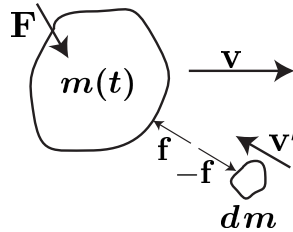


Lecture D27 - Variable Mass Systems: The Rocket Equation

In this lecture, we consider the problem of a body in which the mass of the body changes during the motion, that is, m is a function of t , i.e. $m(t)$. Although there are many cases for which this particular model is applicable, one of obvious importance to us are rockets. We shall see that a significant fraction of the mass of a rocket is the fuel, which is expelled during flight at a high velocity and thus, provides the propulsive force for the rocket.

We shall start by considering a body with velocity \mathbf{v} and external forces \mathbf{F} , gaining mass at a rate $\dot{m} = dm/dt$. Let us look at the process of gaining a small amount of mass dm . Let \mathbf{v}' be the velocity of dm before it is captured by m , and let \mathbf{f} represent the average value of the impulsive forces that dm exerts on m during the short interval dt , in which the capturing takes place. By Newton's third law, dm will experience a force $-\mathbf{f}$, exerted by m , over the same dt .



We can now examine the capture process from the point of view of dm and equate the impulse, $-\mathbf{f}dt$, to the change in linear momentum of dm ,

$$-\mathbf{f}dt = dm(\mathbf{v} + d\mathbf{v} - \mathbf{v}'). \quad (1)$$

Here, $\mathbf{v} + d\mathbf{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m , we write

$$\mathbf{F}dt + \mathbf{f}dt = m(\mathbf{v} + d\mathbf{v}) - m\mathbf{v} = m d\mathbf{v}. \quad (2)$$

The term $dm d\mathbf{v}$ in equation 1 is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining equations 1 and 2,

$$\mathbf{F}dt - dm(\mathbf{v} - \mathbf{v}') = m d\mathbf{v},$$

or, dividing through by dt ,

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F} - (\mathbf{v} - \mathbf{v}') \frac{dm}{dt} = \mathbf{F} + (\mathbf{v}' - \mathbf{v}) \frac{dm}{dt}. \quad (3)$$

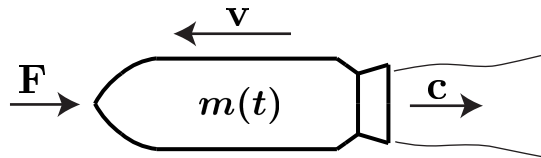
Here, $\mathbf{v}' - \mathbf{v}$ is the velocity of dm relative to m . This expression is valid when $dm/dt > 0$ (mass gain) and when $dm/dt < 0$ (mass loss). The term $(\mathbf{v}' - \mathbf{v})dm/dt$ is an additional force on m which is due to the gain (or loss) of mass. Equation 3 can also be written as

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} + \mathbf{v}' \frac{dm}{dt},$$

which shows that, for systems involving variable mass, the usual expression stating conservation of linear momentum, $d(m\mathbf{v})/dt = \mathbf{F}$, is only applicable when the initial (final) velocity of the captured (expelled) mass, \mathbf{v}' , is zero.

The Rocket Equation

We consider a rocket of mass m , moving at velocity \mathbf{v} and subject to external forces \mathbf{F} (typically gravity and drag). The rocket mass changes at a rate $\dot{m} = dm/dt$, with a velocity \mathbf{c} relative to the rocket. We shall assume that the magnitude of \mathbf{c} is constant.



The velocity of the gas observed from a stationary frame will be $\mathbf{v}' = \mathbf{v} + \mathbf{c}$. Thus,

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{c} \frac{dm}{dt}. \quad (4)$$

The term $\mathbf{T} = \mathbf{c} dm/dt$ is called the *thrust* of the rocket and can be interpreted as an additional force on the rocket due to the gas expulsion.

Equation 4 is a vector equation which can be projected along the direction of \mathbf{v} (tangent to the path). Thus,

$$m \frac{dv}{dt} = F_t - c \frac{dm}{dt} = F_t + T, \quad (5)$$

where F_t is the tangential component of \mathbf{F} , v and c are the magnitudes of \mathbf{v} and \mathbf{c} respectively, and we have assumed that \mathbf{c} is parallel and has opposite direction to \mathbf{v} . The magnitude of the thrust is $T = -c\dot{m}$. Note that for a rocket \dot{m} will be negative (mass is lost). If the force F_t is known, this equation can be integrated in time to yield an expression for the velocity as a function of time.

Let us consider some simple cases:

1) $F_t = 0$

If gravity and drag effects are neglected we have,

$$m \frac{dv}{dt} = -c \frac{dm}{dt}$$

or, integrating between an initial time t_0 , and a final time t ,

$$\Delta v = v - v_0 = -c(\ln m - \ln m_0) = -c \ln \frac{m}{m_0}. \quad (6)$$

Alternatively, this expression can be cast as the well known rocket equation,

$$m = m_0 e^{-\Delta v/c}, \quad (7)$$

which gives the mass of the rocket at a time t , as a function of the mass initial mass m_0 , Δv , and c . The mass of the propellant, $m_{propellant}$, is given by,

$$m_{propellant} = m_0 - m = m_0(1 - e^{-\Delta v/c}).$$

From the above equations, we see that for a given Δv and m_0 , increasing c increases m (payload) and decreases $m_{propellant}$. Unfortunately, we can only choose c as high as the current technology will allow. Note that if \dot{m} is constant, we can write $m(t) = m_0 + \dot{m}t$, and substitute into equation 6 to obtain an expression for v as a function of t ,

$$v = v_0 - c \ln\left(1 + \frac{\dot{m}}{m_0} t\right) .$$

Recall that according to the convention used, \dot{m} is negative as the mass decreases with time.

