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16.07 Dynamics
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Lecture D23 - 3D Rigid Body Kinematics: The Inertia Tensor

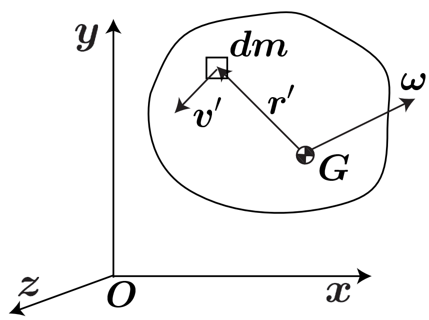
In this lecture, we will derive expressions for the angular momentum and kinetic energy of a 3D rigid body. We shall see that this introduces the concept of the Inertia Tensor.

Angular Momentum

We start form the expression of the angular momentum of a system of particles about the center of mass, $\boldsymbol{H}_G$, derived in lecture D17,

$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times \boldsymbol{v}' \, dm \ .$$

Here, $\boldsymbol{r}'$ is the position vector relative to the center of mass, $\boldsymbol{v}'$ is the velocity relative to the center of mass. We note that, in the above expression, an integral is used instead of a summation, since we are now dealing with a continuum distribution of mass.



For a 3D rigid body, the distance between any particle and the center of mass will remain constant, and the particle velocity, relative to the center of mass, will be given by

$$\boldsymbol{v}' = \boldsymbol{\omega} \times \boldsymbol{r}' \ .$$

Thus, we have,

$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times (\boldsymbol{\omega} \times \boldsymbol{r}') \, dm = \int_m [(\boldsymbol{r}' \cdot \boldsymbol{r}')\boldsymbol{\omega} - (\boldsymbol{r}' \cdot \boldsymbol{\omega})\boldsymbol{r}'] \, dm \ .$$

We note that, for planar bodies undergoing a 2D motion in its own plane, $\boldsymbol{r}'$ is perpendicular to $\boldsymbol{\omega}$, and the term $(\boldsymbol{r}' \cdot \boldsymbol{\omega})$ is zero. In this case, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{H}_G$ are always parallel. In the three dimensional case however, this simplification does not occur, and as a consequence, the angular velocity vector, $\boldsymbol{\omega}$, and the angular momentum vector, $\boldsymbol{H}_G$, are in general, *not parallel*.

In cartesian coordinates, we have, $\boldsymbol{r}' = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}$ and $\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$, and the above expression can be expanded to yield,

$$
\begin{aligned}
\boldsymbol{H}_G &= \left(\omega_x \int_m (x'^2+y'^2+z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')x'\, dm\right)\boldsymbol{i} \\
&+ \left(\omega_y \int_m (x'^2+y'^2+z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')y'\, dm\right)\boldsymbol{j} \\
&+ \left(\omega_z \int_m (x'^2+y'^2+z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')z'\, dm\right)\boldsymbol{k} \\
&= (\;\; I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} \\
&+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} \\
&+ (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}\;. \qquad (1)
\end{aligned}
$$

The quantities I_{xx}, I_{yy}, and I_{zz} are called *moments of inertia* with respect to the x, y and z axis, respectively, and are given by

$$I_{xx} = \int_m (y'^2+z'^2)\, dm\;, \qquad I_{yy} = \int_m (x'^2+z'^2)\, dm\;, \qquad I_{zz} = \int_m (x'^2+y'^2)\, dm\;.$$

We observe that the quantity in the integrand is precisely the square of the distance to the x, y and z axis, respectively. They are analogous to the moment of inertia used in the two dimensional case. It is also clear, from their expressions, that the moments of inertia are always positive. The quantities I_{xy}, I_{xz}, I_{yx}, I_{yz}, I_{zx} and I_{zy} are called *products of inertia*. They can be positive, negative, or zero, and are given by,

$$I_{xy} = I_{yx} = \int_m x'y'\, dm\;, \qquad I_{xz} = I_{zx} = \int_m x'z'\, dm\;, \qquad I_{yz} = I_{zy} = \int_m y'z'\, dm\;.$$

If we are interested in calculating the angular momentum with respect to a fixed point O then, the resulting expression would be,

$$
\begin{aligned}
\boldsymbol{H}_O &= (\;\; (I_{xx})_O\,\omega_x - (I_{xy})_O\,\omega_y - (I_{xz})_O\,\omega_z)\,\boldsymbol{i} \\
&+ (-(I_{yx})_O\,\omega_x + (I_{yy})_O\,\omega_y - (I_{yz})_O\,\omega_z)\,\boldsymbol{j} \\
&+ (-(I_{zx})_O\,\omega_x - (I_{zy})_O\,\omega_y + (I_{zz})_O\,\omega_z)\,\boldsymbol{k}\;. \qquad (2)
\end{aligned}
$$

