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<td>As Published</td>
<td><a href="http://dx.doi.org/10.1109/GLOCOM.2009.5426115">http://dx.doi.org/10.1109/GLOCOM.2009.5426115</a></td>
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<tr>
<td>Publisher</td>
<td>Institute of Electrical and Electronics Engineers</td>
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<tr>
<td>Version</td>
<td>Final published version</td>
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<tr>
<td>Citable link</td>
<td><a href="http://hdl.handle.net/1721.1/60262">http://hdl.handle.net/1721.1/60262</a></td>
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Coherent Optical Communication over the Turbulent Atmosphere with Spatial Diversity and Wavefront Predistortion

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Abstract—Optical communication through the atmosphere has the potential to provide data transmission over a distance of 1 to 100km at very high rates. The deleterious effects of turbulence can severely limit the utility of such a system, however, causing outages of up to 100ms. In this paper, we investigate the use of spatial diversity with wavefront predistortion and coherent detection to overcome these turbulence-induced outages. This system can be realized, especially if the turbulence or the receiver is in the nearfield of the transmitter. New results include closed-form expressions for average bit error rate, outage probability, and power efficiency. Additionally, system performance gains in the presence of a worst-case interferer are presented. These significant results are used to develop design intuition for future system implementation.

I. INTRODUCTION

Current free-space optical systems are unable to support reliable low-cost gigabit class communication over tens of kilometers [1]. The reliability of these systems is considerably reduced by deep fades of 20-30dB of typical duration of 1-100ms [2]. Such fades, caused by microscale atmospheric temperature fluctuation, may result in the corruption of 10^9 bits at 10Gbps. A system engineer typically has four degrees of freedom to mitigate the effects of fading: power, temporal diversity, frequency diversity, and/or spatial diversity. Increasing power to provide 30dB of margin is prohibitively costly. Similarly, increasing temporal and/or spatial diversity by implementing a space-time code is not an attractive solution because it requires a gigabit interleaver and long delays. Frequency-diversity can provide some additional robustness but is costly because of the broadband components required. These components must be broadband because of the large frequency coherence of atmospheric turbulence. Thus we are motivated to explore architectures with a high degree of spatial diversity. This requires systems with many transmit and receive apertures. Such systems can be readily implemented due to the relatively short coherence length of the atmosphere at optical wavelengths.

In this paper, we investigate the performance of spatial diversity systems. We focus on an architecture that employs wavefront predistortion and coherent detection. We find that this architecture effectively mitigates the effects of the turbulent atmosphere. We assume independent control of the phase and magnitude of each transmitter. Through coherent detection, we assume that the receiver can measure the phase and magnitude of the received wave. Thus we can optimally allocate transmit power into the spatial modes with the smallest propagation losses in order to decrease bit errors and mitigate turbulence-induced outages. Additionally, spatial mode moduluation and rejection provides robust communication in the presence of interference by other users with knowledge of the system architecture.

Previous work on sparse aperture coherent detection systems has not included wavefront predistortion, and assumes the turbulence is in the nearfield of only the receiver. Average bit error rate (BER) was studied in [3], [4]. Outage probability was studied in [5]. In contrast, we present performance results for sparse aperture coherent detection systems including wavefront predistortion. We present closed-form expressions for turbulence average BER that apply when there are many transmit and receive apertures. Because the instantaneous BER can vary significantly around the average BER, we also present new outage probability expressions. We quantify the effect of an optimal interferer and transmitter power allocation. We conclude that spatial mode control significantly improves communication performance, especially for nearfield operation. Finally, we provide recommendations for the design of such a system.

II. PROBLEM FORMULATION

We assume transmitters are arranged in the $\rho$-plane and receivers are arranged in $\rho'$-plane. The $\rho$- and $\rho'$-planes are assumed to be parallel, as shown in Fig. 1. The distance from the $\rho$-plane origin to transmitter $i$ is denoted $\rho_i$. Similarly, the distance from the $\rho'$-plane origin to receiver $j$ is denoted $\rho'_j$.

We assume a coherent monochromatic scalar field of wavelength $\lambda$ is transmitted from $n_{tx}$ apertures in the $\rho$-plane. The field propagates $z$ meters through a linear, isotropic, statistically homogeneous medium to the $\rho'$-plane where it
is detected with $n_{rx}$ apertures. Each aperture’s diameter is assumed to be less than a coherence length. Additionally, the minimum distance between each transmitter is at least a coherence length, far enough apart so that the statistics of all $n_{tx}n_{rx}$ links are uncorrelated. We refer to this geometry as a sparse aperture system. In contrast, we refer to a system with one large transmit aperture and one large receive aperture as a single aperture system. We define a pair of ancillary variables, $n_{min} = \min(n_{tx}, n_{rx})$ and $n_{max} = \max(n_{tx}, n_{rx})$. We will see that the system performance depends on $n_{tx}$ and $n_{rx}$ only through these ancillary variables.

