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Topological superconductors as nonrelativistic limits of Jackiw-Rossi and Jackiw-Rebbi models

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We argue that the nonrelativistic Hamiltonian of p_x+ip_y superconductor in two dimensions can be derived from the relativistic Jackiw-Rossi model by taking the limit of large Zeeman magnetic field and chemical potential. In particular, the existence of a fermion zero mode bound to a vortex in the p_x+ip_y superconductor can be understood as a remnant of that in the Jackiw-Rossi model. In three dimensions, the nonrelativistic limit of the Jackiw-Rebbi model leads to a "$p+is$" superconductor in which spin-triplet p-wave and spin-singlet s-wave pairings coexist. The resulting Hamiltonian supports a fermion zero mode when the pairing gaps form a hedgehoglike structure. Our findings provide a unified view of fermion zero modes in relativistic (Dirac-type) and nonrelativistic (Schrödinger-type) superconductors.

I. INTRODUCTION

Fermion zero modes bound to topological defects have been discovered by Jackiw and Rebbi in 1976 (Ref. [1]) and recently received renewed interest in condensed matter physics (see, for example, Ref. [2]). Vortices in a certain class of superconductors in two dimensions (2D) support zero-energy Majorana bound states and obey non-Abelian statistics,[3] which can be potentially used for topological quantum computation.[4] Although vortices in the ordinary nonrelativistic s-wave superconductor do not support Majorana zero modes, the weakly paired phase of the p_x+ip_y superconductor, which is believed to be realized in Sr_2RuO_4,[5] does support Majorana zero modes bound to vortex cores.[3,6,7]

It is also known from the pioneering work by Jackiw and Rossi that the relativistic s-wave superconductor in 2D (Jackiw-Rossi model) has similar properties.[8] Remarkably it has been shown that such a system can be realized on the surface of the three-dimensional (3D) topological insulator in contact with the s-wave superconductor.[9] Besides these examples, there is a number of proposals to realize Majorana zero modes using heterostructures of semiconductor and superconductor,[10–12] superconductor and ferromagnet,[13] and quantum (anomalous) Hall state and superconductor.[14]

Although the nonrelativistic p_x+ip_y superconductor and the relativistic Jackiw-Rossi model share similar properties, the existence of a fermion zero mode bound to a vortex has been discussed separately in the two systems.[8,9,15–19] In this paper (Sec. II), we argue that they are actually linked by showing that the former Hamiltonian can be derived from the latter by taking the limit of large Zeeman magnetic field and chemical potential. In particular, the fermion zero mode bound to a vortex persists under taking this limit.

Then in Sec. III, we turn to the relativistic Jackiw-Rebbi model in 3D, which is known to exhibit a fermion zero mode associated with a pointlike topological defect (hedgehog).[1,18,20,21] The limit of large mass and chemical potential (nonrelativistic limit) leads to a "$p+is$" superconductor in which spin-triplet p-wave and spin-singlet s-wave pairings coexist. We show that the resulting nonrelativistic Hamiltonian supports a fermion zero mode when the pairing gaps form a hedgehoglike structure.

We note that the analysis presented in this paper is largely motivated by the recent paper by Silaev and Volovik:[22] The nonrelativistic Hamiltonian of the Balian-Werthamer (BW) state of the superfluid ^{3}He was derived from the relativistic superconductor with the odd parity pairing[23] and their topological properties were studied. In this paper, we shall broadly use "relativistic" to indicate Dirac-type Hamiltonians and "nonrelativistic" to indicate Schrödinger-type Hamiltonians. For readers' convenience, references to the main results are summarized in Table I.

II. JACKIW-ROSSI MODEL IN 2D AND ITS NONRELATIVISTIC LIMIT

A. Jackiw-Rossi model and fermion zero mode at a vortex

We start with the Hamiltonian describing 2D Dirac fermions coupled with an s-wave pairing gap (Jackiw-Rossi[8] or Fu-Kane[9] model)

$$H = \frac{1}{2}\int d\mathbf{x}\,\Psi^\dagger\mathcal{H}\Psi \tag{1}$$

with $\Psi^\dagger=(\psi^\dagger,-i\psi^T\sigma_2)$ and

TABLE I. References to the equations in which the Hamiltonian, zero-energy solution bound to a defect, and its normalizability condition are shown for the relativistic model and its nonrelativistic descendant both in 2D and 3D.