Here, the moments of products of inertia have expressions which are analogous to those given above but with x', y' and z' replaced by x, y and z. Thus, we have that

$$(I_{xx})_O = \int_m (y^2+z^2)\, dm\;, \qquad (I_{yy})_O = \int_m (x^2+z^2)\, dm\;, \qquad (I_{zz})_O = \int_m (x^2+y^2)\, dm\;,$$

and,

$$(I_{xy})_O = (I_{yx})_O = \int_m xy\, dm\;, \qquad (I_{xz})_O = (I_{zx})_O = \int_m xz\, dm\;, \qquad (I_{yz})_O = (I_{zy})_O = \int_m yz\, dm\;.$$

The Tensor of Inertia

The expression for angular momentum given by equation 1, can be written in matrix form as,

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} ,$$

or,

$$\boldsymbol{H}_G = [I_G]\, \boldsymbol{\omega} \ ,$$

where $[I_G]$ is the tensor of inertia about the center of mass G and with respect to the xyz axes. The tensor of inertia gives us an idea about how the mass is distributed in a rigid body. It follows from the definition of the products of inertia, that the tensor of inertia is always *symmetric*. In addition, if the body has symmetries with respect to some of the axis, then some of the products of inertia become zero. For instance, if the body is symmetric with respect to the plane $x = 0$ then, we will have $I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0$. This can be shown by looking at the definition of the products of inertia. The integral for, say, I_{xy} can be decomposed into two integrals for the two halves of the body at either side of the plane $x = 0$. The integrand on one half, $x'y'$, will be equal in magnitude and opposite in sign to the integrand on the other half (because x' will change sign). Therefore, the integrals over the two halves will cancel each other and the product of inertia I_{xy} will be zero.

Another case of practical importance is when we consider axisymmetric bodies of revolution. In this case, if one of the axis coincides with the axis of symmetry, the tensor of inertia has a simple diagonal form.

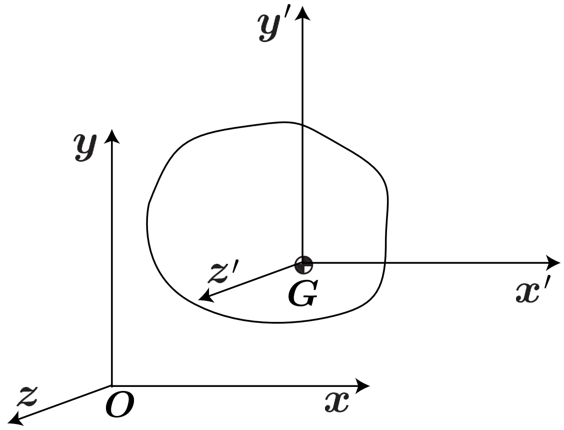
Analogously, we can define the tensor of inertia about point O, by writing expression 2 in matrix form. Thus, we have

$$\boldsymbol{H}_O = [I_O]\, \boldsymbol{\omega} \ ,$$

where the components of $[I_O]$ are the moments and products of inertia about point O given above.

Parallel Axis Theorem

It will often be easier to obtain the tensor of inertia with respect to axis passing through the center of mass. In some problems however, we will need to calculate the tensor of inertia about different axes. The parallel axis theorem introduced in lecture D18 for the two dimensional moments of inertia can be extended and applied to each of the components of the tensor of inertia.



In particular we can write,

$$
\begin{aligned}
(I_{xx})_O &= \int_m (y^2 + z^2)\, dm = \int_m ((y_G + y')^2 + (z_G + z')^2)\, dm \\
&= \int_m (y'^2 + z'^2) + 2y_G \int_m y'\, dm + 2z_G \int_m z'\, dm + (y_G^2 + z_G^2) \int_m dm \\
&= I_{xx} + m(y_G^2 + z_G^2)\ .
\end{aligned}
$$

Here, we have use the fact that y' and z' are the coordinates relative to the center of mass and therefore their integrals over the body are equal to zero. Similarly, we can write,

