J. Peraire
16.07 Dynamics
Fall 2004
Version 1.2

Lecture D16 - 2D Rigid Body Kinematics

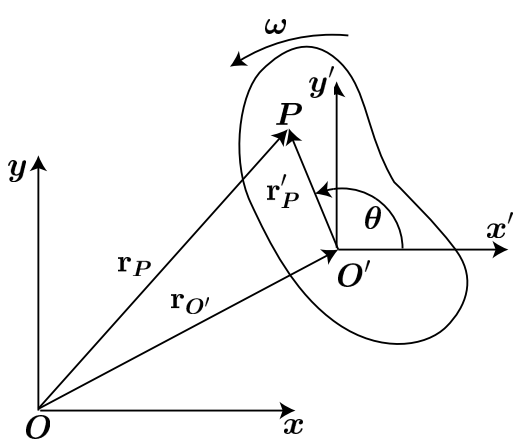
In this lecture, we will start from the general relative motion concepts introduced in lectures D11 and D12, and then apply them to describe the motion of 2D rigid bodies. We will think of a *rigid body* as a system of particles in which the distance between any two particles stays constant. The term *2-dimensional* implies that particles move in parallel planes. This includes, for instance, a planar body moving within its plane, but also a general 3D body rotating about a fixed axis.

Motion Description

Even though a rigid body is composed of an infinite number of particles, the motion of these particles is constrained to be such that the body remains a rigid body during the motion. In particular, the only degrees of freedom of a 2D rigid body are *translation* and *rotation*.

Parallel Axes

Consider a 2D rigid body which is rotating about point O', and, simultaneously, point O' is moving relative to a fixed reference frame O.



In order to determine the motion of a point P in the body, we consider a set of axes $x'y'$, parallel to xy, with origin at O', and write,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O'} + \boldsymbol{r}'_P \tag{1}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + (\boldsymbol{v}_P)_{O'} \tag{2}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + (\boldsymbol{a}_P)_{O'} \ . \tag{3}$$

Here, $\boldsymbol{r}_P$, $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$ are the position, velocity and acceleration vectors of point P, as observed by O; $\boldsymbol{r}_{O'}$ is the position vector of point O'; and $\boldsymbol{r}'_P$, $(\boldsymbol{v}_P)_{O'}$ and $(\boldsymbol{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P, as observed by O'. Relative to point O', all the points in the body describe a circular orbit ($r'_P =$ constant), and hence we can easily calculate the velocity,

$$(v_P)_{O'} = r'_P\dot{\theta} = r\omega \ ,$$

or, in vector form,

$$(\boldsymbol{v}_P)_{O'} = \boldsymbol{\omega} \times \boldsymbol{r}'_P \ ,$$

where $\boldsymbol{\omega}$ is the angular velocity vector. The acceleration has a circumferential and a radial component,

$$((a_P)_{O'})_\theta = r'_P\ddot{\theta} = r'_P\dot{\omega}, \qquad ((a_P)_{O'})_r = -r'_P\dot{\theta}^2 = -r'_P\omega^2 \ .$$

Noting that $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are perpendicular to the plane of motion (i.e. $\boldsymbol{\omega}$ can change magnitude but not direction), we can write an expression for the acceleration vector as,

$$(\boldsymbol{a}_P)_{O'} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ .$$

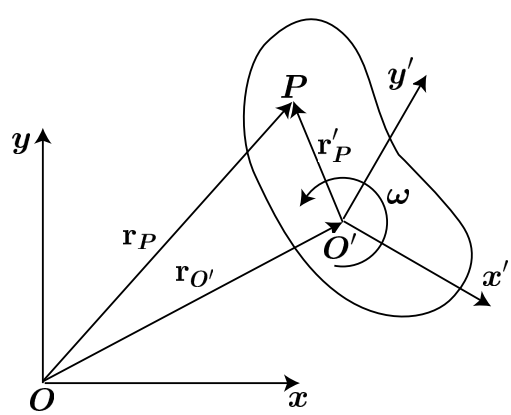
Recall here that for any three vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, we have $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$. Therefore $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) = (\boldsymbol{\omega} \cdot \boldsymbol{r}'_P)\boldsymbol{\omega} - \omega^2\boldsymbol{r}'_P = -\omega^2\boldsymbol{r}'_P$. Finally, equations 2 and 3 become,

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{4}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ . \tag{5}$$

Body Axes

An alternative description can be obtained using body axes. Now, let $x'y'$ be a set of axes which are rigidly attached to the body and have the origin at point O'.