	Hamiltonian	Solution	Normalizability
2D relativistic	(2)	(6)	(7)
2D nonrelativistic	(13)	(17)	(19)
3D relativistic	(29)	(36) or (38)	(39)
3D nonrelativistic	(46)	(50) or (51)	(53)

$$\mathcal{H} = \begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} + \sigma_z h - \mu & \Delta \\ \Delta^* & -\boldsymbol{\sigma} \cdot \boldsymbol{p} + \sigma_z h + \mu \end{pmatrix}. \quad (2)$$

This Hamiltonian can be realized on the surface of the 3D topological insulator in contact with the s-wave superconductor.[9] h is the Zeeman magnetic field and μ is the chemical potential. When the pairing gap Δ is spatially dependent, $\boldsymbol{p} \equiv (p_x, p_y)$ has to be regarded as derivative operators $(-i\partial_x, -i\partial_y)$. The energy eigenvalue problem is

$$\varepsilon \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix}. \quad (3)$$

When h and μ are both zero, the number of fermion zero modes ($\varepsilon = 0$) bound to a vortex formed by $\Delta(x,y) \equiv \Delta_1 + i\Delta_2$ is determined by the winding number of the two scalar fields[8,15,18]

$$\text{Index } \mathcal{H} = \frac{1}{2\pi} \int dl_i \epsilon_{ab} \hat{\Delta}_a \partial_i \hat{\Delta}_b \equiv N_w, \quad (4)$$

where $\hat{\Delta}_a \equiv \Delta_a / \sqrt{\Delta_1^2 + \Delta_2^2}$ and the line integral is taken at spatial infinity. However, in the presence of h and μ, the index theorem is no longer valid: h and μ terms in the Hamiltonian can couple zero modes and they become nonzero energy states so that two states form a pair with opposite energies. Therefore, in general, only one zero mode survives for odd N_w while no zero mode survives for even N_w.[2,18,24]

If we work in polar coordinates (r, θ) with the gap function given by the vortex form

$$\Delta(x,y) = |\Delta(r)| e^{in\theta} \text{ with } |\Delta(\infty)| > 0, \quad (5)$$

it is easy to find the explicit zero-energy solution for odd $N_w = n$ (Ref. 25)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left[\begin{array}{c} \sqrt{\mu + h} J_l(\sqrt{\mu^2 - h^2}\, r) e^{-(\pi/4)i} \\ \sqrt{\mu - h} J_{l+1}(\sqrt{\mu^2 - h^2}\, r) e^{(\pi/4)i + i\theta} \end{array} \right] \times e^{il\theta - \int^r dr' |\Delta(r')|} \quad (6)$$

and $v_1 = -u_1^*$, $v_2 = u_2^*$ with an integer $l \equiv (n-1)/2$. We note that the zero-energy solution, Eq. (6), is normalizable as long as

$$\mu^2 + |\Delta(\infty)|^2 > h^2 \quad (7)$$

is satisfied and there is a topological phase transition at $\mu^2 + |\Delta|^2 = h^2$ [see also Eq. (24) below].

B. Derivation of $p_x + ip_y$ superconductor and fermion zero mode

We now derive a clear connection between the Jackiw-Rossi model and the nonrelativistic $p_x + ip_y$ superconductor. Suppose we are interested in the low-energy spectrum of Hamiltonian (2) in the limit where both $h > 0$ and $\mu > 0$ are equally large

$$\varepsilon, \left| \sqrt{\mu^2 + |\Delta|^2} - h \right| \ll h \sim \mu. \quad (8)$$

The low-energy spectrum in such a limit can be obtained by eliminating small components u_2 and v_2.[26] Substituting the following two equations from Eq. (3):

$$(\varepsilon + h + \mu)u_2 = p_+ u_1 + \Delta v_1$$

$$(\varepsilon - h - \mu)v_2 = -p_- v_1 + \Delta^* u_1 \quad (9)$$

into the remaining two equations, we obtain

$$(\varepsilon - h + \mu)u_1 = \frac{p^2 u_1 + p_- \Delta v_1}{\varepsilon + h + \mu} + \frac{-\Delta p_- v_1 + |\Delta|^2 u_1}{\varepsilon - h - \mu}$$

$$(\varepsilon + h - \mu)v_1 = \frac{p^2 v_1 - p_+ \Delta^* u_1}{\varepsilon - h - \mu} + \frac{\Delta^* p_+ u_1 + |\Delta|^2 v_1}{\varepsilon + h + \mu}. \quad (10)$$

Here we introduced $p_\pm \equiv p_x \pm i p_y$.