We can model the field propagation from the transmit plane to the receive plane as:

$$ y = \frac{1}{\sqrt{n_{max}}} H x + w $$

where $x \in \mathbb{C}^{n_{tx}}$ is a vector representing the transmitted field, $y \in \mathbb{C}^{n_{rx}}$ is a vector representing the received field, and $w \in \mathbb{C}^{n_{rx}}$ represents additive white circularly symmetric complex Gaussian noise. The normalization $(n_{max})^{-1/2}$ is chosen to ensure that adding additional apertures will not increase the system power gain. The normalization has been chosen here to ensure reciprocity is satisfied. Finally, $H \in \mathbb{C}^{n_{rx} \times n_{tx}}$ represents the channel. Entry $h_{ij}$ of $H$ is the gain from the $i$th transmit aperture to the $j$th receive aperture. For free-space propagation, the extended Huygens-Fresnel principle reduces to:

$$ h_{ij} = \frac{1}{j \lambda z} \exp \left[ \frac{j 2 \pi z}{\lambda} \left( 1 + \frac{|\rho_i - \rho'_j|^2}{2z^2} \right) \right] $$

Because we assumed each aperture is smaller than a coherence length, we can model the atmospheric turbulence as piecewise constant over each aperture.

Assuming the effects of the turbulence-induced index of refraction variation on the propagating field can be modeled using Rytov’s method, the fading along the path from a single transmit aperture to a single receive aperture is a multiplicative factor of the form $e^{\chi + j\phi}$.

$$ h_{ij} = \frac{1}{j \lambda z} \exp \left[ \frac{j 2 \pi z}{\lambda} \left( 1 + \frac{|\rho_i - \rho'_j|^2}{2z^2} \right) \right] \times \exp \left[ \chi(\rho_i, \rho'_j) + j\phi(\rho_i, \rho'_j) \right] $$

where the log-amplitude and phase fluctuations, $\chi(\rho_i, \rho'_j)$ and $\phi(\rho_i, \rho'_j)$, are independent jointly Gaussian random scalars. That is $\chi \sim N(m_\chi, \sigma_\chi^2)$ and $\sigma^2_\chi$ is the log-amplitude variance. By conservation of energy, $m_\chi = -\sigma^2_\chi$. $\phi \sim N(m_\phi, \sigma^2_\phi)$ where $m_\phi$ is the phase mean and $\sigma^2_\phi$ is the phase variance. We assume $\sigma^2_\phi \gg 2\pi$ so that the phase probability distribution function is approximately uniform from zero to $2\pi$, $\phi \sim U[0, 2\pi]$.

As with many engineering applications, decoupling the input-output relationship is both powerful and useful. We accomplish this, as is commonly done, with an eigenmode decomposition of $(n_{max})^{-1/2}HH^H$:

$$ \frac{1}{n_{max}} HH^H = \Phi \Gamma \Phi^H $$

where the $i$th column of $\Phi$ is an output eigenmode, and the $i$, $j$th entry of the diagonal matrix $\Gamma$ is the eigenvalue, or diffraction gain, associated with the $i$th eigenmode. For this paper, an eigenmode is a particular spatial field distribution, or spatial mode. These eigenmodes form a complete orthonormal set spanning the space of all possible receive fields. We will denote the diffraction gain associated with the $i$th eigenmode as $\gamma_i$ so that:

$$ \tilde{y}_i = \gamma_i^{1/2} \tilde{x}_i + \tilde{w}_i $$

where $\tilde{x}$, $\tilde{y}$, and $\tilde{w}$ are related to $x$, $y$, and $w$ through the usual eigenmode channel decomposition, such as [6]. Note that $\tilde{w}$ retains its circularly symmetric complex Gaussian distribution. We denote the variance of $\tilde{w}$ as $\sigma^2$.

We assume instantaneous channel state information measured by the receiver is available to the transmitter. Implied in this assumption is a feedback path from the receiver to the transmitter of sufficient rate and delay to allow for some minimum set of channel information to be received at the transmitter before the atmospheric state has changed. The delay is required to be less than an atmospheric coherence time, on the order of 1-100ms, which is reasonable for communication links on the order of tens of kilometers. Additionally, we will invoke the Taylor frozen atmosphere hypothesis [7], assuming the atmosphere will remain approximately constant over the period of a codeword. For gigabit communication, this assumption is easily satisfied.