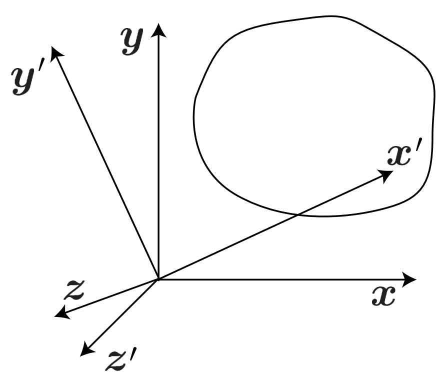
$$
(I_{yy})_O = I_{yy} + m(x_G^2 + z_G^2), \quad (I_{zz})_O = I_{zz} + m(x_G^2 + y_G^2),
$$

and,

$$
(I_{xy})_O = (I_{yx})_O = I_{xy} + m x_G y_G, \quad (I_{xz})_O = (I_{zx})_O = I_{xz} + m x_G z_G, \quad (I_{yz})_O = (I_{zy})_O = I_{yz} + m y_G z_G\ .
$$

Rotation of Axes

In some situations, we will know the tensor of inertia with respect to some axes xyz and, we will be interested in calculating the tensor of inertia with respect to another set of axis $x'y'z'$. We denote by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ the unit vectors along the direction of xyz axes, and by $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ the unit vectors along the direction of $x'y'z'$ axes.



If $[I]$ is the tensor of inertia with respect to the xyz axes (passing through either G or O), then $[I]\boldsymbol{i} = I_{xx}\boldsymbol{i} - I_{xy}\boldsymbol{j} - I_{xz}\boldsymbol{k}$, and $\boldsymbol{i} \cdot [I]\boldsymbol{i} = I_{xx}$. More generally, if $\boldsymbol{n}$ is a unit vector, $\boldsymbol{n} \cdot [I]\boldsymbol{n} = I_{nn}$, where I_{nn} is the moment of inertia about the axis defined by $\boldsymbol{n}$ and passing through the point to which $[I]$ refers. Similarly, we can dot $[I]\boldsymbol{i}$ with $\boldsymbol{j}$ to obtain $\boldsymbol{j} \cdot [I]\boldsymbol{i} = -I_{xy}$, and, more generally, $\boldsymbol{m} \cdot [I]\boldsymbol{n} = -I_{mn}$, where $\boldsymbol{m}$ and $\boldsymbol{n}$ are two orthogonal unit vectors.

Therefore, if we want to calculate the tensor of inertia with respect to axis $x'y'z'$, we will have

$$\begin{pmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{pmatrix} = \begin{pmatrix} \boldsymbol{i}' \cdot \boldsymbol{i} & \boldsymbol{j}' \cdot \boldsymbol{i} & \boldsymbol{k}' \cdot \boldsymbol{i} \\ \boldsymbol{i}' \cdot \boldsymbol{j} & \boldsymbol{j}' \cdot \boldsymbol{i} & \boldsymbol{k}' \cdot \boldsymbol{j} \\ \boldsymbol{i}' \cdot \boldsymbol{k} & \boldsymbol{i}' \cdot \boldsymbol{k} & \boldsymbol{i}' \cdot \boldsymbol{k} \end{pmatrix} \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \boldsymbol{i}' \cdot \boldsymbol{i} & \boldsymbol{i}' \cdot \boldsymbol{j} & \boldsymbol{i}' \cdot \boldsymbol{k} \\ \boldsymbol{j}' \cdot \boldsymbol{i} & \boldsymbol{j}' \cdot \boldsymbol{j} & \boldsymbol{j}' \cdot \boldsymbol{k} \\ \boldsymbol{k}' \cdot \boldsymbol{i} & \boldsymbol{k}' \cdot \boldsymbol{j} & \boldsymbol{k}' \cdot \boldsymbol{k} \end{pmatrix}.$$

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

7/7, 7/8, Appendix B

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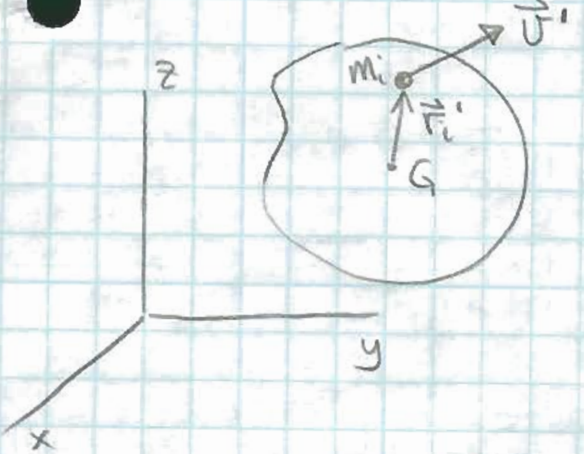
D23 – 3D RIGID BODY DYNAMICS

In lecture D17 we derived expressions for the time variation of the angular momentum of a system of particles. In particular we saw that

$$\dot{\vec{H}}_G = \vec{M}_G$$

We shall start here by particularizing this expression to 3D rigid bodies.

