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16.07 Dynamics
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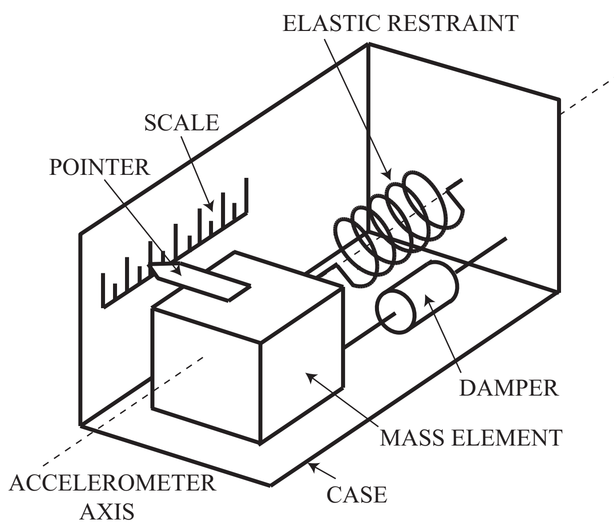
Lecture D14 - Accelerometers. Newtonian Relativity

Accelerometers

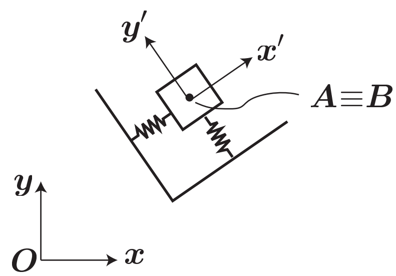
An accelerometer is a device used to measure linear acceleration without an external reference. The main idea has already been illustrated in the previous lecture with the example of the boy in the elevator. Clearly, if we know the weight of the boy when the acceleration is zero, we can determine from the reading on the scale the value of the acceleration. In summary, the acceleration will produce an inertial force on a test mass, and this force can be nulled and measured with precision. Below we have a sketch of a very simple one axis accelerometer.



In addition to the restraining spring, the accelerometer has a damper which quickly eliminates the oscillations that would otherwise occur due to rapid changes in acceleration. The force of the damper is proportional to the velocity and opposes the direction of motion, so that when the mass settles to a fixed position, the effect of the damper force disappears.

If we are able to obtain the value of the acceleration at every instant, then, given the initial conditions, two time integrations can be done in real time to yield the *position*. This is the basis of *inertial navigation*, a topic of enormous technical significance in aerospace. The problem with this simple strategy is that the test mass is also acted upon by gravity, in addition to the inertial force. The navigation problem can be resolved if we know our location exactly, and the direction in which the accelerometer's measurement axis is pointing (in practice one would use a combination of accelerometers to obtain readings in three orthogonal directions). The direction is measured independently using gyroscopes, which we will study in future lectures; the location (required to extract the value of the gravity force) is tracked by integrating the acceleration. The final ingredient is an accurate map, or model, for the gravity field as a function of position.

Let us consider an accelerometer, or system of accelerometers, capable of providing readings along two (or three, when we consider a 3D space) orthogonal directions $x'y'$.



The external force on the test mass, m, will be $\boldsymbol{F} = \boldsymbol{R} + m\boldsymbol{g}$, where $\boldsymbol{R}$ is the total force exerted by the springs, and $\boldsymbol{g}$ is the gravity acceleration vector. If the mass has an acceleration $\boldsymbol{a}$, the inertial force for an observer B moving with the mass will be simply $\boldsymbol{F}_{inertial} = -m\boldsymbol{a}$. Thus,

$$\boldsymbol{F} + \boldsymbol{F}_{inertial} = \boldsymbol{R} + m\boldsymbol{g} - m\boldsymbol{a} = \boldsymbol{0} ,$$

or,

$$\frac{\boldsymbol{R}}{m} = -(\boldsymbol{g} - \boldsymbol{a}) = (\boldsymbol{a} - \boldsymbol{g}) .$$

The quantity $(\boldsymbol{a} - \boldsymbol{g})$ is called the specific force and is the actual measurement that we will obtain from the accelerometer. Note that if we are dealing with a single axis accelerometer, with the axis parallel to the unit vector $\boldsymbol{e}_a$, then the accelerometer will only provide the component of $\boldsymbol{R}$ along that axis, i.e. $R_a = \boldsymbol{e}_a \cdot \boldsymbol{R}$. Thus, from the above equation, we have

$$\frac{R_a}{m} = \boldsymbol{e}_a \cdot (\boldsymbol{a} - \boldsymbol{g}) = (a_a - g_a) .$$

Newtonian Relativity (from Hollister)

