \delta$ and finally we have

$$h \in \mathscr{A}^{\frac{n-1}{2}+\delta, \frac{n-1}{2}, \frac{n-1}{2}+\delta}(X).$$

See [MW] for details.

Consider the Cauchy problem for linear equation with fixed h and $g = m + h$

$$\Box_g k_{\mu\nu} = F_{\mu\nu}(h)(\partial k, \partial h), \quad k_{t=0} = k^0, \quad \partial_t k|_{t=0} = k^1.$$

Then $k = h$ is a solution if $(k^0, k^1) = (h^0, h^1)$. Denote by $\tilde{k} = \tilde{\rho}^{\frac{1-n}{2}} k$ and define for $\tau \in \mathbb{C}, \Re\tau \in (\frac{n-1}{2}, n-1)$

$${}^g\mathcal{R}_{\mathcal{F}} : \rho_0^\tau C^\infty \times \rho_0^{\tau+1} C^\infty \ni (k^0, k^1) \to \tilde{k}|_{S_1} \in (\rho_0\rho_2)^{\tau - \frac{n-1}{2}} C^\infty(S_1).$$

Lemma 3.3.3. *For $\Re\tau \in (\frac{n-1}{2}, n-1)$, ${}^g\mathcal{R}_{\mathcal{F}}$ and ${}^m\mathcal{R}_{\mathcal{F}}$ have the same boundary operator.*

Proof. With $(k^0, k^1) \in \rho_0^\tau C^\infty \times \rho_0^{\tau+1} C^\infty$, we first apply $\Box_m$ as above and have

$$k \in \mathscr{A}^{\tau, \frac{n-1}{2}, \tau}.$$

Consider the equation

$$\Box_m k'_{\mu\nu} = 0, \quad k'_{t=0} = k^0, \quad \partial_t k'|_{t=0} = k^1.$$

Then

$$\Box_m(k'-k)=\Box_g k-\Box_m k+F(h)(\partial h,\partial k)\in\mathcal{A}^{\frac{n-1}{2}+\delta+\tau+2,n-1,\frac{n-1}{2}+\delta+\tau+2}.$$

Hence $k'-k\in\mathcal{A}^{\frac{n-1}{2}+\delta+\tau,\frac{n-1}{2},\frac{n-1}{2}+\delta+\tau}+\mathcal{A}^{\frac{n-1}{2}+\delta+\tau+2,n-1,n-1}$. If $\tau<n-1$, this term contributes zero to the boundary operator. □

By the same proof for the mapping property of ${}^m\mathcal{R}_{\mathcal{F}}$ in [MW], we show that ${}^g\mathcal{R}_{\mathcal{F}}$ defines a continuous map for $\delta\in(0,1)$

$$\begin{aligned}{}^g\mathcal{R}_{\mathcal{F}}:\rho_0^{\frac{n-1}{2}+\delta}H_b^{N+1}(\Sigma_0)\times\rho_0^{\frac{n+1}{2}+\delta}H_b^{N+1}(\Sigma_0)\\ \longrightarrow(\rho_0\rho_2)^{\delta}[H_b^{\frac{1}{2}-\delta}(\overline{\mathbb{R}};H^{N+\frac{1}{2}+\delta}(\mathbb{S}^{n-1}))\cap L^2(\mathbb{S}^{n-1};H_b^{N+1}(\overline{\mathbb{R}}))]\end{aligned}$$

with norm bounded by constant C independent of $g=m+h$ for any h as a solution with Cauchy data $(h^0,h^1)\in\widetilde{\mathcal{U}}_\epsilon^{N,\delta}\cap\mathscr{A}^{\frac{n-1}{2}+\delta}(\Sigma_0)$ since they all give same boundary operator. Here

$${}^g\mathcal{R}_{\mathcal{F}}(h^0,h^1)=\mathscr{R}_{\mathscr{F}}(h^0,h^1).$$

Hence by density argument, we showed that

⟨cauchy.29⟩ **Proposition 3.3.4.** *The nonlinear Møller wave operator defines a continuous map*

$$\mathscr{R}_{\mathscr{F}}:\mathcal{U}_\epsilon^{N,\delta}\longrightarrow(\rho_0\rho_2)^{\delta}[H_b^{\frac{1}{2}-\delta}(\overline{\mathbb{R}};H^{N+\frac{1}{2}+\delta}(\mathbb{S}^{n-1}))\cap L^2(\mathbb{S}^{n-1};H_b^{N+1}(\overline{\mathbb{R}}))]$$

for $N>n+6$, $\delta\in(0,1)$ and $\epsilon>0$ small.