ANGULAR MOMENTUM AND INERTIA TENSOR



The angular momentum about G is defined as

$$\vec{H}_G = \int_m \vec{r}' \times \vec{v}\, dm = \int_m \vec{r}' \times \vec{v}'\, dm$$

Here, $\vec{r}'$ is the position vector relative to G, $\vec{v}$ is the absolute velocity and $\vec{v}'$ is the velocity relative to G. (see D17 for details).

Since we are dealing with a rigid body

$$\vec{v}' = \vec{\omega} \times \vec{r}'$$

and

$$\vec{H}_G = \int_m \vec{r}' \times (\vec{\omega} \times \vec{r}')\, dm$$

Now, we can use the vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$\vec{H}_G = \int_m [(\vec{r}' \cdot \vec{r}')\vec{\omega} - (\vec{r}' \cdot \vec{\omega})\vec{r}\,]\, dm$$

We note that in 2D, $\vec{r}'$ is perpendicular to $\vec{\omega}$ and therefore $\vec{H}_G$ is always parallel to $\vec{\omega}$. In 3D however, $\vec{H}_G$ is in general <u>not parallel</u> to $\vec{\omega}$. Expanding, we have

$$\vec{H}_G = \left[\omega_x \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')x'\, dm\right]\vec{i}$$

$$+ \left[\omega_y \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')y'\, dm\right]\vec{j}$$

$$+ \left[\omega_z \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')z'\, dm\right]\vec{k}$$

$$= (I_{xx}\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z)\,\vec{i}$$

$$(-I_{yx}\,\omega_x + I_{yy}\,\omega_y - I_{yz}\,\omega_z)\,\vec{j}$$

$$(-I_{zx}\,\omega_x - I_{zy}\,\omega_y + I_{zz}\,\omega_z)\,\vec{k}$$

Here, we have used $\vec{r}' = x'\vec{i} + y'\vec{j} + z'\vec{k}$ and $\vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$

In matrix form we have

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \text{or} \quad \vec{H}_G = \underline{\underline{I}}_G \, \vec{\omega}$$

$\underline{\underline{I}}_G$ is known as the <u>inertia tensor</u>

* I_{xx}, I_{yy}, I_{zz} are the <u>mass moments of inertia</u> about the x, y, z axis.

$$I_{xx} = \int_m (y'^2 + z'^2)\,dm \,, \quad I_{yy} = \int_m (x'^2 + z'^2)\,dm \,, \quad I_{zz} = \int (x'^2 + y'^2)\,dm$$

They are always positive - (similar to 2D moments of inertia)

* I_{xy}, I_{xz}, I_{yz} etc.. are the <u>products of inertia</u> they can be positive, negative or zero. They are given by:

$$I_{xy} = I_{yx} = \int_m x'y'\,dm \,, \quad I_{xz} = I_{zx} = \int_m x'z'\,dm \,, \quad I_{yz} = I_{zy} = \int_m y'z'\,dm$$

We see that the inertia tensor is always <u>symmetric</u>

Also, if there are symmetries with respect to a plane the corresponding products of inertia are zero. For example if a body is symmetric with respect to the plane $x = 0$

Then $I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0$ and the inertia tensor has the form

$$\underline{\underline{I_G}} = \begin{vmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & -I_{yz} \\ 0 & -I_{zy} & I_{zz} \end{vmatrix}$$

If there are two orthogonal planes of symmetry which are orthogonal to the coordinate axis, then the tensor of inertia is diagonal.

* We can <u>always</u> find a set of axis in which the inertia tensor is <u>diagonal</u>. These are called the <u>principal axes</u> of inertia. On these axes

$$\underline{\underline{I_G}} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

where we use $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$

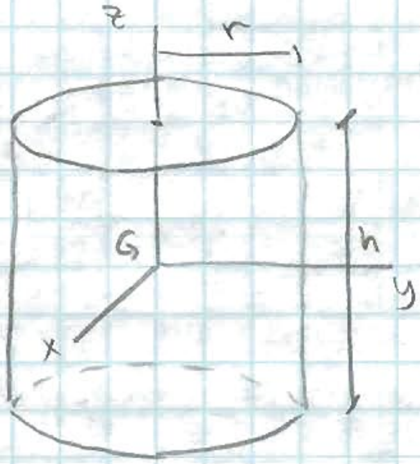
Two of the diagonal terms are the maximum and minimum moments of inertia of that body about an axis.