The received field is detected coherently with either a heterodyne or a homodyne receiver. Wavefront predistortion is implemented by controlling the amplitude and phase at each transmit aperture independently. However, we do not allow the amplitude or phase to vary within an individual aperture. We will limit the total power transmitted at a given instant to $P_t$, independent of the number of transmitters:

$$ E[||x||^2] = P_t $$

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2009 proceedings.
Throughout this paper, we will often assert that the number of transmit and receive apertures is large. For mild turbulence and a range of 10km, the coherence length is about 1cm: 100 transmitters can be placed in a 10cm-by-10cm patch. Thus the assertion that the number of apertures is large is valid for useful systems.

III. PREVIOUS RESULTS

In this section, we review results presented in a companion paper [8], which form the basis for new results presented in subsequent sections. In Section II of this paper we developed an eigen-decomposition of the system, where the input-output relationship is decoupled. The performance of the communication system is then fully governed by the diffraction gains, \( \gamma_i \). The probability density function (pdf) of the diffraction gains for a single atmospheric state, as the number of transmit and receive apertures asymptotically approaches infinity, converges, almost surely, to the Marcenko-Pastur density [8]:

\[
f_\beta(\gamma) = (1 - \beta)^+ \delta(\gamma) + \frac{\sqrt{(\gamma - (1-\sqrt{\beta})^2)^+ ((1+\sqrt{\beta})^2 - \gamma)^+}}{2\pi\gamma}
\]

where \((x)^+ = \max(x,0)\), \(\delta(\cdot)\) is the Dirac impulse, and \(\beta = n_{\text{min}}/n_{\text{max}}\). The validity of this density to optical systems is explored in [8]. The paper concludes that the result is valid for a wide range of turbulence strengths and ranges. Fig. 2 shows the Marcenko Pastur density for various system geometries. As a corollary, the number of eigenmodes corresponding to non-zero eigenvalues converges, almost surely, to \(n_{\text{min}}\). Additionally, the maximum eigenvalue converges, almost surely, to \(\gamma_{\text{max}} = (1 + \sqrt{\beta})^2\).

IV. PERFORMANCE OF SPARSE APERTURE SYSTEMS

In this section we present the performance of sparse aperture systems with wavefront control and coherent detection.

A. Asymptotic Bit Error Rate

Binary phase shift keying (BPSK) is a common modulation scheme for optical communication systems with coherent detection. Though results developed here may easily be applied to other modulation schemes (e.g. binary on-off keying, frequency shift keying, etc.), we will limit ourselves to the discussion of BPSK systems for brevity. The optimal scheme to minimize BER using BPSK is to allocate all power to the eigenmode associated with the maximum eigenvalue of the channel gain matrix \(H\). To transmit a bit \(C[n] \in \{0,1\}\) at time \(n\) with power \(P_t\), the optimal input field is simply:

\[
x[n] = \sqrt{P_t \nu_{\text{max}}} e^{j\pi C[n]}
\]

where \(\nu_{\text{max}}\) is the input eigenmode associated with the largest eigenvalue of \(H\).

For this modulation scheme, the turbulence average BER is:

\[
\Pr(\text{error}) = \int_0^\infty Q \left( \frac{\sqrt{2\gamma_{\text{max}} E_b}}{\sigma^2} \right) f_\beta'(\gamma_{\text{max}}) d\gamma_{\text{max}}
\]

where \(f_\beta'(\gamma_{\text{max}})\) is the pdf of the largest eigenvalue and \(E_b\) is the received energy per bit when wavefront predistortion is unavailable. This result is general for any sparse aperture optical communication system, but depends on an unknown pdf, \(f_\beta'(\gamma_{\text{max}})\). However the pdf for the largest eigenvalue is known in the asymptotic case. As the number of apertures grows large, the pdf of the largest eigenvalue converges, almost surely, to:

\[
\lim_{n_{\text{min}} \to \infty} f_\beta'(\gamma_{\text{max}}) = \delta \left( \gamma_{\text{max}} - \left(1 + \sqrt{\beta}\right)^2 \right)
\]

where \(\delta(\cdot)\) is the Dirac delta. Using (10) to evaluate (9) provides a closed-form expression for the probability of error:

\[
\lim_{n_{\text{min}} \to \infty} \Pr(\text{error}) = \int_0^\infty Q \left( \frac{\sqrt{2\gamma_{\text{max}} E_b}}{\sigma^2} \right) \delta \left( \gamma_{\text{max}} - \left(1 + \sqrt{\beta}\right)^2 \right) d\gamma_{\text{max}}
\]

\[
= Q \left( \sqrt{\frac{2}{\sigma^2}} \left(1 + \sqrt{\beta}\right)^2 E_b \right)
\]

