Name:...**Solutions**.....

Massachusetts Institute of Technology

16.07 Dynamics

Midterm Exam

October 28, 2004

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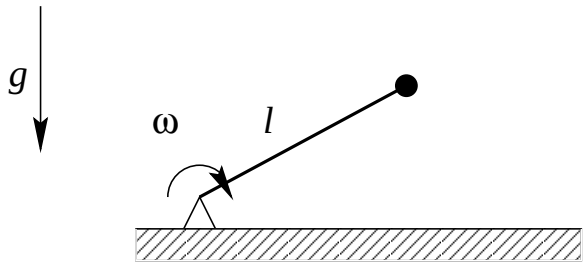
1. (10 points) A massless rod of length l with a mass attached at its end has angular velocity ω at the instant when it makes an angle θ_0 with the horizontal. What is the maximum angle the rod will make with the horizontal after the mass has rebounded? Assume an elastic collision.



Solution:
Energy conservation

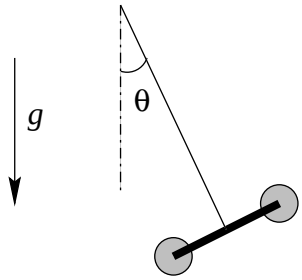
$$\frac{1}{2}m(\omega l)^2 + mg(l\sin\theta_0) = mg(l\sin\theta_{max}) \quad (1)$$

$$\sin\theta_{max} = \sin\theta + \frac{\omega^2 l}{2g} \quad (2)$$

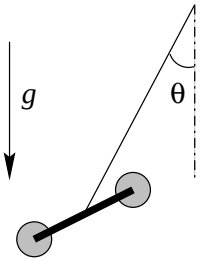
$$\theta_{max} = \sin^{-1}(\sin\theta + \frac{\omega^2 l}{2g}) \quad (3)$$

Note that the height of the pivot does not matter (the zero level for the potential energy was placed at the top of the pivot).

2. (10 points) A pendulum consists of a rigid massless rod and two equal masses. The rope is attached at the middle of the rod and at the initial instant when the pendulum is released from $\theta = 25°$, **the masses are at rest** and the rod is perpendicular to the rope. What is the angle between the rope and the rod when $\theta = -20°$? No credit will be given without explanation.

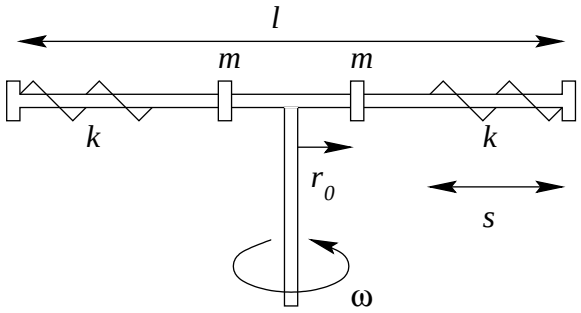


Solution:
The rod is suspended from its center of gravity and due to symmetry there is no moment acting on it. The force of gravity acting on one of th masses generates a moment which is exactly balanced by the force of gravity acting on the other mass.



A quick calculation of the angle geometry gives the angle to be 45°.

3. (10 points) The two sliders which have mass m are initially latched at distance r_0 from the axis of rotation of a horizontal rod with length l. The horizontal rod is initially rotating with angular velocity ω. The two springs at both ends have uncompressed length of s and stiffness k. What is the maximum distance from the axis of rotation which the two sliders will reach after they are released? Neglect friction effects.



Solution:
Angular momentum conservation

$$2mr_0^2\omega = 2mr^2\Omega \tag{4}$$

$$\Omega = (\frac{r_0}{r})^2\omega \tag{5}$$

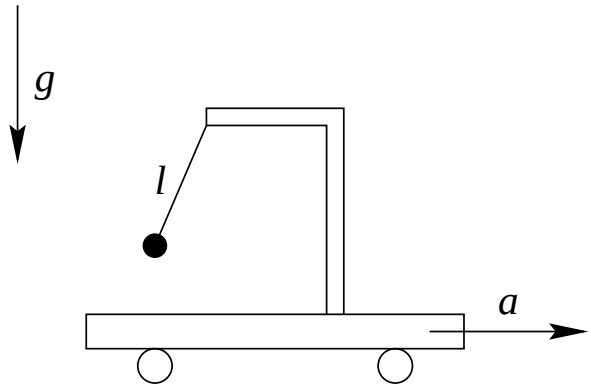
Energy conservation

$$2 \times \frac{1}{2}m(\omega r_0)^2 = 2 \times \frac{1}{2}m(\Omega r)^2 + 2k(r + s - \frac{l}{2})^2 \tag{6}$$

$$m(\omega r_0)^2 = m(\frac{\Omega}{r})^2 + 2k(r + s - \frac{l}{2})^2 \tag{7}$$

The last equation cannot be solved explicitly for r.