3.3.2 Constraint Condition for Characteristic Data.

For $h=\tilde{\rho}^{\frac{n-1}{2}}\tilde{h}$ with $\tilde{h}$ $C^{0,\delta'}$ to S_1 for some $\delta'>0$, the harmonic gauge condition gives at $\rho_1=0$

$$\begin{aligned}\partial_\tau(\tilde{h}_{00}+\tilde{h}_{0i}\theta^i)-\tfrac{1}{2}\partial_\tau\mathrm{tr}_m\tilde{h}=0\\ \partial_\tau(\tilde{h}_{j0}+\tilde{h}_{ji}\theta^i)+\tfrac{1}{2}\theta_j\partial_\tau\mathrm{tr}_m\tilde{h}=0\end{aligned}\qquad(3.18)\ \boxed{\texttt{constraint.4}}$$

This implies on S_1,

$$\partial_\tau\tilde{h}_{\rho\rho}=0.$$

3.3.3 Linearization.

The linearization of $\mathscr{R}_{\mathscr{F}}$ at $(0,0)$, denoted by $\mathcal{R}_{\mathcal{F}}$ for simplicity, is the Møller wave operator for linear equation:

$$\Box_{\tilde{m}} h_{\mu\nu} = 0$$

which is studied intensively in [MW].

Theorem 3.3.5. *For $\sigma \in (0,1)$, the map*

$$\mathcal{R}_{\mathcal{F}} : \rho_0^{\frac{n-1}{2}+\delta} H_b^{N+1}(\Sigma_0) \times \rho_0^{\frac{n}{2}+\delta} H_b^{N+1}(\Sigma_0)$$
$$(\rho_0\rho_2)^{\delta}[H_b^{\frac{1}{2}-\delta}(\overline{\mathbb{R}}; H^{N+\frac{1}{2}+\delta}(\mathbb{S}^{n-1})) \cap L^2(\mathbb{S}^{n-1}; H_b^{N+1}(\overline{\mathbb{R}}))]$$

is an isomorphism.

Definition 3.3.6. Define $\widetilde{\mathcal{W}}_{\epsilon}^{N,\delta}$ the space consisting of the elements $\tilde{h}^{S_1}$ which satisfies the constraint condition (3.18) such that

$$\|\tilde{h}^{S_1}\|_{(\rho_0\rho_2)^{\delta}[H_b^{\frac{1}{2}-\delta}(\overline{\mathbb{R}};H^{N+\frac{1}{2}+\delta}(\mathbb{S}^{n-1}))\cap L^2(\mathbb{S}^{n-1};H_b^{N+1}(\overline{\mathbb{R}}))]} \leq C\epsilon$$

Then the implicit function theorem and Proposition 3.3.4 show Theorem 1.0.1.

Chapter 4

Characteristic Initial Value Problem for Einstein Vacuum Equations with $n \geq 5$.

In this chapter, we consider the Characteristic initial value problem for Einstein vacuum equations: given $\tilde{h}^{S_1}$ a symmetric $(n+1)\times(n+1)$ matrix on S_1 satisfying the constraint condition

$$\partial_\tau(\tilde{h}^{S_1}_{00} + \tilde{h}^{S_1}_{0i}\theta^i) - \tfrac{1}{2}\partial_\tau \mathrm{tr}_m \tilde{h}^{S_1} = 0,$$
$$\partial_\tau(\tilde{h}^{S_1}_{j0} + \tilde{h}^{S_1}_{ji}\theta^i) + \tfrac{1}{2}\theta_j\partial_\tau \mathrm{tr}_m \tilde{h}^{S_1} = 0, \quad j = 1, ..., n,$$

we want to find out a solution $g = m + h$ to $R_{\mu\nu} = 0$ such that $\tilde{\rho}^{\frac{1-n}{2}} h|_{S_1} = \tilde{h}^{S_1}$. The uniqueness theorem for Characteristic initial value problem allows us only considering the reduced Einstein equations (1.3), or equivalently their conformal transformation (3.3).