Then, the motion of an arbitrary point P can be expressed in terms of the general expressions for relative motion derived in lectures D11 and D12. Recall that,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O'} + \boldsymbol{r}'_P \tag{6}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + (\boldsymbol{v}_P)_{O'} + \boldsymbol{\Omega} \times \boldsymbol{r}'_P \tag{7}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + (\boldsymbol{a}_P)_{O'} + 2\boldsymbol{\Omega} \times (\boldsymbol{v}_P)_{O'} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}'_P + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}'_P) \ . \tag{8}$$

Here, $\boldsymbol{r}_P$, $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$ are the position, velocity and acceleration vectors of point P as observed by O; $\boldsymbol{r}_{O'}$ is the position vector of point O'; $\boldsymbol{r}'_P$, $(\boldsymbol{v}_P)_{O'}$ and $(\boldsymbol{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P as observed by O'; and $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}$ are the body angular velocity and acceleration.
Since we only consider 2D motions, the angular velocity vector, $\boldsymbol{\Omega}$, and the angular acceleration vector, $\dot{\boldsymbol{\Omega}}$, do not change direction. Furthermore, because the body is rigid, the relative velocity and acceleration of any point in the body, as observed by the body axes, is zero. Thus, equations 7 and 8 simplify to,

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{9}$$

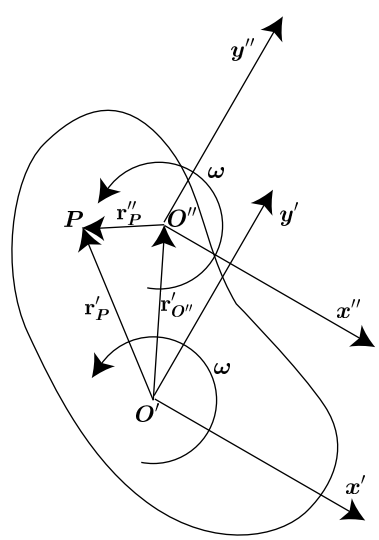
$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ , \tag{10}$$

which are identical to equations 4 and 5, as expected.

Invariance of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$

The angular velocity, $\boldsymbol{\omega}$, and the angular acceleration, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, are invariant with respect to the choice of the reference point O'. In other words, this means that an observer using parallel axes situated *anywhere* in the rigid body will observe all the other points of the body turning around, in circular paths, with the same angular velocity and acceleration. Mathematically, this can be seen by considering an arbitrary point in the body O'' and writing,

$$\boldsymbol{r}'_P = \boldsymbol{r}'_{O''} + \boldsymbol{r}''_P \ .$$



Substituting into equations 6, 9 and 10, we obtain,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O'} + \boldsymbol{r}'_{O''} + \boldsymbol{r}''_P \tag{11}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_{O''} + \boldsymbol{\omega} \times \boldsymbol{r}''_P \tag{12}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_{O''} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}''_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_{O''}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}''_P) \ . \tag{13}$$

From equations 6, 9 and 10, replacing P with O'', we have that $\boldsymbol{r}_{O''} = \boldsymbol{r}_{O'} + \boldsymbol{r}'_{O''}$, $\boldsymbol{v}_{O''} = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_{O''}$, and $\boldsymbol{a}_{O''} = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_{O''} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_{O''})$. Therefore, we can write,

$$\boldsymbol{r}_P = \boldsymbol{r}_{O''} + \boldsymbol{r}''_P \tag{14}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_{O''} + \boldsymbol{\omega} \times \boldsymbol{r}''_P \tag{15}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O''} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}''_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}''_P) \ . \tag{16}$$

These equations show that if the velocity and acceleration of point P are referred to point O'' rather than point O', then $\boldsymbol{r}'_P \neq \boldsymbol{r}''_P$, $\boldsymbol{v}_{O'} \neq \boldsymbol{v}_{O''}$, and $\boldsymbol{a}_{O'} \neq \boldsymbol{a}_{O''}$, although the angular velocity and acceleration vectors, $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, remain unchanged.