While this result is only exact in the asymptotic case, it provides a very good approximation for a finite but large number of apertures. The \((1 + \sqrt{\beta})^2\) term is the power gain over a system without the wavefront predistortion. This power gain term results from the ability to allocate all of the system transmit power into the spatial mode with the best propagation performance. Essentially, we select the mode with the best constructive interference for the particular receiver aperture geometry and atmospheric state.

Using the monotonicity of the Q-function, it is easy to prove that the optimum system is balanced, using the same number.
of transmit and receive apertures. For a balanced system, setting \( \beta = 1 \) yields a power gain of four. This result is intuitively satisfying. Consider two scenarios: first, if there are more transmit apertures than receive apertures, the system can form more spatial modes than it can resolve and degrees of freedom are unused. Second, if there are more receive apertures than transmit apertures, the system can resolve more spatial modes than it can form and, again, degrees of freedom are unused. As a result, our intuition suggests a balanced system is optimal.

As the number of receive apertures becomes much larger than the number or transmit apertures, \( \beta \rightarrow 0 \), the system performance approaches that of the system without wavefront predistortion. This is also an expected result. As the system becomes very asymmetric, the ability to predistort the wavefront is lost.

The gain, in terms of probability of error, of moving to a diversity system with wavefront predistortion is:

\[
\Pr(\text{error} | \text{sparse aperture}) = e^{-3SNR}
\]

At high SNR, using the sparse aperture system provides a large gain in BER compared to the no diversity system. At low SNR, the advantage of the more sophisticated system is less pronounced.

It is clear, in the asymptotic case, that the average BER does not depend on turbulence strength. Effectively, the many apertures act to average out the spatial variation induced by the atmospheric turbulence. Turbulence strength does factor into the system design; in stronger turbulence, apertures may be placed closer together while in weaker turbulence, they must be placed farther apart. Further, stronger turbulence causes slower convergence to the Marcenko-Pastur density; which means more apertures are required for (11) to be valid.

Lastly, as the total aperture size increases for a single aperture system, the power gain saturates as the aperture size approaches the coherence length. We have shown that the sparse aperture system, however, does not saturate with total aperture size. Indeed, the number of apertures used is only limited by form factor constraints.

A Monte-Carlo simulation was performed to validate the theory presented in (11). In the simulation, we assumed that the instantaneous atmospheric state was available at the transmitter. For a single atmospheric state, an equiprobable binary source was encoded according to (8), transmitted through the simulated atmosphere, detected coherently, and the number of raw bit errors recorded. This process was repeated many times with independent realizations of the atmosphere to arrive at the average BER presented in Fig. 3. In the figure, we show theory and simulation versus SNR, \( Eb/\sigma^2 \). The number of transmitters was 100, 100, 200 and the number of receivers was 100, 50, 20 giving \( \beta = 1 \), \( \beta = 0.5 \), and \( \beta = 0.1 \).

From the figure, we see very good agreement between theory and simulation. As we stated earlier, the theory provides an approximate solution to any system with a large but finite number of apertures. Here, we see the approximation is very close to the theory.

B. Outage Probability and Power Efficiency

Thus far, we have been concerned with performance metrics with asymptotically infinite transmit and receive apertures. The performance of these systems is well characterized by the metrics we developed; turbulence-induced fading is completely averaged out by the many independent spatial modes seen by the system. We then showed that the asymptotic performance metrics were very good approximations to the physically realizable finite aperture case. This average performance is very important, but it does not tell the entire story for the finite aperture case: turbulence-induced constructive and destructive interference will cause significant variation in the metric around its mean performance. Here we wish to quantify this variation by developing closed-form expressions for outage probability and power efficiency.