In the limit under consideration, Eq. (8), we can neglect ε compared to $h + \mu$ and approximate $\sqrt{\mu^2 + |\Delta|^2}$ by h. The remaining components u_1 and v_1 obey the new energy eigenvalue problem

$$\varepsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \dfrac{p^2}{2m} - \mu_{nr} & \dfrac{1}{2}\{p_-, \Delta_{nr}\} \\ \dfrac{1}{2}\{p_+, \Delta_{nr}^*\} & -\dfrac{p^2}{2m} + \mu_{nr} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad (11)$$

where we defined the nonrelativistic mass, chemical potential, and pairing gap as

$$m \equiv h, \quad \mu_{nr} \equiv \sqrt{\mu^2 + |\Delta|^2} - h, \quad \Delta_{nr} \equiv \frac{\Delta}{h}. \quad (12)$$

The resulting Hamiltonian

$$\mathcal{H}_{nr} = \begin{pmatrix} \dfrac{p^2}{2m} - \mu_{nr} & \dfrac{1}{2}\{p_-, \Delta_{nr}\} \\ \dfrac{1}{2}\{p_+, \Delta_{nr}^*\} & -\dfrac{p^2}{2m} + \mu_{nr} \end{pmatrix} \quad (13)$$

describes the nonrelativistic $p_x + ip_y$ superconductor. We note that when $h < 0$, one obtains the Hamiltonian of the $p_x - ip_y$ superconductor where p_+ and p_- are exchanged in Eq. (13).

The first nontrivial check of this correspondence is the comparison of spectrum in a uniform space where Δ is constant. The relativistic Hamiltonian (2) has the energy eigenvalues

$$\varepsilon^2 = p^2 + h^2 + \mu^2 + |\Delta|^2 \pm 2\sqrt{p^2\mu^2 + h^2(\mu^2 + |\Delta|^2)}. \quad (14)$$

Its low-energy branch (lower sign) at small p is correctly reproduced by the energy eigenvalue of the nonrelativistic Hamiltonian (13)

$$\varepsilon_{nr}^2 = \left(\frac{p^2}{2m} - \mu_{nr} \right)^2 + p^2 |\Delta_{nr}|^2 \quad (15)$$

under the assumptions in Eq. (8).

Because the above "nonrelativistic limit" does not rely on the spatial independence of Δ, the fermion zero mode found

TABLE II. Properties under the time-reversal operator $\mathcal{T}$. τ-matrices act on the particle-hole space and $\circ$ ($\times$) indicates even (odd) under $\mathcal{T}$. Replacement of τ_0 by τ_3 exchanges the roles of Δ_1 and Δ_2.

$\mathcal{T}$	$\mathcal{T}^T/\mathcal{T}$	h	μ	Δ_1	Δ_2
$\sigma_2 \otimes \tau_0$	-1	$\times$	$\circ$	$\circ$	$\times$
$\sigma_1 \otimes \tau_1$	$+1$	$\times$	$\times$	$\circ$	$\circ$

in Eq. (6) persists into the p_x+ip_y superconductor, Eq. (13). In order to demonstrate it, we consider the simplified vortex configuration with a constant $|\Delta_{\mathrm{nr}}|>0$

$$\Delta_{\mathrm{nr}}(x,y) = e^{in\theta}|\Delta_{\mathrm{nr}}|. \qquad (16)$$

When n is odd, we can find the explicit zero-energy solution ($\varepsilon=0$) to Eq. (11) (Ref. 17)

$$u_1 = J_l[\sqrt{2m\mu_{\mathrm{nr}} - (m|\Delta_{\mathrm{nr}}|)^2}r]e^{-(\pi/4)i+il\theta - m|\Delta_{\mathrm{nr}}|r} \qquad (17)$$

and $v_1 = -u_1^*$. One can see that this zero-energy solution is the direct consequence of that in Eq. (6) because Eqs. (8) and (12) lead to

$$\mu^2 - h^2 = (\mu+h)(\mu-h) \approx 2m\left(\mu_{\mathrm{nr}} - \frac{m|\Delta_{\mathrm{nr}}|^2}{2}\right). \quad (18)$$

Thus we have established that the existence of a fermion zero mode bound to a vortex in the p_x+ip_y superconductor, Eq. (13), is a remnant of that in the Jackiw-Rossi model, Eq. (2). In particular, the condition for the normalizability of the zero-energy solution, Eq. (7), is translated into

$$\mu_{\mathrm{nr}} > 0 \qquad (19)$$

which coincides with the well-known topological phase transition in the p_x+ip_y superconductor existing at $\mu_{\mathrm{nr}}=0$ (Refs. 3, 17, and 27) [see also Eq. (25) below]. Our finding also clarifies why a vortex with winding number N_w in the p_x+ip_y superconductor cannot support $|N_w|$ zero modes in contrast to in the Jackiw-Rossi model with $h=\mu=0$.[16,17] In order to derive the p_x+ip_y superconductor as a nonrelativistic limit of the Jackiw-Rossi model, one needs to introduce h and μ which split an even number of zero modes into positive- and negative-energy states. Therefore, only one zero mode survives for odd N_w in the p_x+ip_y superconductor.