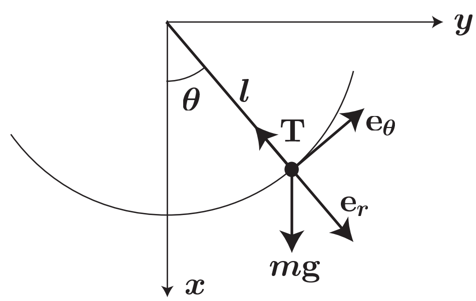
Although relativistic corrections are not necessary for most aerospace engineering applications (with the possible exception of the very large scale GPS systems), one concept of Einstein's General Relativity is helpful in understanding gravity. It is called the *Principle of Equivalence*. The concept is that, since the specific force $(\boldsymbol{a} - \boldsymbol{g})$ is the only quantity that can be measured, gravity and inertial acceleration cannot be distinguished by measurement only. Once again, this can be illustrated with the example of the boy in the elevator. If the boy did not know his weight before getting into the elevator, once inside the elevator he would not be able to tell the difference between gravitational acceleration and inertial acceleration. That is, for an accelerated observer, inertial forces and gravity appear to be of exactly the same form. It is then appropriate to think that the effect of being accelerated is equivalent to that of modifying the gravity field to an effective gravitational acceleration given by,

$$\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a} \ . \tag{1}$$

We will illustrate this idea in the example below.

Example ***Pendulum***

We now consider a simple pendulum consisting of a mass, m, suspended from a string of length l and negligible mass.



We can formulate the problem in polar coordinates, and, noting that $r = l$ (constant), write for the r and θ components,

$$\begin{aligned} mg\cos\theta - T &= -ml\dot{\theta}^2 \\ -mg\sin\theta &= ml\ddot{\theta} \ , \end{aligned} \tag{2}$$

where T is the tension on the string. If we restrict the example to the small oscillations case, then $\sin\theta \approx \theta$, and the θ-equation

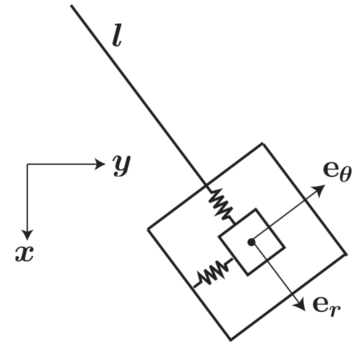
$$\ddot{\theta} + \frac{g}{l}\theta = 0 \ ,$$

can be integrated to yield

$$\theta(t) = C_1\cos(\sqrt{\frac{g}{l}}\,t) + C_2\sin(\sqrt{\frac{g}{l}}\,t) \ ,$$

where the constants C_1 and C_2 are determined by the initial conditions.

Let us now consider a non inertial observer who is rigidly attached to the oscillating mass, and try to determine the effective gravity that this observer will experience. One possible way to do this would be to install a small accelerometer capable of providing specific forces in the local r and θ directions. You could try to compute the forces that the springs exert on the test mass, using both, the inertial and the accelerated reference frames.



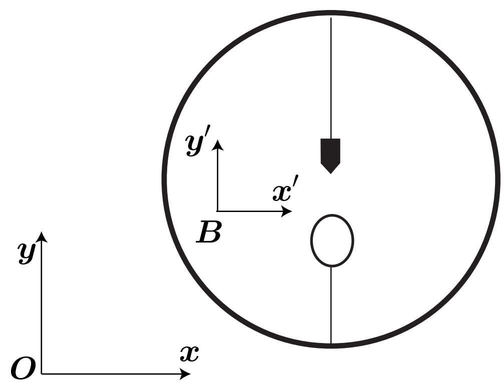
Using the expression (1) for the effective acceleration, we have,

$$\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a} = (g\cos\theta\boldsymbol{e}_r - g\sin\theta\boldsymbol{e}_\theta) - (-l\dot{\theta}^2\boldsymbol{e}_r + l\ddot{\theta}\boldsymbol{e}_\theta) = (g\cos\theta + l\dot{\theta}^2)\boldsymbol{e}_r \ ,$$

where we have used the fact that $g\sin\theta + l\ddot{\theta} = 0$, from (2). Thus, we see that the effective gravity points in the radial direction. This explains why when we swing a bottle with fluid inside, the surface of the fluid remains normal to the string.