1) Outage Probability: There are many ways to measure the variability in system performance due to fading. Outage probability defined in terms of BER is particularly useful because it guarantees at least some minimum performance some fraction of the time. Formally, the outage probability associated with some BER, \( P^* \), is the probability that any given atmospheric state will yield an instantaneous BER more than \( P^* \):

\[
P_{\text{outage}}(P^*) = \Pr(\text{BER} \geq P^*) = 1 - F_{\text{BER}}(P^*)
\]

where \( P_{\text{outage}} \) is the outage probability, \( P_i \) is the instantaneous probability of bit-error, \( P^* \) is the minimum performance we wish to guarantee in terms of BER, and \( F_{\text{BER}}(\cdot) \) is the BER cumulative distribution function (cdf). In earlier sections, we showed that the balanced system is optimal in terms of average performance. To simply the outage capacity analysis, we will assume the balanced system and define \( n = n_{tx} = n_{rx} \).
Because the system performance is fully determined by the maximum eigenvalue, we begin by finding the cdf of the maximum eigenvalue.

**Theorem 4.1:** The cdf of the maximum eigenvalue of a balanced sparse aperture system is:

\[
F_z(z) = \left(\frac{\sqrt{z(4-z)} + 4\sin^{-1}(\sqrt{z/4})}{2\pi}\right)^n
\]

**Proof.** Define \(z\) to be the maximum eigenvalue of a given atmospheric state:

\[
z = \max(\gamma_1, \gamma_2, ..., \gamma_n)
\]

Where we have used that, almost surely, there will be \(n\) non-zero eigenvalues. Assuming that the system is large enough for the eigenvalue distribution to be approximated by the Marcenko-Pastur distribution and that the eigenvalues are drawn independently, the cdf of \(z\) is:

\[
F_z(z) = \left(\int_0^z f_\beta(z')dz'\right)^n
= \left(\int_0^z \sqrt{z'+4\sin^{-1}(\sqrt{z'/4})} dz'\right)^n
= \left(\frac{\sqrt{z(4-z)} + 4\sin^{-1}(\sqrt{z/4})}{2\pi}\right)^n
\]

\[\square\]

We have assumed the eigenvalues to be drawn independently from the Marcenko-Pastur distribution. We do not expect this assumption to hold strictly; however by making this assumption we provide an upper-bound to the outage probability for outage probabilities less than half. With the maximum eigenvalue distribution, we are now in a position to present the probability of outage.

**Theorem 4.2:** The probability of outage, \(P_{\text{outage}}\), associated with a desired probability of error, \(P^*\) is:

\[
P_{\text{outage}}(P^*) = \Pr(P_i \geq P^*)
= \left(\frac{\sqrt{\theta(4-\theta)} + 4\sin^{-1}\sqrt{\theta}}{2\pi}\right)^n
\]

where \(\theta = \frac{\sigma^2}{2E_b} \left(Q^{-1}(P^*)\right)^2\).

**Proof.** Before proceeding to the proof, we note that the probability of error is related to SNR through the Q-function:

\[
\Pr(\text{error}) = Q\left(\sqrt{\frac{2\gamma_{\text{max}} E_b}{\sigma^2}}\right)
\]

Now we can express the probability of outage in terms of the maximum eigenvalue, the SNR and the desired BER:

\[
P_{\text{outage}} = \Pr(P(\text{error}) \geq P^*)
= \Pr\left(Q\left(\sqrt{\frac{2\gamma_{\text{max}} E_b}{\sigma^2}}\right) \geq P^*\right)
= \Pr\left(\gamma_{\text{max}} \leq \frac{\sigma^2}{2E_b} \left(Q^{-1}(P^*)\right)^2\right)
= F_z\left(\frac{\sigma^2}{2E_b} \left(Q^{-1}(P^*)\right)^2\right)
\]

where we have used the maximum eigenvalue distribution, \(F_z\), derived in Theorem 4.1. \(\square\)

Fig. 4 shows the probability of outage versus desired BER for various numbers of apertures. While this is a closed-form expression, it is not in terms of elementary functions. Corollary 4.1 provides a bound in terms of elementary functions.

**Corollary 4.1:** An upper-bound on the probability of outage, \(P_{\text{outage}}\), associated with a desired probability of error, \(P^*\) is:

\[
P_{\text{outage}}(P^*) \leq \left(\frac{\sqrt{\psi(4-\psi)} + 4\sin^{-1}(\sqrt{\psi})}{2\pi}\right)^n
\]

where \(\psi = \frac{\sigma^2}{2E_b} \log 2P^*\)

**Proof.** To prove this upper bound, we use a well-known upper bound to the Q-function:

\[
P_{\text{outage}} = \Pr(P(\text{error}) \geq P^*)
= \Pr\left(Q\left(\sqrt{\frac{2\gamma_{\text{max}} E_b}{\sigma^2}}\right) \geq P^*\right)
\leq \Pr\left(\frac{1}{2} \exp\left(-\frac{\gamma_{\text{max}} E_b}{\sigma^2}\right) \geq P^*\right)
\leq F_z\left(\frac{\sigma^2}{2E_b} \log(2P^*)\right)
\]

\[\square\]