In principal axes

$$\vec{H}_G = I_x \omega_x \vec{i} + I_y \omega_y \vec{j} + I_z \omega_z \vec{k}$$

Examples:

- Cylinder



$$I_z = \frac{1}{2} m r^2$$

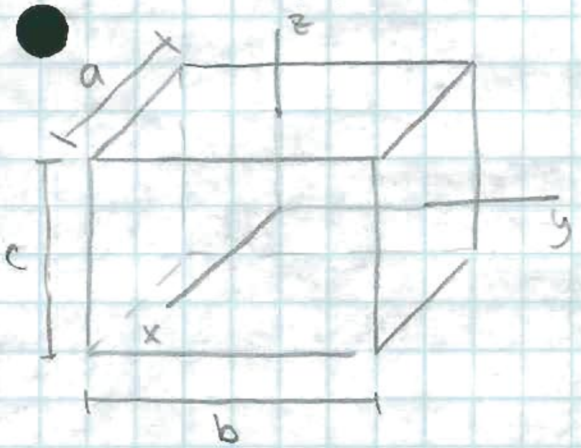
$$I_x = I_y = \frac{1}{12} m (3r^2 + h^2)$$

Disk-like $h \to 0$ $\frac{I_z}{I_x} = 2$

Rod-like $r \to 0$ $I_z = 0$, $I_x = I_y = \frac{1}{12} m h^2$

Sphere-like $h^2 = 3r^2$ $I_x = I_y = I_z$

- Parallelipiped



$$I_x = \frac{1}{12} m (b^2 + c^2)$$

$$I_y = \frac{1}{12} m (a^2 + c^2)$$

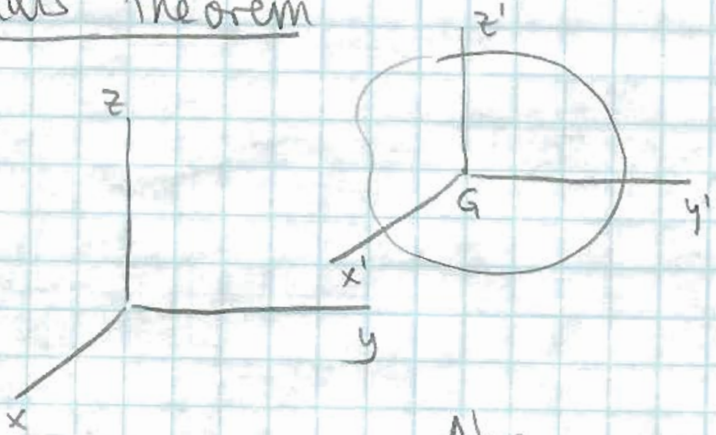
$$I_z = \frac{1}{12} m (a^2 + b^2)$$

Rod-like $a^2 + b^2 = 0$, $I_z = 0$, $I_x = I_y = \frac{1}{12} m c^2$

In order to compute the angular momentum about a fixed point O we could repeat the same process to end up with

$$\vec{H} = \underline{\underline{I}}_O \, \vec{\omega}$$

where $\underline{\underline{I}}_O$ is the tensor of inertia about O. In general the components of the tensor of inertia satisfy the <u>parallell axis theorem</u>

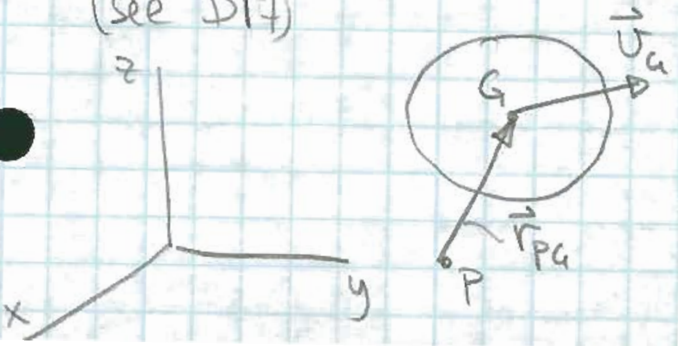


$$I_{xx} = (I_{xx})_G + m(y_G^2 + z_G^2)$$
$$I_{yy} = (I_{yy})_G + m(x_G^2 + z_G^2)$$
$$I_{zz} = (I_{zz})_G + m(x_G^2 + y_G^2)$$

Also

$$I_{xy} = (I_{xy})_G + m x_G y_G$$
$$I_{xz} = (I_{xz})_G + m x_G z_G$$
$$I_{yz} = (I_{yz})_G + m y_G z_G$$

Finally, the angular momentum of a rigid body with respect to an fixed arbitrary point P will be given by (see D17)



$$\vec{H}_P = \vec{H}_G + \vec{r}_{PG} \times m \vec{v}_G$$