First, the isomorphism property of nonlinear Møller wave operator $\mathscr{R}_{\mathscr{F}}$ shows the existence and uniqueness theorem for the characteristic initial value problem directly:

Theorem 4.0.7. *Given $\tilde{h}^{S_1} \in \widetilde{W}^{N,\sigma}_\epsilon$ for $0 < \sigma < \frac{1}{2}$, $N > n+6$, $\epsilon > 0$ small, there exists a unique solution h to the Einstein vacuum equations with Cauchy data $(h^0, h^1) \in \widetilde{U}^{N,\sigma}_{C\epsilon}$ such that $\tilde{h}^{S_1}$ provides the radiation field of h.*

However, we also can prove the global existence and uniqueness theorem by energy estimates for $\tilde{h}$ in a dense subset of $\widetilde{W}_\epsilon^{N,\sigma}$.

4.1 Backward Energy Estimates for $n \geq 5$.

In this section, we prove the the following theorem by energy estimates:

⟨thmchar⟩
Theorem 4.1.1. *Given $\tilde{h}^{S_1} \in \widetilde{W}_\epsilon^{N+1,\sigma}$ satisfying $\|\tilde{h}^{S_1}\|_{\rho_0^\sigma \rho_2^\delta H_b^{N+1}} < \epsilon$ for some $0 < \sigma < \frac{1}{2} < \delta$, $N > n+6$ and $\epsilon > 0$ small , the Einstein vacuum equation has a unique solution h with Cauchy data $(h^0, h^1) \in \widetilde{U}_{C\epsilon}^{N,\sigma}$ such that $\tilde{h}^{S_1}$ provides the radiation field of h.*

Now assume $\tilde{h}^{S_1} \in \widetilde{\mathcal{W}}_\epsilon^{N+1,\sigma}$ for some $N > n+6$, $\frac{1}{2} > \sigma > 0$, $\epsilon > 0$ small and moreover for some $\delta > \frac{1}{2}$

$$\|\tilde{h}^{S_1}\|_{\rho_0^\sigma \rho_2^\delta H_b^{N+1}(S_1)} \leq \epsilon.$$

Then the constraint condition gives

$$\tilde{h}_{\rho\rho}^{S_1} = 0$$

and for all $0 \leq k \leq N-1$

$$\|\partial_{\rho_1} \tilde{\partial}^k \tilde{h}_{\rho\rho}\|_{L^\infty(S_1)} \leq C \|\tilde{h}\|_{(\rho_0\rho_2)^\delta H_b^{k+2}}.$$

Here C only depend on $\delta > \frac{1}{2}$. The global existence theorem and uniqueness hold for linear equation. We prove the existence theorem by iteration and backward energy estimates. Then the uniqueness theorem follows automatically.

Consider the a sequence of linear equations and their conformal transformation for $l \geq 0$ starting with $h^{-1} = 0$:

$$\Box_{g^l} h_{\mu\nu}^{l+1} = F(\mu\nu)(\partial h^l, \partial h^{l+1}),$$
$$(\Box_{\tilde{g}^l} + \gamma(\tilde{h}^l))\tilde{h}^{l+1} = \tilde{f}^l(\tilde{h}^{l+1}).$$

Here $g^l = m + h^l, \tilde{g}^l = \tilde{\rho}^2 g^l$ and h^l all have same characteristic data $\tilde{h}$. We also study the equation for $h^{l+1} - h^l$ to get the convergence of h^l.

$$\Box_{g^l}(h^{l+1} - h^l) = (\Box_{g^{l-1}} - \Box_{g^l})h^l + (F(\partial h^l, \partial h^{l+1}) - F(\partial h^{l-1}, \partial h^l))$$

$$(\Box_{\tilde{g}^l} + \gamma(\tilde{h}^l))(\tilde{h}^{l+1} - \tilde{h}^l) = \tilde{f}^l(\tilde{h}^{l+1}) - \tilde{f}^{l-1}(\tilde{h}^l) + (\Box_{\tilde{g}^{l-1}} - \Box_{\tilde{g}^l})\tilde{h}^l + (\gamma(\tilde{h}^{l-1}) - \gamma(\tilde{h}^l))\tilde{h}^l,$$

We basically apply the energy estimates backwards in each domain, which means we have to change the sign of divergence terms and rebound them.