From Corollary 4.1, it is easy to prove the outage probability approaches a step function as the number of apertures goes to infinity. This corollary shows that a system designer can achieve any desired non-zero outage probability by adding apertures.
2) Power Efficiency: Here we define power efficiency in two equivalent ways. The first way defines power efficiency as a comparison between the finite aperture system and the asymptotic system: power efficiency is the multiplicative power gain required for the finite sparse aperture system to perform at least as well as the infinite sparse aperture system, at least $P_{outage}$ fraction of the time. The second, equivalent way, defines power efficiency as a comparison between the sparse aperture system in and out of a fading environment: power efficiency is the power gain required to overcome fading, at least $P_{outage}$ fraction of the time. Mathematically, power efficiency is:

$$m = \arg_{\gamma_{max}} \left\{ \Pr \left[ Q \left( \sqrt{\frac{2\gamma_{max} E_b}{\sigma^2}} m \right) \geq Q \left( \sqrt{2 \left( 1 + \sqrt{\beta} \right)^2 \frac{E_b}{\sigma^2}} \right) \right] = P_{outage} \right\}$$  \hspace{1cm} (13)$$

where $P_{outage}$ is the desired outage probability. In general, the power efficiency will be a function of outage probability; requiring a smaller outage probability will increase the required power efficiency.

**Theorem 4.3:** For a large number of apertures, the power efficiency $m$ for a balanced sparse aperture system is the solution to the nonlinear equation:

$$\sqrt{\frac{4}{m(4 - 4/m)}} + 4 \sin^{-1}(\sqrt{1/m}) = P_{outage}^{1/n}$$

where, by definition, $m \geq 1$.

**Proof.** To prove this theorem, we start with the definition of power efficiency and solve:

$$P_{outage} = \Pr \left[ Q \left( \sqrt{\frac{2\gamma_{max} E_b}{\sigma^2}} m \right) \geq Q \left( \sqrt{\frac{8 E_b}{\sigma^2}} \right) \right]$$

$$= \Pr \left[ \gamma_{max} \leq \frac{4}{m} \right]$$

$$= F_{\gamma_{max}} \left( \frac{4}{m} \right)$$

$$= \left( \frac{\sqrt{4/m(4 - 4/m)} + 4 \sin^{-1}(\sqrt{1/m})}{2\pi} \right)^n \hspace{1cm} (14)$$

where we have used the maximum eigenvalue distribution, $F_{\gamma_{max}}(z)$, derived in Theorem 4.1. \hfill \square

As we would expect, power efficiency is not a function of SNR. This implies that the amount of power gain required to achieve turbulence-free performance does not depend on the SNR, only the number of apertures and the desired outage probability. Fig. 5 shows the numerical solution to Theorem 4.3 as a function of the number of receivers for various outage probabilities. From the figure, it is evident that increasing the number of receivers beyond 100 provides limited additional protection against fading.

While there is no closed-form solution to Theorem 4.3, an asymptotically tight upper-bound is presented in the following corollary.

**Corollary 4.2:** An upper bound on the power efficiency $m$ for a balanced sparse aperture system is:

$$m \leq \frac{1}{1 - \sqrt{1 - P_{outage}^{2/n}}}$$

**Proof.** Starting with Theorem 4.3:

$$P_{outage}^{1/n} = \sqrt{\frac{4}{m(4 - 4/m)} + 4 \sin^{-1}(\sqrt{1/m})}$$

$$\leq \sqrt{1 - (1/m - 1)^2}$$

where we use the inequality here, without proof. Solving for $m$ gives Corollary 4.2. \hfill \square

The power efficiency bound presented in Corollary 4.2 provides an asymptotically tight bound, which can be used to gain design insight. With the bound, it is easy to prove that the power efficiency approaches 0dB as the number of apertures gets large.

C. System Performance in the Presence of an Interferer

Any deployed system will be affected by interference. In a densely populated urban area, other systems might inadvertently couple power into the receiver. There are other situations where a hacker might couple power into the receiver and deny service. In this section, we will investigate the worst-case effects of an interferer. Because we conduct a worst-case analysis, any other interferer with similar geometry and power constraints will have a smaller negative impact on communication performance. As a result, the analysis presented in this section provides a bound on performance impairment from an interferer.

