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16.07 Dynamics
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Lecture D22 - 3D Rigid Body Kinematics

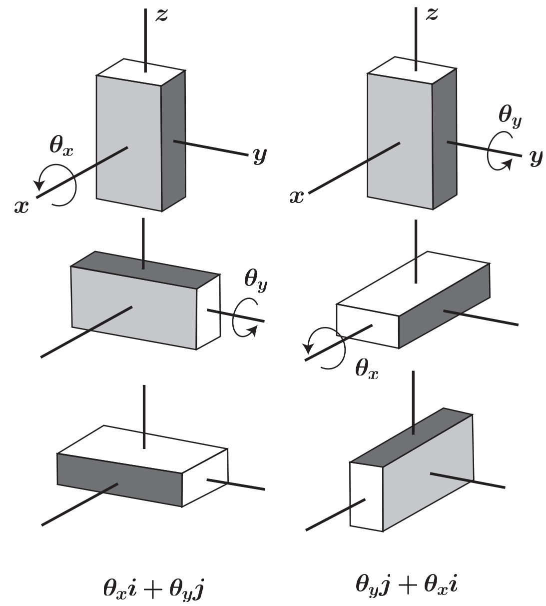
In this lecture, we consider the motion of a 3D rigid body. We shall see that in the general three dimensional case, the angular velocity of the body can change in magnitude as well as in direction, and, as a consequence, the motion is considerably more complicated than that in two dimensions.

Rotation About a Fixed Point

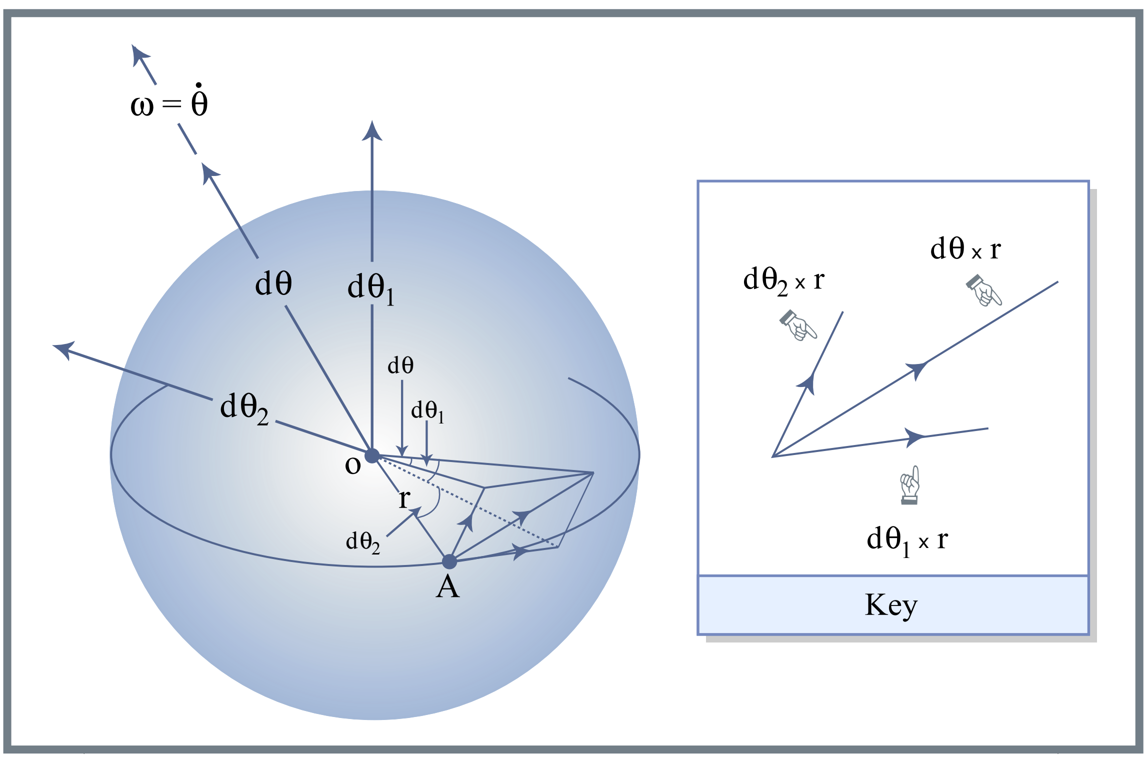
We consider first the simplified situation in which the 3D body moves in such a way that there is always a point, O, which is fixed. It is clear that, in this case, the path of any point in the rigid body which is at a distance r from O will be on a sphere of radius r that is centered at O. We point out that the fixed point O is not necessarily a point in rigid body (the second example in this notes illustrates this point).