4.1.1 In $\overline{\Omega}_4$.

We solve the equations up to Σ_4. First on S_1

$$\begin{aligned} |\partial_{\bar{a}}\tilde{\partial}^k\tilde{h}(\bar{b})| =& |\int_0^{\bar{b}} \bar{b}\partial_{\bar{b}}\partial_{\bar{a}}\tilde{\partial}^k\tilde{h}\frac{d\bar{b}}{\bar{b}}| \\ \leq& (\int_0^{\bar{b}} |\bar{b}'|^{2\delta}\frac{d\bar{b}'}{\bar{b}})^{\frac{1}{2}} C\|\tilde{h}\|_{\rho_2^\delta H_b^{k+2}(S_1\cap\Omega_4)} \leq C\epsilon\bar{b}^\delta. \end{aligned}$$

Here C only depend on k and $\delta > 0$ and hence

$$\|\partial_{\bar{a}}\tilde{\partial}^k\tilde{h}(\bar{b})\|_{\rho_2^{\delta'} H_b^{k+2}(S_1\cap\Omega_4)} \leq C'C\epsilon,$$

where C' only depend on $\frac{1}{2} < \delta' < \delta$. We choose $1 < \alpha < \frac{n-1}{2}$ for the time-like function T_4' in this case. Define

$$M_4^N(T_4', \tilde{h}^{l+1}) = (\int_{\Sigma_{T_4'}} e^{2\delta' T_4} \sum_{k\leq N} \langle F(T_4, \tilde{\partial}\tilde{h}^{l+1}), \nabla T_4'\rangle_{\tilde{g}^l} d\mu^l)^{\frac{1}{2}},$$

$$K_4^N(T_4', \tilde{h}^{l+1}) = (\int_{\Sigma_4\backslash\Sigma_4^{T_4'}} e^{2\delta' T_4} \sum_{k\leq N} \langle F(T_4, \tilde{\partial}\tilde{h}^{l+1}), \nabla T_4'\rangle_{\tilde{g}^l} d\mu^l)^{\frac{1}{2}},$$

where the covariant derivatives are w.r.t. $\tilde{g}^l$ and volume form is chosen such that

$$d\mu^l \wedge dT_4' = dvol_{\tilde{g}^l}.$$

If $\sum_{|k|\le\frac{N}{2}+1}\|\rho_2^{\frac{n-1}{2}}\rho_1^{\frac{n-5}{2}}\tilde{\partial}^k\tilde{h}^l\|_\infty \le \epsilon_4$ and $M_4^N(T_4',\tilde{h}^l) \le \epsilon_4$ for some $\epsilon_4 > 0$ small such that $1+\epsilon_4-2\delta' < 0$ and $\epsilon_4 < \delta'$, then the harmonic gauge condition gives

$$|\rho_2^{\frac{n-1}{2}}\tilde{h}^l_{\rho\rho}| \le \epsilon_4\rho_1^{\frac{1}{2}}.$$

Apply the energy estimates for $\tilde{h}^{l+1}$ backwards, we have

$$-\partial_{T_4'}(M_4^N(T_4',\tilde{h}^{l+1}))^2 - \partial_{T_4'}(K_4^N(T_4',\tilde{h}^{l+1}))^2 \le \ \epsilon_4(-T_4')^{-\frac{1}{2}}(M_4^N(T_4',\tilde{h}^{l+1}))^2.$$

Hence for all l

$$M_4^N(T_4',\tilde{h}^{l+1}) + K_4^N(T_4',\tilde{h}^{l+1}) \le C\epsilon,$$

where C is a constant independent of ϵ_4 if it is small. If ϵ small enough, then

$$\sum_{|k|\le\frac{N}{2}+1}\|\rho_1^{\frac{n-5}{2}}\rho_2^{\frac{n-1}{2}}\tilde{\partial}^k\tilde{h}^{l+1}\|_\infty \le \epsilon_4 \quad \text{and} \quad M_4^N(T_4',\tilde{h}^{l+1}) \le \epsilon_4$$

hold and hence hold for all l. Moreover, the converges of $\tilde{h}$ follows the energy estimates for $\tilde{h}^{l+1}-\tilde{h}^l$.

$$\begin{aligned}-\partial_{T_4'}(M_4^N(T_4',\tilde{h}^{l+1}-\tilde{h}^l))^2 \le& \epsilon_4(-T_4')^{-\frac{1}{2}}M_4^N(T_4',\tilde{h}^{l+1}-\tilde{h}^l)M_4^N(T_4',\tilde{h}^l-\tilde{h}^{l-1}),\\ M_4^N(T_4',\tilde{h}^{l+1}-\tilde{h}^l) \le& \int_0^{-T_4'}\epsilon_4(-T_4')^{-\frac{1}{2}}M_4^N(T_4',\tilde{h}^{l+1}-\tilde{h}^l)dT_4'.\end{aligned}$$