To conduct a worst-case analysis, we will assume the interferer has instantaneous knowledge of channel state, i.e. knowledge of the channel eigenmodes with their associated...
eigenvalues. Further, we will assume the interferer has knowledge of the modulation scheme and is synchronized with the transmitter.

In this section, we will again choose BPSK as the modulation scheme. For an optical system, transmitting using BPSK, with transmit energy in a bit period of \( E_{b,t} \) and interference energy in a bit period of \( I_{b,t} \) the average worst-case BER is:

\[
E[P(error)] = \max_{I_{b,t}} \left( \min_{p_i} \mathbb{E} \left[ Q \left( \sqrt{\frac{2\gamma_i E_{b,t}}{\sigma^2 + I_{b,i}}} \right) \right] \right)
\]  

where the expectation is over the atmospheric turbulence and the transmitter power allocation pdf. \( I_{b,i} \) is the interference energy per bit allocated to the \( i \)th eigenmode. \( p_i \) is the probability that \( E_{b,t} \) energy is allocated to the \( i \)th eigenmode. In this formulation the interferer is able to shape its waveform to couple an arbitrary, but limited, amount of energy into each eigenmode. Similarly, the transmitter is able to couple an arbitrary, but limited amount of energy into each eigenmode. By including the expectation we have assumed that the transmitter can change its spatial mode much faster than the interferer can measure the transmitter’s spatial mode and adapt. Alternatively, we have assumed that the interferer does not have knowledge of the transmitter’s power allocation scheme, which may be changed dynamically. This assumption is required for convergence. Thus the optimization can be interpreted as follows. For a given distribution of interference power, \( I_{b,i} \), the transmitter allocates power to minimize BER.

For a given average distribution of transmit power \( E_{b,t} \), the interferer allocates power to maximize BER.

The solution (15) is presented in Theorem 4.4.

**Theorem 4.4:** For the problem setup in (15), the optimal interference power allocation is:

\[
I_{b,i} = \left( \frac{\gamma_i}{\mu} - \sigma^2 \right)^+
\]

where \( \mu \) is chosen to satisfy the total power constraint:

\[
\sum_i \left( \frac{\gamma_i}{\mu} - \sigma^2 \right)^+ = I_{b,t}
\]

The optimal transmitter power allocation is then uniform:

\[
p_i = \frac{1}{|S|}, \quad \forall i \in S
\]

\[S = \left\{ i \mid i = \arg \max_k \frac{\gamma_k}{\sigma^2 + I_{b,k}} \right\}\]

where \( |S| \) is the cardinality of the set \( S \). The associated BER is then:

\[
E[P(error)] = Q \left( \sqrt{\frac{2\gamma_{\min} E_{b,t}}{\sigma^2 + I_{b,t}}} \right)
\]

**Proof.** The proof is completed by enforcing the Karush-Kuhn-Tucker conditions, but is not included here for brevity. \( \square \)

The interference power allocation stated in Theorem 4.4 is much like water-filling; we plot the values of \( \sigma^2/\gamma_i \) versus the eigenvalue index, \( i \), and imagine the line traced out as a vat which may hold water. Interference power is allocated to eigenmodes such that the water level on the graph (which represents the inverse of the signal-to-interference noise ratio) is \( 1/\mu \). Interference power is first allocated to the eigenmodes with the largest eigenvalue. As interference power is increased, it is allocated to weaker and weaker eigenmodes. Thus, as expected, the optimal weak interferer will degrade the channel by allocating its total power to the strongest eigenmode. A strong interferer will allocate power to all eigenmodes, effectively creating a channel where all non-zero eigenmodes are equal.

The transmit power hops randomly among the eigenmodes with maximum signal to interference noise power \( \gamma_i/(\sigma^2 + I_{b,i}) \). The frequency at which the transmit power hops eigenmodes is governed by the ability of the interferer to measure the transmit waveform. If the interferer can measure the waveform quickly, the transmitter must mode hop faster. In the limiting case where the interferer cannot measure the transmit waveform, the transmitter does not need to mode hop.

The Marcenko-Pastur density of eigenvalues has not been used in the formulation or proof of Theorem 4.4; in fact, the theorem is valid for even a small number of apertures. Unfortunately, \( \mu \) in Theorem 4.4 must be solved for numerically. In the case of a strong interferer with a large number of apertures however, we can evaluate BER in the presence of an interferer in closed-form.