Euler's theorem states that the general displacement of a rigid body, with one fixed point is a rotation about some axis. This means that any two rotations of arbitrary magnitude about different axes can always be combined into a single rotation about some axis.

At first sight, it seems that we should be able to express a rotation as a vector which has a direction along the axis of rotation and a magnitude that is equal to the angle of rotation. Unfortunately, if we consider two such rotation vectors, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, not only would the combined rotation $\boldsymbol{\theta}$ be different from $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$, but in general $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$. This situation is illustrated in the figure below, in which we consider a 3D rigid body undergoing two 90^o rotations about the x and y axis. It is clear that the result of applying the rotation in x first and then in y is different from the result obtained by rotating first in y and then in x. Therefore, it is clear that *finite rotations* cannot be treated as vectors, since they do not satisfy simple vector operations such as the parallelogram vector addition law.



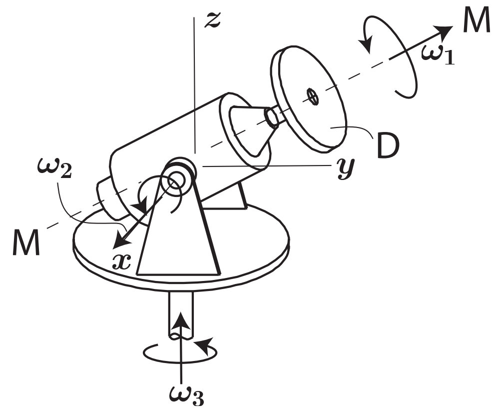
On the other hand, if we consider *infinitesimal rotations* only, it is not difficult to verify that they do indeed behave as vectors. This is illustrated in the figure below, which considers the effect of two combined infinitesimal rotations, $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$, on point A.



(Figure adapted from: Meriam, J. L., and K. L. Kraige. *Dyanmics*. 5th edition, Wiley.)

As a result of $d\boldsymbol{\theta}_1$, point A has a displacement $d\boldsymbol{\theta}_1 \times \boldsymbol{r}$, and, as a result of $d\boldsymbol{\theta}_2$, point A has a displacement $d\boldsymbol{\theta}_2 \times \boldsymbol{r}$. The total displacement of point A can then be obtained as $d\boldsymbol{\theta} \times \boldsymbol{r}$, where $d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$. Therefore, it follows that angular velocities $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$ can be added vectorially to give $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. This means that if at any instant the body is rotating about a given axis with angular velocity $\boldsymbol{\omega}_1$ and at the same time this axis is rotating about another axis with angular velocity $\boldsymbol{\omega}_2$, the total angular velocity of the body will be simply $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

We want to determine the angular velocity of the disc D.



First, we note that the disc is rotating with angular velocity $\boldsymbol{\omega}_1$ about the axis MM'. In turn, this axis is rotating with angular velocity $\boldsymbol{\omega}_2$ about the horizontal axis, which is at this instant aligned with the x axis. At the same time, the whole assembly is rotating about the z axis with angular velocity $\boldsymbol{\omega}_3$. Therefore, the total angular velocity of the disc is

$$\boldsymbol{\omega} = \omega_2 \, \boldsymbol{i} + \omega_1 \cos\phi \, \boldsymbol{j} + (\omega_1 \sin\phi + \omega_3) \, \boldsymbol{k} \ .$$

Here, ϕ is the angle between MM' and the y axis.

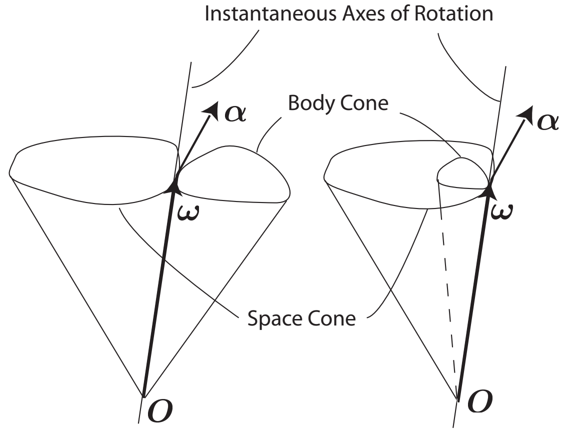
Instantaneous Axis of Rotation

Once the instantaneous angular velocity, $\boldsymbol{\omega}$, has been determined, the velocity of any point in the rigid body is simply

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \ , \tag{1}$$

where $\boldsymbol{r}$ is the position vector of the point considered with respect to the fixed point O. It follows that for any point which is on the line passing through O and parallel to $\boldsymbol{\omega}$, the velocity will be zero. This line is therefore the *Instantaneous Axis of Rotation*.