4. (10 points) A pendulum with length l is mounted on a cart which is accelerating on a flat road with constant acceleration a. What is the period of oscillation of the pendulum?



Solution:
Using the effective gravity concept,

$$\vec{g}_{eff} = \vec{g} - \vec{a} \tag{8}$$

$$g_{eff} = \sqrt{g^2 + a^2} \tag{9}$$

because $\vec{a}$ and $\vec{g}$ are prependicular.

The period is therefore equal to

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}} \tag{10}$$

5. (10 points) An airplane flying with velocity v is executing a horizontal turn to the left of the pilot with radius R. The propeller of the airplane is spinning counterclockwise with constant angular velocity p. Find the magnitude and the direction of the angular acceleration of the propeller.

Solution:
Using a coordinate system centered at the pilot and having its x axis aligned with the direction of propeller rotation and y axis pointing to the right of the pilot, we have:

$$\vec{\omega} = p\vec{i} + \frac{v}{R}\vec{k} \tag{11}$$

$$\vec{\Omega} = \frac{v}{R}\vec{k} \tag{12}$$

Applying Coriolis's Theorem

$$\vec{\alpha} = \dot{\vec{\omega}} = \dot{\vec{\omega}}_{xyz} + \vec{\Omega} \times \vec{\omega} \tag{13}$$

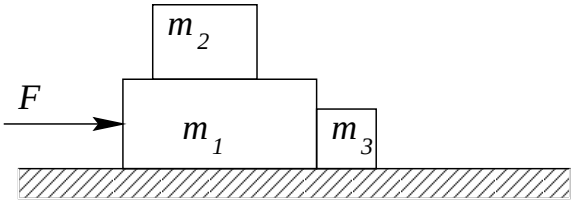
$$\vec{\alpha} = \frac{pv}{R}\vec{k} \times \vec{i} = \frac{pv}{R}\vec{j} \tag{14}$$

Therefore the magnitude of the angular acceleration is $\frac{pv}{R}$ and it points to the right of the pilot.

6. (10 points) Three blocks are positioned as shown in the figure and a force F is applied to one of them. If the coefficient of friction **between any two surfaces** is μ, answer the following two questions:

   a) What is the maximum force which can be applied without m_2 slipping over m_1?

   b) For the force calculated in a) what is the contact force between m_1 and m_3?



Solution:
The maximum force on m_2 is $F_{fr} = \mu g m_2$, therefore the maximum acceleration it can have is $a = \mu g$.

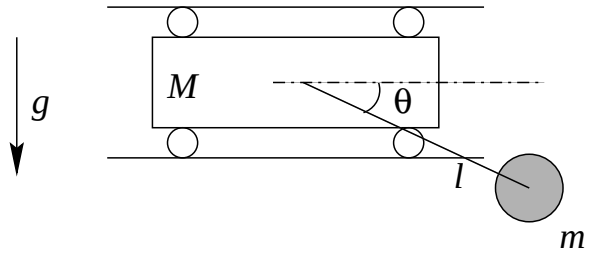
a) The maximum total force is therefore

$$\begin{aligned} F &= a(m_1 + m_2 + m_3) + (F_{fr})_{all} = \\ &= \mu g(m_1 + m_2 + m_3) + \mu g(m_1 + m_2 + m_3) = 2\mu g(m_1 + m_2 + m_3) \end{aligned}$$

b) The contact force has to counteract the friction which block m_3 feels and to provide the required acceleration of m_3, therefore

$$F_{cont} = am_3 + (F_{fr})_3 = am_3 + \mu g m_3 = 2\mu g m_3 \qquad (15)$$

7. (12 points) A ball with mass m is attached via a rigid rod to the four wheel trolley which has mass M. If the ball is released at $\theta = 0°$ what is the magnitude of the velocity of the ball relative to the trolley at the instant when $\theta = 90°$. The length of the rod is l.



Solution: Linear momentum conservation

$$Mv_M + mv_m = 0 \tag{16}$$

Energy conservation

$$mgl = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2 \tag{17}$$

Solving the two equations simultaneously for the velocities gives

$$v_m = -\sqrt{\frac{2gl}{1+\frac{m}{M}}} \tag{18}$$

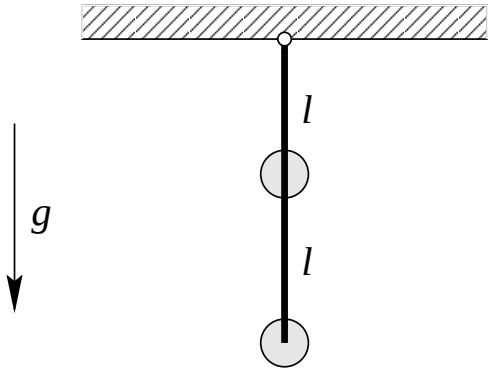
$$v_M = \frac{m}{M}\sqrt{\frac{2gl}{1+\frac{m}{M}}} \tag{19}$$

From the equation for relative velocity

$$v_{m/M} = v_m - v_M \tag{20}$$

$$v_{m/M} = \sqrt{(1+\frac{m}{M})2gl} \tag{21}$$

8. (12 points) A pendulum is constructed from two equal masses and a massless rod of total length $2l$. The rod can freely rotate about its top end. What is the period T of oscillation of the pendulum?