C. Altland-Zirnbauer symmetry class (Refs. 28 and 29) and topological invariant

Finally, we note the Altland-Zirnbauer symmetry class of the Hamiltonians that we have investigated in this section. Hamiltonian (2) with spatially dependent $\Delta_{1,2} \neq 0$ has the charge conjugation symmetry

$$\mathcal{C}^{-1}\mathcal{H}\mathcal{C} = -\mathcal{H}^* \quad \text{with} \quad \mathcal{C} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \qquad (20)$$

The properties of each term under the time-reversal operator $\mathcal{T}$ ($\mathcal{T}^{-1}\mathcal{H}\mathcal{T}=\mathcal{H}^*$ at $h=\mu=\Delta_{1,2}=0$) are summarized in Table II. In particular, the so-called chiral symmetry,

$$\chi^{-1}\mathcal{H}\chi = -\mathcal{H} \quad \text{with} \quad \chi = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad (21)$$

is present only if $h=\mu=0$ and essential for the index theorem, Eq. (4). Therefore, the relativistic Hamiltonian (2) with $h,\mu \neq 0$ and thus the resulting nonrelativistic Hamiltonian (13) belong to the symmetry class D.

According to Refs. 30 and 31, the class D Hamiltonians defined in compact 2D momentum spaces can be classified by an integer-valued topological invariant, which is the first Chern number[32–34]

$$C_1 \equiv \frac{-i}{2\pi} \int d\boldsymbol{p} \left(\frac{\partial a_y}{\partial p_x} - \frac{\partial a_x}{\partial p_y} \right) \qquad (22)$$

with

$$a_i(\boldsymbol{p}) \equiv \sum_{\varepsilon_a<0} \langle \varepsilon_a, \boldsymbol{p} | \frac{\partial}{\partial p_i} | \varepsilon_a, \boldsymbol{p} \rangle. \qquad (23)$$

We shall use Eqs. (22) and (23) as a definition of the topological invariant C_1 even for relativistic (Dirac-type) Hamiltonians while C_1 in this case can be a half integer. However, for superconductors, C_1 is always an integer because of the Nambu-Gor'kov doubling. We find that the topological invariant for the relativistic Hamiltonian (2) is given by

$$C_1 = \begin{cases} 0 & \text{for } \mu^2 + |\Delta|^2 > h^2 \\ -\operatorname{sgn}(h) & \text{for } \mu^2 + |\Delta|^2 < h^2 \end{cases} \qquad (24)$$

while the topological invariant for the nonrelativistic Hamiltonian (13) is given by

$$C_1 = \begin{cases} 1 & \text{for } \mu_{\mathrm{nr}} > 0 \\ 0 & \text{for } \mu_{\mathrm{nr}} < 0. \end{cases} \qquad (25)$$

Therefore, in general, the topological invariant of the momentum space Hamiltonian is not preserved by the nonrelativistic limit.[35]

Nevertheless, both values of C_1 computed for the relativistic and nonrelativistic Hamiltonians are consistent with recent conjectures relating the topological invariant of a momentum space Hamiltonian to the number of fermion zero modes bound to a vortex.[2,36] For class D superconductors defined in compact momentum spaces (as is the case for nonrelativistic Hamiltonians), Teo and Kane in Ref. 2 conjecture that the number of fermion zero modes is

$$\nu = C_1 N_w \bmod 2. \qquad (26)$$

This formula gives $\nu=1$ for $\mu_{\mathrm{nr}}>0$ and $\nu=0$ for $\mu_{\mathrm{nr}}<0$ for an odd winding number N_w. On the other hand, Santos *et al.* in Ref. 36 do not constrain Hamiltonians to be defined in compact momentum spaces, allowing for relativistic (Dirac-

type) Hamiltonians, and conjecture that the number of fermion zero modes is

$$\nu = (C_1 + N_f)N_w \bmod 2, \qquad (27)$$

where N_f is the number of Dirac flavors [$N_f = 1$ for the Jackiw-Rossi model, Eq. (2), and $N_f = 0$ for the $p_x + ip_y$ superconductor, Eq. (13)]. For an odd winding number N_w, their formula gives $\nu = 1$ for $\mu^2 + |\Delta|^2 > h^2$ and $\mu_{nr} > 0$ and $\nu = 0$ for $\mu^2 + |\Delta|^2 < h^2$ and $\mu_{nr} < 0$. Therefore, the conjectured counting of fermion zero modes in terms of the momentum space topological invariant works both in the relativistic and nonrelativistic Hamiltonians, even though the value of C_1 is not preserved by the nonrelativistic limit.