Notice $M_4^N(T_4',\tilde{h}^0-\tilde{h}^{-1}) \le C\epsilon$ small and T_4' is bounded on $\overline{\Omega}_4$, hence

$$M_4^N(T_4',\tilde{h}^{l+1}-\tilde{h}^l) \le C\epsilon\epsilon_4^l.$$

Hence we can choose ϵ small enough such that

$$M_4^N(T_4',\tilde{h}^{l+1}) \le C\epsilon(\sum_0^l\epsilon_4^{2l}) \le C\epsilon(1-\epsilon_4^2)^{\frac{1}{2}} < \epsilon_4.$$

Therefore $\tilde{h}^l$ converges to $\tilde{h}$ such that

$$(K_4^N(T_4', \tilde{h}))^2 + M_4^N(T_4', \tilde{h}) \leq C\epsilon < \epsilon_4$$

is valid for all T_4'. Note T_4' is bounded above on $\overline{\Omega}_4$ and

$$L_4^N(\rho_1, \tilde{h}) < C\epsilon < \epsilon_4, \quad \forall \rho_1 \in [0, \infty).$$

4.1.2 In $\overline{\Omega}_3$.

We solve the equation up to Σ_3 here. Since τ is bounded in this domain on S_1,

$$|\partial_\rho \tilde{\partial}^k \tilde{h}(\bar{b})| \leq |\tilde{\partial}^k \tilde{h}(\tau_0)| + |\int_\tau^{\tau_0} \partial_\tau \partial_\rho \tilde{\partial}^k \tilde{h} d\tau'| \leq C\|\tilde{h}\|_{H_b^{k+2}(S_1 \cap \Omega_3)} \leq C\epsilon.$$

Here C only depend on τ_0 which is fixed in our setting. Hence

$$\|\partial_\rho \tilde{\partial}^k \tilde{h}\|_{H_b^{k+2}(S_1 \cap \Omega_3)} \leq C' C\epsilon.$$

Define

$$\begin{aligned} M_3^N(T_3', \tilde{h}^{l+1}) =& \Big(\int_{\Sigma_{T_3'}} \sum_{k \leq N} e^{\lambda T_3'} \langle F(T_3, \tilde{\partial}^k \tilde{h}^{l+1}), \nabla T_3' \rangle d\mu^l\Big)^{\frac{1}{2}}, \\ K_3^N(T_3, T_3', \tilde{h}^{l+1}) =& \Big(\int_{\Sigma_{T_3} \cap \{\rho(2\tau_0 - \tau) \leq -T_3'\}} \sum_{k \leq N} e^{\lambda T_3} \langle F(T_3, \tilde{\partial}^k \tilde{h}^{l+1}), \nabla T_3 \rangle d\mu^l\Big)^{\frac{1}{2}}, \end{aligned}$$

where $d\mu^l$ is the volume form on $\Sigma_{T_3'}$ such that

$$d\mu^l \wedge dT_3' = dvol_{\tilde{g}}^l.$$

Here $\tilde{h}^l$ all have same characteristic data on $S_1 \cap \Omega_3$ and Σ_4 with

$$K_3^N(\tau_0, \tau_0 \rho_1, \tilde{h}) = K_4^N(\rho_1, \tilde{h}) < C\epsilon, \quad M_3^N(0, \tilde{h}) \leq \epsilon.$$

Choose λ large enough. Then for $\sum_{|k|\leq \frac{N}{2}+1} \|\rho_1^{\frac{n-5}{2}} \tilde{\partial}^k \tilde{h}^l\|_\infty \leq \epsilon_3$ and $M_3^N(-T_3', \tilde{h}^l) \leq \epsilon_3$ for some $\epsilon_3 > 0$ small enough, by harmonic gauge condition we have

$$|\tilde{h}^l_{\rho\rho}| \leq \int_0^\rho |\partial_\rho \tilde{h}^l_{\rho\rho}| d\rho \ \leq C\epsilon_3 \rho_1^{\frac{1}{2}}.$$