Solution: We shall use $\dot{H} = M$. The moment M is:

$$M = -mg(l\sin\theta) - mg(2l\sin\theta) \approx -3mgl\theta, \tag{22}$$

if θ is a small angle. The angular momentum H is found to be

$$\begin{aligned} H &= ml(l\dot{\theta}) + m2l(2l\dot{\theta}) && (23)\\ H &= 5ml^2\dot{\theta} && (24)\\ \dot{H} &= 5ml^2\ddot{\theta}. && (25) \end{aligned}$$

Therefore the equation of motion is:

$$\begin{aligned} 5ml^2\ddot{\theta} + 3mgl\theta &= 0 && (26)\\ \ddot{\theta} + \frac{3g}{5l}\theta &= 0 && (27) \end{aligned}$$

and

$$\begin{aligned} \omega &= \sqrt{\frac{3g}{5l}} && (28)\\ T &= 2\pi\sqrt{\frac{5l}{3g}} && (29) \end{aligned}$$

9. (16 points) A particle moves along a parabola $x = at,\ y = bt^2$ with $a = 1$ m/s, $b = 1$ m/s^2 and t being the time in seconds. For $t = 1$ s give the velocity and the acceleration vectors in:

   a) Cartesian coordinates

   b) Intrinsic coordinates

   c) Polar coordinates

   d) Find $\dot{\theta}$, $\ddot{\theta}$, $\dot{r}$ and $\ddot{r}$

Solution:

a)

$$\begin{aligned} v_x &= a, v_y = 2bt, a_x = 0, a_y = 2b && (30)\\ v_x(t=1) &= 1 \text{ m/s}, v_y(t=1) = 2 \text{ m/s} && (31)\\ a_x(t=1) &= 0 \text{ m/s}^2, a_y(t=1) = 2 \text{ m/s}^2 && (32) \end{aligned}$$

b)

$$\begin{aligned} v(t=1) &= \sqrt{v_x^2 + v_y^2} = \sqrt{5} \text{ m/s} && (33)\\ \vec{v} &= v\vec{e}_t + 0\vec{e}_n = \sqrt{5}\vec{e}_t + 0\vec{e}_n \text{ m/s} && (34)\\ \vec{e}_t &= (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) && (35)\\ \vec{e}_n &= (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) && (36)\\ a_t &= \vec{a} \cdot \vec{e}_t = \frac{4}{\sqrt{5}} \text{ m/s}^2 && (37)\\ a_n &= \vec{a} \cdot \vec{e}_n = \frac{2}{\sqrt{5}} \text{ m/s}^2 && (38)\\ \vec{a} &= \frac{4}{\sqrt{5}}\vec{e}_t + \frac{2}{\sqrt{5}}\vec{e}_n \text{ m/s}^2 && (39) \end{aligned}$$

c)

$$\begin{aligned} \vec{e}_r &= (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) && (40)\\ \vec{e}_\theta &= (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) && (41) \end{aligned}$$

$$v_r = \vec{v} \cdot \vec{e}_r = \frac{3}{\sqrt{2}} \text{ m/s} \quad (42)$$

$$v_\theta = \vec{v} \cdot \vec{e}_\theta = \frac{1}{\sqrt{2}} \text{ m/s} \quad (43)$$

$$\vec{v} = \frac{3}{\sqrt{2}}\vec{e}_r + \frac{1}{\sqrt{2}}\vec{e}_\theta \text{ m/s} \quad (44)$$

$$a_r = \vec{a} \cdot \vec{e}_r = \frac{2}{\sqrt{2}} \text{ m/s}^2 \quad (45)$$

$$a_\theta = \vec{a} \cdot \vec{e}_\theta = \frac{2}{\sqrt{2}} \text{ m/s}^2 \quad (46)$$

$$\vec{a} = \frac{2}{\sqrt{2}}\vec{e}_r + \frac{2}{\sqrt{2}}\vec{e}_\theta \text{ m/s}^2 \quad (47)$$

d)

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \quad (48)$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \quad (49)$$

From the previous two equations and the velocity/acceleration found earlier we can find ($r = \sqrt{2}$m):

$$\dot{r} = \frac{3}{\sqrt{2}} \text{ m/s} \quad (50)$$

$$\dot{\theta} = \frac{1}{2} \text{ rad/s} \quad (51)$$

$$\ddot{r} = \frac{5\sqrt{2}}{4} \text{ m/s}^2 \quad (52)$$

$$\ddot{\theta} = -\frac{1}{2} \text{ rad/s}^2 \quad (53)$$