III. JACKIW-REBBI MODEL IN 3D AND ITS NONRELATIVISTIC LIMIT

A. Jackiw-Rebbi model and fermion zero mode at a hedgehog

In this section, we extend the above developed analysis to three dimensions. For this purpose, we consider the following Hamiltonian describing 3D Dirac fermions coupled with three real scalar fields ($\Delta \equiv \Delta_1 + i\Delta_2$ and Δ_3):

$$H = \frac{1}{2} \int d\boldsymbol{x}\, \Psi^\dagger \mathcal{H} \Psi \qquad (28)$$

with $\Psi^\dagger = (\psi^\dagger, -i\psi^T \alpha_2)$ and

$$\mathcal{H} = \begin{pmatrix} \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m - \mu - i\gamma^5 \beta \Delta_3 & \Delta \\ \Delta^* & -\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m + \mu + i\gamma^5 \beta \Delta_3 \end{pmatrix}. \qquad (29)$$

This Hamiltonian with zero mass m and zero chemical potential μ, after an appropriate unitary transformation and renamings ($\beta \leftrightarrow i\gamma^5 \beta$, $\Delta_1 \to \phi_1$, $\Delta_2 \to -\phi_2$, $\Delta_3 \to -\phi_3$), was studied initially by Jackiw and Rebbi[1] and recently by Teo and Kane[21] in the context of ordinary and topological insulators coexisting with superconductivity. When the scalar fields $\Delta_{1,2,3}$ are spatially dependent, $\boldsymbol{p} \equiv (p_x, p_y, p_z)$ has to be regarded as derivative operators ($-i\partial_x, -i\partial_y, -i\partial_z$). The energy eigenvalue problem is

$$\varepsilon \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix}, \qquad (30)$$

where $u_{1,2}$ and $v_{1,2}$ are two-component fields. Here we employ the standard representation of Dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (31)$$

and hence

$$i\gamma^5 \beta = \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix}. \qquad (32)$$

When m and μ are both zero, the number of fermion zero modes ($\varepsilon = 0$) bound to a hedgehog formed by $\Delta_{1,2,3}(x,y,z)$

is determined by the winding number of the three scalar fields[1,18,20]

$$\text{Index } \mathcal{H} = \frac{1}{8\pi} \int dS_i \epsilon_{ijk} \epsilon_{abc} \hat{\Delta}_a \partial_j \hat{\Delta}_b \partial_k \hat{\Delta}_c \equiv N_w, \qquad (33)$$

where $\hat{\Delta}_a \equiv \Delta_a / \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$ and the surface integral is taken at spatial infinity. However, in the presence of m and μ, the index theorem is no longer valid: m and μ terms in the Hamiltonian can couple zero modes and they become nonzero energy states so that two states form a pair with opposite energies. Therefore, in general, only one zero mode survives for odd N_w while no zero mode survives for even N_w.[2,18,37]

Here, instead of the symmetric hedgehog ($\Delta_i \propto \hat{x}_i$), we assume the hedgehoglike configuration in which $\Delta_{1,2}$ depend only on (x,y) and form a vortex and Δ_3 depends only on z and forms a kink. They have the same winding number but the latter has the advantage that an analytic solution can be found even with $m, \mu \neq 0$. If we work in cylindrical coordinates (r, θ, z) with the gap functions given by the forms

$$\Delta(x,y) = |\Delta(r)| e^{in\theta} \text{ with } |\Delta(\infty)| > 0 \qquad (34)$$

and

$$\Delta_3(z \to \pm\infty) \to \pm|\Delta_3|, \qquad (35)$$

it is easy to find the explicit zero-energy solution for odd $N_w = n$ (Ref. 38)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} \sqrt{\mu+m}\, J_l(\sqrt{\mu^2 - m^2}\, r) e^{-(\pi/4)i} \\ 0 \\ 0 \\ \sqrt{\mu-m}\, J_{l+1}(\sqrt{\mu^2 - m^2}\, r) e^{(\pi/4)i + i\theta} \end{bmatrix} \times e^{il\theta - \int^r dr' |\Delta(r')| - \int^z dz' \Delta_3(z')} \qquad (36)$$