2) $F_t = -mg$

A constant gravitational field acting in the opposite direction to the velocity vector can be easily incorporated. In this case, equation 5 becomes,

$$m \frac{dv}{dt} = -mg - c \frac{dm}{dt},$$

which can be integrated to give

$$v = v_0 - c \ln \frac{m}{m_0} - gt = v_0 - c \ln \frac{m}{m_0} - g \frac{m - m_0}{\dot{m}}.$$

3) $F_t = -mg - D$

The effect of the drag force, D , is harder to quantify. It turns out that for many important applications drag effects are very small. The drag force is characterized in terms of a drag coefficient, C_D . Thus,

$$D = \frac{1}{2} \rho v^2 A C_D ,$$

where A is the cross-sectional area of the rocket. The air density changes with altitude z , and may be approximated by

$$\rho = \rho_0 e^{-z/H} ,$$

where $H \approx 8000$ m is the so-called “scale height” of the atmosphere, and ρ_0 is the air density at sea level. It turns out that the differential equation that results for the velocity cannot be integrated explicitly and, in practice, needs to be integrated numerically. It is interesting to note, however, that the effect of drag losses is usually quite small, and it is often reasonable to ignore it in a first calculation. In order to see the importance of D versus the the effect of gravity, we can estimate the value of the ratio D/mg . At conditions typical for maximum drag, $\rho \approx 0.25$ kg/m³ and $v = 700$ m/s. Considering a rocket of 12,000 kg with a cross section of $A = 1$ m² and $C_D = 0.2$, we have,

$$\frac{\rho AC_D v^2}{2mg} = \frac{0.25 \times 1 \times 0.2 \times 700^2}{2 \times 12,000 \times 9.8} = 0.021 ,$$

which indicates that the drag force is only about 2% of the gravity force.

Example

Single vs. Two Stage Rockets [1]

Single stage

To achieve an orbital speed of $v = 7600$ m/s, we require an *ideal* Δv of about 9000 m/s where the extra velocity is needed to overcome gravity and drag. Chemical rockets produce exhausts jets at velocities of $c \approx 2500 - 4500$ m/s. Using the higher c , if we wish to place a payload in orbit with a single stage rocket, we have a mass ratio (mass at burn out, m_f , divided by initial mass, m_0) of

$$\frac{m_f}{m_0} = e^{-9000/4500} = e^{-2} = 0.135 .$$

The mass m_f must include all components of the rocket infrastructure, including the engine, empty tank, guidance equipment, etc., as well as the payload. The mass of the propellant will be $m_0 - m_f = (1 - 0.135)m_0 = 0.865m_0$. If we assume that the tank plus engine are the main contributors to the weight of the rocket and weigh 10% of the propellant, $0.087m_0$, we have,

$$m_{payload}^{1stage} = (0.135 - 0.087)m_0 = 0.048m_0 .$$

Clearly, there is not much margin here, and, in fact, single-stage-to-orbit vehicles have yet to be engineered successfully. The alternative solution is to subdivide the rocket, so that empty tanks are dropped when they are no longer needed.

Two stages

Consider now a two stage vehicle. We shall again assume that each empty tank plus its engine weighs 10% of the propellant it carries. The required Δv is now subdivided into two Δv 's of 4500 m/s each.

First stage

If m_0 is the initial mass and m_1 is the mass after burn out, we have,

$$m_1 = e^{-4500/4500}m_0 = e^{-1}m_0 = 0.368m_0$$

The fuel burnt will be $m_0 - m_1 = (1 - 0.368)m_0 = 0.632m_0$, and the weight of the tank and engine to be dropped will be $0.0632m_0$, leaving an initial mass for the second stage, m_2 , of

$$m_2 = 0.368m_0 - 0.063m_0 = 0.295m_0 .$$

Second stage

After burn out,

$$m_f = e^{-1}m_2 = e^{-1}0.295m_0 = 0.109m_0.$$

The fuel burnt in this stage will be $m_2 - m_f = (0.295 - 0.109)m_0 = 0.186m_0$ and the weight of the tank plus engine will be $0.019m_0$, leaving for the payload

$$m_{payload}^{2stage} = (0.109 - 0.019)m_0 = 0.090m_0,$$

which is still very small but about *twice* the size of the payload obtained for the single stage rocket.

Note

Specific Impulse

The specific impulse I_{sp} , with units of seconds, is often used in practice to characterize the performance of a rocket engine. The definition of the specific impulse is the magnitude of the thrust divided by the propellant weight flow rate,

$$I_{sp} = \frac{T}{\dot{m}g} = \frac{c}{g} .$$

Typical values of I_{sp} are around 300 s for solid propellants and up to 500 s for higher energy fuels.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

4/6, 4/7

References

- [1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.