Example ***Plumb and balloon on an aircraft***

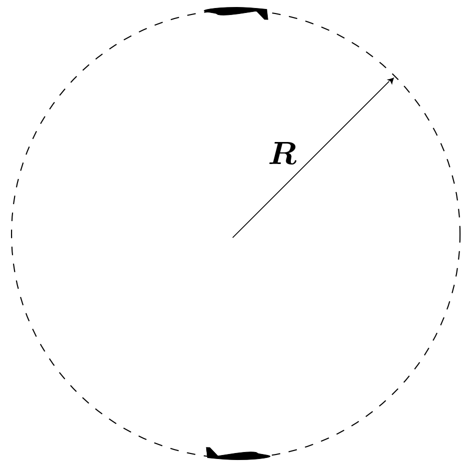
Inside an aircraft there is a plumb suspended from the ceiling and a helium filled balloon attached to the floor as shown in the sketch. When the aircraft is flying at a horizontal constant velocity, the tension in the string holding the plumb is 2 N, and the tension in the string holding the balloon is 1 N.



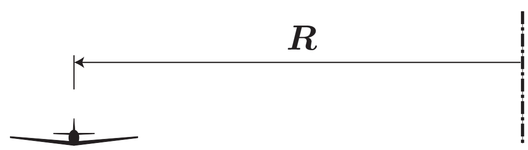
We want to calculate the angle that the strings form with the vertical and the tension in the strings when:

a) the aircraft is at the bottom of a vertical circular loop of radius R, and is flying at a constant speed of $v = \sqrt{gR}$.

b) the aircraft is at the top of a vertical circular loop of radius R, and is flying at a constant speed of $v = \sqrt{gR}$.



c) the aircraft is doing a horizontal loop of radius R, at a constant speed of $v = \sqrt{gR}$. During the loop, the aircraft is in an unbanked position (zero roll).



We will also neglect dynamic effects. That is, we will consider that the plumb and the balloon are at rest relative to the aircraft and in equilibrium under the instantaneous forces.

Solution:

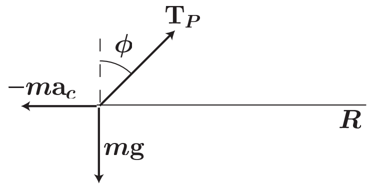
The mass of the plumb is $m_P = (2/g)$ kg. For the balloon, we have that the weight of the volume of air displaced is $\rho_{air} V_B g$, and the weight of the helium inside the balloon is $\rho_{helium} V_B g$, where V_B is the volume of the balloon. If we neglect the weight of the balloon itself, we have $-(\rho_{helium} - \rho_{air})V_B = (1/g)$ kg.

a) At the bottom of the vertical loop, the acceleration will be $a_c = v^2/R = g$. Therefore, a vertical axis accelerometer will show an acceleration of $2g$. The effective gravity will be $\boldsymbol{g}_{eff} = -2g\boldsymbol{j}$. The strings for the plumb and for the balloon will remain vertical. The tension in the plumb line will be $T_P = m_P(2g) = 4$ N, and for the balloon $T_B = -(\rho_{helium} - \rho_{air})V_B(2g) = 2$ N.

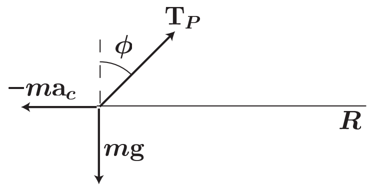
b) At the top of the loop, a vertical axis accelerometer will show a zero reading (i.e. $\boldsymbol{a} = \boldsymbol{g}$). Therefore, the effective gravity will be zero, and we will have, $T_P = 0$ and $T_B = 0$.

c) In this case, the centripetal acceleration, $a_c = v^2/R = g$, is in the horizontal direction.

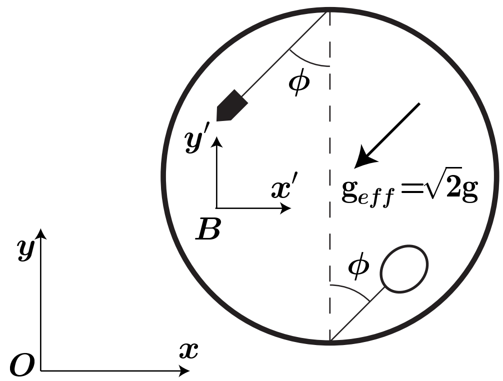
   The vector $\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a} = -\sqrt{2}g(\boldsymbol{i} + \boldsymbol{j})$ has a magnitude of $\sqrt{2}g$ and forms an angle of 45^o with the vertical. Therefore, the tension in the plumb line will be $T_P = m_P(\sqrt{2}g) = 2\sqrt{2}$ N,



and for the balloon $T_B = -(\rho_{helium} - \rho_{air})V_B(\sqrt{2}g) = \sqrt{2}$ N.



The plumb and balloon strings will form an angle of $\phi = 45^o$ with the vertical.



References

[1] W.H. Hollister, *Unified Engineering Notes*, Course 93-94.