and $v_1 = i\sigma_2 u_1$, $v_2 = i\sigma_2 u_2$ with an integer $l \equiv (n-1)/2$. On the other hand, when

$$\Delta_3(z \to \pm\infty) \to \mp|\Delta_3| \qquad (37)$$

with the same $\Delta(x,y)$ in Eq. (34), we have $N_w = -n$ and the zero-energy solution in Eq. (36) is replaced by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 0 \\ \sqrt{\mu+m}\, J_{l+1}(\sqrt{\mu^2 - m^2}\, r) e^{(\pi/4)i + i\theta} \\ \sqrt{\mu-m}\, J_l(\sqrt{\mu^2 - m^2}\, r) e^{-(\pi/4)i} \\ 0 \end{bmatrix} \times e^{il\theta - \int^r dr' |\Delta(r')| + \int^z dz' \Delta_3(z')}. \qquad (38)$$

We note that the zero-energy solution, Eq. (36) or (38), is normalizable as long as

$$\mu^2 + |\Delta(\infty)|^2 > m^2 \qquad (39)$$

is satisfied.

B. Derivation of $p + is$ superconductor and fermion zero mode

We now study the nonrelativistic limit of the above Jackiw-Rebbi model. Suppose we are interested in the low-

energy spectrum of Hamiltonian (29) in the limit where both $m > 0$ and $\mu > 0$ are equally large

$$\varepsilon, |\sqrt{\mu^2 + |\Delta|^2} - m| \ll m \sim \mu. \tag{40}$$

The low-energy spectrum in such a limit can be obtained by eliminating small components u_2 and v_2.[26] Substituting the following two equations from Eq. (30):

$$(\varepsilon + m + \mu)u_2 = (\boldsymbol{\sigma} \cdot \boldsymbol{p} - i\Delta_3)u_1 + \Delta v_1$$

$$(\varepsilon - m - \mu)v_2 = -(\boldsymbol{\sigma} \cdot \boldsymbol{p} + i\Delta_3)v_1 + \Delta^* u_1 \tag{41}$$

into the remaining two equations, we obtain

$$(\varepsilon - m + \mu)u_1 = \frac{[p^2 + \Delta_3^2 - \boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)]u_1 + (\boldsymbol{\sigma} \cdot \boldsymbol{p} + i\Delta_3)\Delta v_1}{\varepsilon + m + \mu}$$

$$+ \frac{-\Delta(\boldsymbol{\sigma} \cdot \boldsymbol{p} + i\Delta_3)v_1 + |\Delta|^2 u_1}{\varepsilon - m - \mu}$$

$$(\varepsilon + m - \mu)v_1 = \frac{[p^2 + \Delta_3^2 + \boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)]v_1 - (\boldsymbol{\sigma} \cdot \boldsymbol{p} - i\Delta_3)\Delta^* u_1}{\varepsilon - m - \mu}$$

$$+ \frac{\Delta^*(\boldsymbol{\sigma} \cdot \boldsymbol{p} - i\Delta_3)u_1 + |\Delta|^2 v_1}{\varepsilon + m + \mu}. \tag{42}$$

Here the derivative operator $\boldsymbol{\partial}$ in $\boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)$ acts only on Δ_3.

In the limit under consideration, Eq. (40), we can neglect ε compared to $m + \mu$ and approximate $\sqrt{\mu^2 + |\Delta|^2}$ by m. The remaining components u_1 and v_1 obey the new energy eigenvalue problem

$$\varepsilon\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{bmatrix} \dfrac{p^2}{2m} - \mu_{\mathrm{nr}} - \dfrac{\boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)}{2m} & \dfrac{1}{2}\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \Delta_t\} + i\Delta_s \\ \dfrac{1}{2}\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \Delta_t^*\} - i\Delta_s^* & -\dfrac{p^2}{2m} + \mu_{\mathrm{nr}} - \dfrac{\boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)}{2m} \end{bmatrix}$$

$$\times\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{43}$$

where we defined the nonrelativistic chemical potential as

$$\mu_{\mathrm{nr}} \equiv \sqrt{\mu^2 + |\Delta|^2} - m - \frac{\Delta_3^2}{2m} \tag{44}$$

and the spin-triplet p-wave and spin-singlet s-wave pairing gaps as

$$\Delta_t \equiv \frac{\Delta}{m} \quad \text{and} \quad \Delta_s \equiv \frac{\Delta_3\Delta}{m}. \tag{45}$$