Instantaneous Center of Rotation

We have established that the motion of a solid body can be described by giving the position, velocity and acceleration of *any* point in the body, plus the angular velocity and acceleration of the body. It is clear that if we could find a point, C, in the body for which the instantaneous velocity is zero, then the velocity of the body at that particular instant would consist only of a rotation of the body about that point (no translation). If we know the angular velocity of the body, $\boldsymbol{\omega}$, and the velocity of, say, point O', then we could determine the location of a point, C, where the velocity is zero. From equation 9, we have,

$$0 = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_C.$$

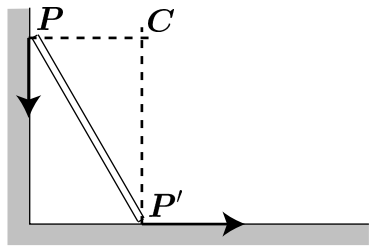
Point C is called the *instantaneous center of rotation.* Multiplying through by $\boldsymbol{\omega}$, we have $-\boldsymbol{\omega} \times \boldsymbol{v}_{O'} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_C)$, and, re-arranging terms, we obtain,

$$\boldsymbol{r}'_C = \frac{1}{\omega^2}(\boldsymbol{\omega} \times \boldsymbol{v}_{O'}) \ ,$$

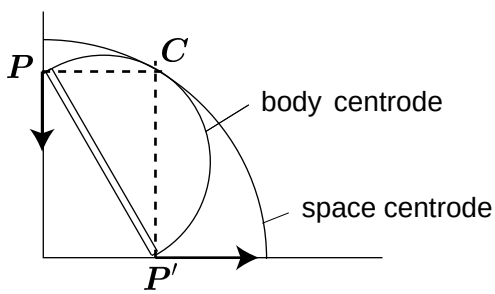
which shows that $\boldsymbol{r}'_C$ and $\boldsymbol{v}_{O'}$ are perpendicular, as we would expect if there is only rotation about C. Alternatively, if we know the velocity at two points of the body, P and P', then the location of point C can be determined geometrically as the intersection of the lines which go through points P and P' and are perpendicular to $\boldsymbol{v}_P$ and $\boldsymbol{v}_{P'}$. From the above expression, we see that when the angular velocity, ω, is very small, the center of rotation is very far away, and, in particular, when it is zero (i.e. a pure translation), the center of rotation is at infinity.

Example ***Sliding bar***

Consider a bar leaning against the wall and slipping downward. It is clear that while the bar is in contact with the wall and the floor, the velocity at point P will be in the vertical direction, whereas the velocity at point P' will be in the horizontal direction. Therefore, drawing the perpendicular lines to $\boldsymbol{v}_P$ and $\boldsymbol{v}_{P'}$ through points P and P', we can determine the instantaneous center of rotation C.



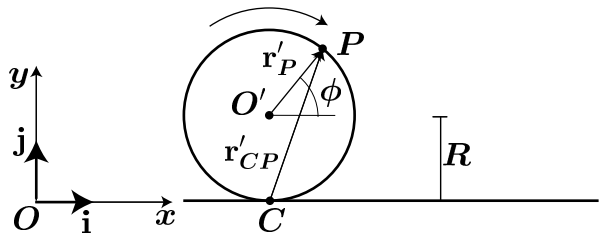
It should be noted that, for a general motion, the location of the center of rotation will change in time. The path described by the instantaneous center of rotation is called the *space centrode*, and the locus of the positions of the instantaneous centers on the body is called the *body centrode*. At a given instant, the space centrode and the body centrode curves are tangent. The tangency point is precisely the instantaneous center of rotation, C. It is not difficult to show that, for the above example, the space and body centrodes are circular arcs, assuming that the points P and P' remain in contact with the walls at all times (as you will see in the homework problem, this requires some friction).



From this example, it should be clear that although we think about the instantaneous center of rotation as a point attached to the body, it need not be a material point. In fact, it can be a point "outside" the body. It is also possible to consider the instantaneous center of acceleration as the point at which the instantaneous acceleration is zero.

Example ***Rolling Cylinder***

Consider a cylinder rolling on a flat surface, without sliding, with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. We want to determine the velocity and acceleration of point P on the cylinder. In order to illustrate the various procedures described, we will consider three different approaches.