As the direction of the instantaneous axis of rotation (or the line passing through O parallel to $\boldsymbol{\omega}$) changes in space, the locus of points defined by the axis generates a fixed *Space Cone*. If the change in this axis is viewed with respect to the rotating body, the locus of the axis generates a *Body Cone*. At any given instant, these two cones are tangent along the instantaneous axis of rotation. When the body is in motion, the body cone appears to roll either on the inside or the outside of the fixed space cone. This situation is illustrated in the figure below.



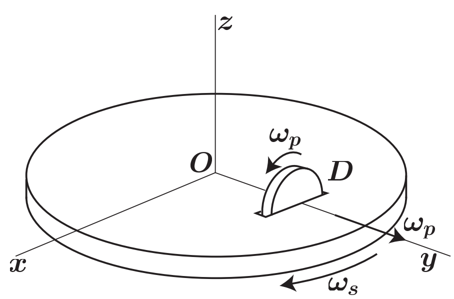
The acceleration of any point in the rigid body is obtained by taking the derivative of expression 1. Thus,

$$\boldsymbol{a} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \ . \tag{2}$$

Here, $\boldsymbol{\alpha}$ is the angular acceleration vector and is locally tangent to both the Space and the Body Cones.

Example ***Instantaneous Center of Rotation***

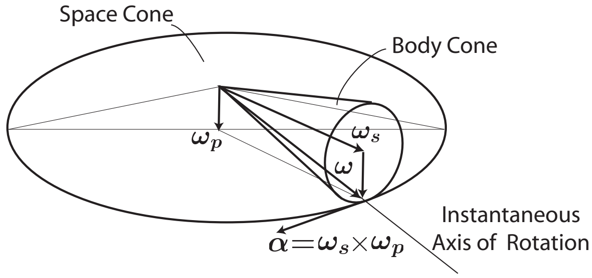
The small disc rotates with a constant angular velocity $\boldsymbol{\omega}_p$ and is mounted on the larger disc which is spinning about the vertical axis with constant angular velocity $\boldsymbol{\omega}_s$.



We note that point O is a fixed point. That is, the motion of the small disc is such that the distance from any point in the disc to the point O remains constant during the motion. The instantaneous angular velocity of the disc is given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_s = \omega_p \boldsymbol{j} - \omega_s \boldsymbol{k} \ .$$

The instantaneous axis of rotation, as well as the Space and Body Cones, are shown in the figure.



The angular acceleration can be determined using Coriolis' theorem, and is given by $\boldsymbol{\alpha} = \boldsymbol{\omega}_s \times \boldsymbol{\omega}_p$ (see lecture D12 for details).

General Motion

In the general case, the displacement of a rigid body is determined by a translation plus a rotation about some axis. This result is a generalization of Euler's theorem, which is sometimes known as Chasles' theorem. In practice, this means that six parameters are needed to define the position of a 3D rigid body. For instance, we could choose three coordinates to specify the position of the center of mass, two angles to define the axis of rotation and an additional angle to determine the magnitude of the rotation.

Unlike the motion about a fixed point, it is not always possible to define an instantaneous axis of rotation. Consider, for instance, a body which is rotating with angular velocity $\boldsymbol{\omega}$ and, at the same time, has a translational velocity parallel to $\boldsymbol{\omega}$. It is clear that, in this case, all the points in the body have a non-zero velocity, and therefore an instantaneous center of rotation cannot be defined.

It turns out that, in some situations, the motion of the center of mass of a 3D rigid body can be determined independent of the orientation. Consider, for instance, the motion of an orbiting satellite in free flight. In this situation, the sum of all external forces on the satellite does not depend on the satellite's attitude, and, therefore, it is possible to determine the position without knowing the attitude. In more complex situations, however, it may be necessary to solve simultaneously for both the position of the center of mass and the attitude.

The velocity, $\boldsymbol{v}_P$, and acceleration, $\boldsymbol{a}_P$, of a point, P, in the rigid body can be determined if we know the velocity, $\boldsymbol{v}_{O'}$, and acceleration, $\boldsymbol{a}_{O'}$, of a point in the rigid body, O', as well as the body's angular velocity, $\boldsymbol{\omega}$, and acceleration, $\boldsymbol{\alpha}$. The corresponding expressions, given below, are particular cases of the relative motion expressions derived in lecture D12,

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{3}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \; . \tag{4}$$

Here, $\boldsymbol{r}'_P$ is the position vector of the point, P, relative to O'. We point out that the angular velocity and angular acceleration are the same for all the points in the rigid body.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

7/1, 7/2, 7/3, 7/4, 7/5, 7/6 (review)