The resulting Hamiltonian

$$\mathcal{H}_{\mathrm{nr}} = \begin{bmatrix} \dfrac{p^2}{2m} - \mu_{\mathrm{nr}} - \dfrac{\boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)}{2m} & \dfrac{1}{2}\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \Delta_t\} + i\Delta_s \\ \dfrac{1}{2}\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \Delta_t^*\} - i\Delta_s^* & -\dfrac{p^2}{2m} + \mu_{\mathrm{nr}} - \dfrac{\boldsymbol{\sigma} \cdot (\boldsymbol{\partial}\Delta_3)}{2m} \end{bmatrix} \tag{46}$$

describes the $p + is$ superconductor in which spin-triplet p-wave and spin-singlet s-wave pairings coexist. Δ_s can be

complex but its phase is locked to the phase of Δ_t [see Eq. (45)] and thus there are three independent degrees of freedom. The last term in the diagonal elements resembles the Zeeman coupling $\boldsymbol{\sigma} \cdot \boldsymbol{B}$ with "magnetic field" $B_i = -\partial_i\Delta_3/(2m)$ generated by the gradient of $\Delta_3 = \Delta_s/\Delta_t$. We note that the nonrelativistic Hamiltonian (46) in the absence of Δ_s is the BW state of the superfluid ^{3}He and studied in Ref. 22.

The first nontrivial check of this correspondence is the comparison of spectrum in a uniform space where Δ and Δ_3 are constant. The relativistic Hamiltonian (29) has the energy eigenvalues

$$\varepsilon^2 = p^2 + m^2 + \mu^2 + |\Delta|^2 + \Delta_3^2$$

$$\pm 2\sqrt{p^2\mu^2 + m^2(\mu^2 + |\Delta|^2) + \mu^2\Delta_3^2}. \tag{47}$$

Its low-energy branch (lower sign) at small p and Δ_3 is correctly reproduced by the energy eigenvalue of the nonrelativistic Hamiltonian (46)

$$\varepsilon_{\mathrm{nr}}^2 = \left(\frac{p^2}{2m} - \mu_{\mathrm{nr}}\right)^2 + p^2|\Delta_t|^2 + |\Delta_s|^2 \tag{48}$$

under the assumptions in Eq. (40).

Because the above nonrelativistic limit does not rely on the spatial independence of Δ and Δ_3, the fermion zero mode found in Eq. (36) or (38) persists into the $p + is$ superconductor, Eq. (46). In order to demonstrate it, we consider the simplified hedgehoglike configuration resulting from Eqs. (34), (35), and (37) with constant $|\Delta_t| > 0$ and $|\Delta_s| > 0$:

$$\Delta_t(x,y) = e^{in\theta}|\Delta_t| \quad \text{and} \quad \Delta_s(x,y,z) = \pm e^{in\theta}\,\mathrm{sgn}(z)|\Delta_s|. \tag{49}$$

When n is odd, we can find the explicit zero-energy solution ($\varepsilon = 0$) to Eq. (43)

$$u_1 = \left\{ \begin{array}{c} J_l\left[\sqrt{2m\mu_{\mathrm{nr}} - (m|\Delta_t|)^2 + \left|\dfrac{\Delta_s}{\Delta_t}\right|^2}\, r\right] \\ 0 \end{array} \right\}$$

$$\times e^{-(\pi/4)i + il\theta - m|\Delta_t|r - |\Delta_s/\Delta_t||z|} \tag{50}$$

corresponding to the upper sign in Eq. (49), or

$$u_1 = \left\{ \begin{array}{c} 0 \\ J_{l+1}\left[\sqrt{2m\mu_{\mathrm{nr}} - (m|\Delta_t|)^2 + \left|\dfrac{\Delta_s}{\Delta_t}\right|^2}\, r\right] \end{array} \right\}$$

$$\times e^{(\pi/4)i + i(l+1)\theta - m|\Delta_t|r - |\Delta_s/\Delta_t||z|} \tag{51}$$

corresponding to the lower sign in Eq. (49), and $v_1 = i\sigma_2 u_1^*$. One can see that this zero-energy solution is the direct consequence of that in Eq. (36) or (38) because Eqs. (40), (44), and (45) lead to

$$\mu^2 - m^2 \approx 2m\left[\mu_{\mathrm{nr}} - \frac{m|\Delta_t|^2}{2} + \frac{1}{2m}\left(\frac{\Delta_s}{\Delta_t}\right)^2\right]. \tag{52}$$

Thus we have established that the existence of a fermion zero mode bound to the hedgehoglike structure, Eq. (49), formed by Δ_t and Δ_s/Δ_t in the $p + is$ superconductor, Eq. (46), is a remnant of that in the Jackiw-Rebbi model, Eq.