Direct Method :

Here, we find an expression for the position of P as a function of time. Then, the velocity and acceleration are obtained by simple differentiation. Since there is no sliding, we have,

$$\boldsymbol{v}_{O'} = -\omega R\,\boldsymbol{i}, \qquad \boldsymbol{a}_{O'} = -\alpha R\,\boldsymbol{i},$$

and,

$$\boldsymbol{r}_P \;=\; \boldsymbol{r}_{O'} + R\cos\phi\,\boldsymbol{i} + R\sin\phi\,\boldsymbol{j}\ .$$

Therefore,

$$\begin{aligned}
\boldsymbol{v}_P = \dot{\boldsymbol{r}}_P &= \boldsymbol{v}_{O'} - R\dot{\phi}\sin\phi\,\boldsymbol{i} + R\dot{\phi}\cos\phi\,\boldsymbol{j} \\
&= -\omega R(1+\sin\phi)\,\boldsymbol{i} + \omega R\cos\phi\,\boldsymbol{j} \\
\boldsymbol{a}_P = \ddot{\boldsymbol{r}}_P &= \boldsymbol{a}_{O'} - R(\ddot{\phi}\sin\phi + \dot{\phi}\cos\phi)\,\boldsymbol{i} + R(\ddot{\phi}\cos\phi - \dot{\phi}\sin\phi)\,\boldsymbol{j} \\
&= [-\alpha R(1+\sin\phi) - \omega^2 R\cos\phi]\,\boldsymbol{i} + [\alpha R\cos\phi - \omega^2 R\sin\phi]\,\boldsymbol{j}
\end{aligned}$$

Relative motion with respect to O' :

Here, we can directly use either set of the expressions given previously, 4 and 5, or, 9 and 10, with,

$$\boldsymbol{v}_P \;=\; \boldsymbol{v}_{O'} + \boldsymbol{\omega}\times\boldsymbol{r}'_P \tag{17}$$

$$\boldsymbol{a}_P \;=\; \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}'_P + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}'_P)\ , \tag{18}$$

$$\boldsymbol{r}'_P = R\cos\phi\,\boldsymbol{i} + R\sin\phi\,\boldsymbol{j}\ , \quad \boldsymbol{\omega} = \omega\boldsymbol{k}, \quad \boldsymbol{\alpha} = \alpha\boldsymbol{k}\ , \quad \boldsymbol{v}_{O'} = -\omega R\,\boldsymbol{i}\ , \quad \boldsymbol{a}_{O'} = -\alpha R\,\boldsymbol{i},$$

and $\boldsymbol{k} = \boldsymbol{i}\times\boldsymbol{j}$. Thus,

$$\begin{aligned}
\boldsymbol{v}_P &= -\omega R(1+\sin\phi)\,\boldsymbol{i} + \omega R\cos\phi\,\boldsymbol{j} \\
\boldsymbol{a}_P &= [-\alpha R(1+\sin\phi) - \omega^2 R\cos\phi]\,\boldsymbol{i} + [\alpha R\cos\phi - \omega^2 R\sin\phi]\,\boldsymbol{j}\ .
\end{aligned}$$

Relative motion with respect to C :

Here, we use expressions 4 and 5, or 9 and 10, with O' replaced by C.

$$\boldsymbol{v}_P \;=\; \boldsymbol{\omega}\times\boldsymbol{r}'_{CP} \tag{19}$$

$$\boldsymbol{a}_P \;=\; \boldsymbol{a}_C + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}'_{CP} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}'_{CP})\ . \tag{20}$$

In the above expressions, we have already used the fact that $\boldsymbol{v}_C = \mathbf{0}$. Now,

$$\boldsymbol{r}'_{CP} = R\cos\phi\,\boldsymbol{i} + R(1+\sin\phi)\,\boldsymbol{j}\ , \qquad \boldsymbol{\omega} = \omega\boldsymbol{k}\ ,$$

and,

$$\boldsymbol{v}_P = -\omega R(1+\sin\phi)\,\boldsymbol{i} + \omega R\cos\phi\,\boldsymbol{j}\ .$$

The calculation of $\boldsymbol{a}_P$, in this case, requires knowing $\boldsymbol{a}_C$. In the no sliding case, $\boldsymbol{a}_C$ can be shown to be equal to $R\omega^2\,\boldsymbol{j}$, i.e., it only has a vertical component. With this, after some algebra, we obtain,

$$\boldsymbol{a}_P = [-\alpha R(1+\sin\phi) - \omega^2 R\cos\phi]\,\boldsymbol{i} + [\alpha R\cos\phi - \omega^2 R\sin\phi]\,\boldsymbol{j}\ .$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

5/1, 5/2, 5/3, 5/4 (review) , 5/5, 5/6 (review)