**Theorem 4.5:** For a sparse aperture system with a large number of apertures, the expected BER in the presence of a strong interferer is:

\[
E[P(error)] = Q \left( \sqrt{\frac{2n_{\min}\beta E_{b,t}}{I_{b,t} + n_{\min}\sigma^2}} \right)
\]

Provided the interferer has sufficient total power:

\[
I_{b,t} \geq n_{\min}\sigma^2 \left( \frac{\beta}{(1 - \sqrt{\beta})^2} - 1 \right)
\]

**Proof.** To prove this theorem, we begin with the optimal interference power allocation given in Theorem 4.4 and assume the interferer has enough power to interfere with each non-zero eigenmode:

\[
I_{b,t} = \sum \left( \frac{\gamma_i}{\mu} - \sigma^2 \right)
\]

\[
= \sum \frac{\gamma_i}{\mu} - n_{\min}\sigma^2
\]

\[
= \frac{n_{\min}\beta}{\mu} - n_{\min}\sigma^2
\]

where we have used, for a large number of apertures, the fact that the number of non-zero eigenvalues converges, almost surely, to \( n_{\min} \) and the average eigenvalue converges, almost surely, to \( \beta \). Solving for \( \mu \) gives:

\[
\mu = \frac{n_{\min}\beta}{I_{b,t} + n_{\min}\sigma^2}
\]
We assumed at the outset of this proof that the interferer has enough power to interfere with each non-zero eigenmode. For this to be true, the power allocated to the minimum eigenmode must be non-negative for the $\mu$ we just derived:

$$\left(\frac{\gamma_{\text{min}}}{\mu} - \sigma^2\right) \geq 0$$

$$\left(\frac{(1 - \sqrt{\beta})^2 (I_{b,t} + n_{\text{min}} \sigma^2)}{n_{\text{min}} \beta} - \sigma^2\right) \geq 0$$

where we have used that the minimum non-zero eigenvalue is, almost surely, $(1 - \sqrt{\beta})^2$. Solving this expression for $I_{b,t}$ gives the minimum power constraint in the theorem. Finally, substituting $\mu$ into the optimal power allocation and expected BER given in Theorem 4.4 completes the proof. □

This result is intuitively satisfying. As the number of system apertures is increased, the interferer must spread its power among more spatial modes thus reducing its impact. Indeed, as the number of apertures becomes large the interferer is completely rejected. Physically, the condition on the interference power guarantees that some interference power is allocated to each non-zero eigenmode. Put into water-filling terms, the interferer has enough water to completely fill the vat.

Again, we see that $\beta = 1$ yields the best performance.

**Corollary 4.3:** For the case of the strong interferer, a balanced system, $\beta = 1$, provides the best interference rejection.

**Proof.** Its clear from the BER expression given in Theorem 4.5 that setting $\beta = 1$ gives the lowest BER. □

**V. CONCLUSION**

Optical communication over the turbulent atmosphere has the potential to provide reliable rapidly-reconfigurable multi-gigabit class physical links over tens of kilometers. Such systems, however, are prone to long (up to 100ms) and deep (10-20dB) fades and are susceptible to interference. In this paper, we have shown that a sparse aperture system with spatial mode control provides protection against fading and interference in addition to better performance (average BER) compared with a single aperture system if the transmitter and receiver are both in the atmosphere.

From a system design perspective, we found that a balanced system, with an equal number of transmit and receive apertures, provides the best performance if the propagation medium is statistically homogenous; giving the lowest average BER and providing the best protection against fading and interference. We showed that, in contrast with a single aperture system, if average BER needs improvement, the total aperture area (i.e. the sum of the sub-aperture areas) can be increased without saturation. Either adding additional apertures of the same size, or increasing the area of existing apertures, up to the coherence area, can increase the total aperture area. Interference rejection or outage performance can be improved by adding additional apertures. Finally, we showed that the protection against fading in terms of power efficiency, provided by increasing the number of apertures, diminishes greatly after about 100 apertures. These significant performance gains result from spatial mode control.

Optical communication generally provides such high data rates that the added complexity involved in implementing a system that communicates over multiple spatial modes simultaneously is not typically justified by the added rate. As such, this paper has focused on metrics related to communicating on one spatial mode at any given instant, such as BER and outage probability. The results presented for expected BER and expected BER in the presence of an interferer can be easily extended to permit the use of expected channel capacity as the performance metric. Closed form results for outage probability defined in terms of channel capacity have yet to be discovered.

**ACKNOWLEDGMENT**

This work was supported by the Defense Advanced Research Projects Agency (DARPA) under the Technology for Agile Coherent Transmission Architecture (TACOTA) program and by the Office of Naval Research (ONR) under the Free Space Heterodyne Communications Program.

**REFERENCES**