TABLE III. Properties under the time-reversal operator $\mathcal{T}$. τ-matrices act on the particle-hole space and $\circ$ ($\times$) indicates even (odd) under $\mathcal{T}$. Replacement of τ_0 by τ_3 exchanges the roles of Δ_1 and Δ_2.

$\mathcal{T}$	$\mathcal{T}^T/\mathcal{T}$	μ	m	Δ_3	Δ_1	Δ_2
$\alpha_2 \otimes \tau_0$	-1	$\circ$	$\times$	$\circ$	$\circ$	$\times$
$\gamma^5 \alpha_2 \otimes \tau_0$	-1	$\circ$	$\circ$	$\times$	$\circ$	$\times$
$\beta \alpha_2 \otimes \tau_1$	$+1$	$\times$	$\times$	$\circ$	$\circ$	$\circ$
$\gamma^5 \beta \alpha_2 \otimes \tau_1$	-1	$\times$	$\circ$	$\times$	$\circ$	$\circ$

(29). In particular, the condition for the normalizability of the zero-energy solution, Eq. (39), is translated into

$$\mu_{\mathrm{nr}} + \frac{1}{2m}\left(\frac{\Delta_s}{\Delta_t}\right)^2 > 0. \tag{53}$$

C. Altland-Zirnbauer symmetry class (Refs. 28 and 29)

Finally, we note the Altland-Zirnbauer symmetry class of the Hamiltonians that we have investigated in this section. Hamiltonian (29) with spatially dependent $\Delta_{1,2,3} \neq 0$ has the charge conjugation symmetry

$$\mathcal{C}^{-1}\mathcal{H}\mathcal{C} = -\mathcal{H}^* \quad \text{with} \quad \mathcal{C} = \begin{pmatrix} 0 & -i\alpha_2 \\ i\alpha_2 & 0 \end{pmatrix}. \tag{54}$$

The properties of each term under the time-reversal operator $\mathcal{T}$ ($\mathcal{T}^{-1}\mathcal{H}\mathcal{T}=\mathcal{H}^*$ at $m=\mu=\Delta_{1,2,3}=0$) are summarized in Table III. In particular, the so-called chiral symmetry,

$$\chi^{-1}\mathcal{H}\chi = -\mathcal{H} \quad \text{with} \quad \chi = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \tag{55}$$

is present only if $m=\mu=0$ and essential for the index theorem, Eq. (33). Therefore, the relativistic Hamiltonian (29) with $m, \mu \neq 0$ and thus the resulting nonrelativistic Hamiltonian (46) belong to the symmetry class D. There is no topological classification of class D Hamiltonians in 3D momentum spaces.[30,31]

IV. SUMMARY

We have studied the nonrelativistic limit of the Jackiw-Rossi model in 2D and the Jackiw-Rebbi model in 3D, both of which are known to exhibit fermion zero modes associated with pointlike topological defects (vortex and hedgehog). We showed that the nonrelativistic limit of the 2D Jackiw-Rossi model leads to the p_x+ip_y superconductor. Because the fermion zero mode persists under taking this limit, we obtain a clear understanding of the existence of a fermion zero mode bound to a vortex in the p_x+ip_y superconductor as a remnant of that in the Jackiw-Rossi model. Similarly, the nonrelativistic limit of the 3D Jackiw-Rebbi model leads to the $p+is$ superconductor in which the spin-triplet p-wave pairing gap Δ_t and the spin-singlet s-wave pairing gap Δ_s coexist. We showed that the resulting Hamiltonian supports a fermion zero mode when Δ_t and Δ_s/Δ_t form a hedgehoglike structure. Fermion zero modes in the superconductors studied in this paper correspond to Majorana fermions and the associated pointlike defects obey non-Abelian statistics both in 2D (Refs. 3 and 7) and 3D.[21,39]

Our findings provide a unified view of Majorana zero modes in relativistic (Dirac-type) and nonrelativistic (Schrödinger-type) superconductors. It should be possible to generalize our analysis to other interesting cases and find new examples of nonrelativistic Hamiltonians, which are more common in condensed matter systems, with topological properties that descend from Dirac-type Hamiltonians, which are generally easier to analyze.

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