And apply the energy estimates backwards,

$$\begin{aligned} & -\partial_{T_3'}(M_3^N(T_3', \tilde{h}^{l+1}))^2 - \partial_{T_3'}(K_3^N(-\tau_0, T_3', \tilde{h}^{l+1}))^2 \\ \leq & -\partial_{T_3'} K_3^N(\tau_0, T_3', \tilde{h}^{l+1}) + \epsilon_3(-T_3')^{-\frac{1}{2}}(M_3^N(T_3', \tilde{h}^{l+1}))^2. \end{aligned}$$

which implies

$$M_3^N(T_3', \tilde{h}^l) + K_3^N(-\tau_0, T_3', \tilde{h}^l) \leq C\epsilon.$$

Here C is independent of ϵ_3 if it is small enough. If ϵ is small enough, then

$$\sum_{|k|\leq \frac{N}{2}+1} \|\rho_1^{\frac{n-5}{2}} \tilde{\partial}^k \tilde{h}^{l+1}\|_\infty \leq \epsilon_3 \quad \text{and} \quad M_3^N(T_3', \tilde{h}^{l+1}) \leq \epsilon_3$$

hold and hence hold for all l. The convergence of $\tilde{h}^l$ follows the energy estimates for $\tilde{h}^{l+1} - \tilde{h}^l$ similar like in $\overline{\Omega}_4$. The limit $\tilde{h}$ satisfies

$$K_3^N(-\tau_0, \infty, \tilde{h}) \leq C_3\epsilon.$$

4.1.3 In $\overline{\Omega}_2$.

We solve the equation up to Σ_1 and then it follows directly from Σ_1 to Σ_0 since that is a Cauchy problem with inverse time direction. First notice here

$$\begin{aligned} |\partial_a \tilde{\partial}^k \tilde{h}(b)| =& |\int_0^b b\partial_b \partial_a \tilde{\partial}^k \tilde{h} \frac{db}{b}| \\ \leq & (\int_0^b b^{2\sigma} \frac{db}{b})^{\frac{1}{2}} C\|\tilde{h}\|_{b^\sigma H_b^{k+2}(S_1\cap\Omega_2)} \leq C\epsilon b^\sigma. \end{aligned}$$

Here C only depend on $\delta > 0$ and hence

$$\|\partial_a \tilde{\partial}^k \tilde{h}\|_{(\rho_0)^{\sigma'} H_b^{k+2}(S_1 \cap \Omega_2)} \leq C' C \epsilon,$$

where C' only depend on $0 < \sigma' < \sigma$. Define

$$M_2^N(T_2', \tilde{h}^{l+1}) = (\int_{\Sigma_{T_4'}} e^{2\delta' T_2} \sum_{k \leq N} \langle F(T_2, \tilde{\partial}\tilde{h}^{l+1}), \nabla T_2' \rangle_{\tilde{g}^l} d\mu^l)^{\frac{1}{2}}.$$

where the volume form is chosen such that

$$d\mu^l \wedge dT_2' = dvol_{\tilde{g}^l}.$$

If $\sum_{|k| \leq \frac{N}{2}+1} \|\rho_2^{\frac{n-1}{2}} \rho_1^{\frac{n-5}{2}} \tilde{\partial}^k \tilde{h}^l\|_\infty \leq \epsilon_2$ and $M_2^N(T_2', \tilde{h})^l \leq \epsilon_2$ for some $\epsilon_2 > 0$ small such that $\epsilon_2 + 2\sigma' < 1$, then the harmonic gauge condition gives

$$|\rho_0^{\frac{n-1}{2}} \tilde{h}^l_{\rho\rho}| \leq \int_0^{\rho_1} a^{-\frac{1}{2}} M_2^N(-a, \tilde{h}^l) da \leq \epsilon_2 \rho_1^{\frac{1}{2}}.$$

Here $-T_2' = \rho_1 = a$. Apply the energy estimates for $\tilde{h}^{l+1}$ backwards, we have

$$-\partial_{T_2'}(M_2^N(T_2', \tilde{h}^{l+1}))^2 \leq (\epsilon_2(-T_2')^{-\frac{1}{2}} + C)(M_2^N(0, \tilde{h}^{l+1}))^2.$$

Hence

$$M_2^N(T_2', \tilde{h}^{l+1}) \leq C\epsilon.$$

If ϵ is small enough,

$$\sum_{|k| \leq \frac{N}{2}+1} \|\rho_1^{\frac{n-5}{2}} \rho_0^{\frac{n-1}{2}} \tilde{\partial}^k \tilde{h}^{l+1}\|_\infty \leq \epsilon_2 \quad \text{and} \quad M_2^N(T_2', \tilde{h}^{l+1}) \leq \epsilon_2$$

hold and hence hold for all l. The converges of $\tilde{h}$ follows the energy estimates for $\tilde{h}^{l+1} - \tilde{h}^l$.

We finish proving Theorem 4.1.1 by energy estimates.

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