

## Lecture D4 - Intrinsic Coordinates

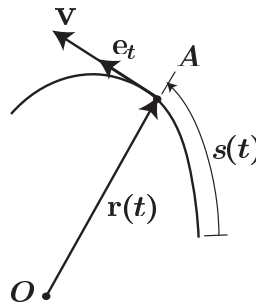
In lecture D2 we introduced the position, velocity and acceleration vectors and referred them to a fixed cartesian coordinate system. While it is clear that the choice of coordinate system does not affect the final answer, we shall see that, in practical problems, the choice of a specific system may simplify the calculations considerably. In previous lectures, all the vectors at all points in the trajectory were expressed in the same frame. When using intrinsic coordinates (as well as the other coordinate systems presented in the next lecture), we shall see that the reference frame is a function of the current position of the particle. This means that, the position, velocity and acceleration vectors at different points are expressed in different frames.

### Intrinsic coordinates: Tangential, Normal and Binormal components.

From the previous lecture, we know that the velocity vector,  $\mathbf{v}$ , can be expressed as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v\mathbf{e}_t, \quad (1)$$

where  $\mathbf{r}(t)$  is the position vector,  $v = \dot{s}$  is the speed,  $\mathbf{e}_t$  is the unit tangent vector to the trajectory, and  $s$  is the path coordinate along the trajectory.



Also, recall that the unit tangent vector can be written as,

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds}. \quad (2)$$

The acceleration vector is the derivative of the velocity vector with respect to time, and therefore we can write, from (1),

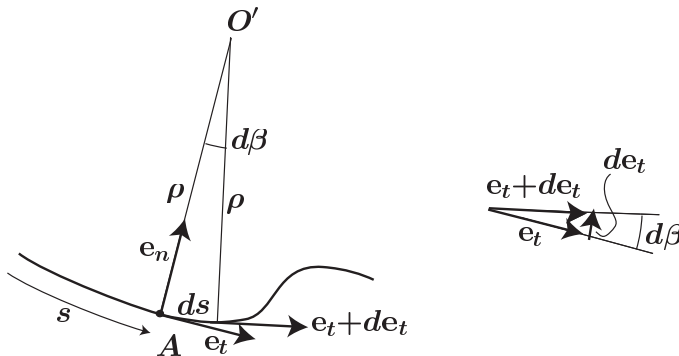
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt}. \quad (3)$$

The vector  $\mathbf{e}_t$  is the local unit tangent vector to the curve which changes from point to point. Consequently, the time derivative of  $\mathbf{e}_t$  will, in general, be nonzero.

The time derivative of  $\mathbf{e}_t$  can be written as,

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\mathbf{e}_t}{ds} \frac{ds}{dt} = \frac{d\mathbf{e}_t}{ds} v . \quad (4)$$

In order to calculate the derivative of  $\mathbf{e}_t$ , we note that, since the magnitude of  $\mathbf{e}_t$  is constant and equal to one, the only changes that  $\mathbf{e}_t$  can have are due to rotation, or *swinging*.



When we move from  $s$  to  $s + ds$ , the tangent vector changes from  $\mathbf{e}_t$  to  $\mathbf{e}_t + d\mathbf{e}_t$ . The change in direction can be related to the angle  $d\beta$ .

The direction of  $d\mathbf{e}_t$ , which is perpendicular to  $\mathbf{e}_t$ , is called the *normal* direction. On the other hand, the magnitude of  $d\mathbf{e}_t$  will be equal to the length of  $\mathbf{e}_t$  (which is one), times  $d\beta$ . Thus, if  $\mathbf{e}_n$  is a *unit normal vector* in the direction of  $d\mathbf{e}_t$ , we can write

$$d\mathbf{e}_t = d\beta \mathbf{e}_n . \quad (5)$$

Dividing by  $ds$  yields,

$$\frac{d\mathbf{e}_t}{ds} = \frac{d\beta}{ds} \mathbf{e}_n = \kappa \mathbf{e}_n = \frac{1}{\rho} \mathbf{e}_n . \quad (6)$$

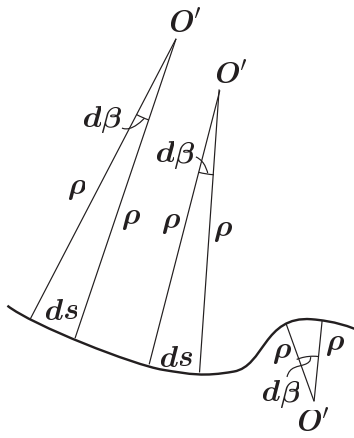
Here,  $\kappa = d\beta/ds$  is a local property of the curve, called the *curvature*, and  $\rho = 1/\kappa$  is called the *radius of curvature*.

Note that in the picture, the sizes of  $d\mathbf{e}_t$ ,  $ds$ , and  $d\beta$  are exaggerated for illustration purposes and actually represent the changes in the limit as  $ds$  (and also  $dt$ ) approach zero.

### Note

### Curvature and radius of curvature

We consider here two tangent vectors  $\mathbf{e}_t$  and  $\mathbf{e} + d\mathbf{e}_t$ , separated by a small  $ds$  and having an angle between them of  $d\beta$ . If we draw perpendiculars to these two vectors, they will intersect at a point, say,  $O'$ . Because the two lines meeting at  $O'$  are perpendicular to each of the tangent vectors, the angle between them will be the same as the angle between  $\mathbf{e}_t$  and  $\mathbf{e} + d\mathbf{e}_t$ ,  $d\beta$ . The point  $O'$  is called the *center of curvature*, and the distance,  $\rho$ , between  $O'$  and  $A$  is the radius of curvature. Thus, from the sketch, we have that  $ds = \rho d\beta$ , or  $d\beta/ds = \kappa = 1/\rho$ .



Intuitively, we can see that each infinitesimal arc,  $ds$ , can be represented by a circle segment of radius  $\rho$  having its center at the center of curvature. It is clear that both the radius of curvature and the center of curvature are functions of  $s$ , and consequently they change from point to point.

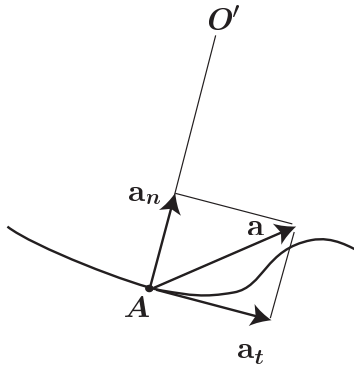
There are two limiting cases which are of interest. When the trajectory is a circle, the center of curvature does not change and coincides with the center of the circle, and the radius of curvature is equal to the radius of the circle. On the other hand, when the trajectory is a straight line, the curvature is zero, and the radius of curvature is infinite. Note also, that, in this case, the derivative of  $\mathbf{e}_t$  is always zero, and the normal direction is not defined.

Going back to expression (4), we have that

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\beta}{ds} v \mathbf{e}_n = \dot{\beta} \mathbf{e}_n = \frac{v}{\rho} \mathbf{e}_n . \quad (7)$$

Finally, we have that the acceleration, can be written as

$$\mathbf{a} = \frac{dv}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n = a_t \mathbf{e}_t + a_n \mathbf{e}_n . \quad (8)$$



Here,  $a_t = \dot{v}$ , is the *tangential* component of the acceleration, and  $a_n = v^2/\rho$ , is the *normal* component of the acceleration. Since  $a_n$  is the component of the acceleration pointing towards the center of curvature, it is sometimes referred to as *centripetal* acceleration. When  $a_t$  is nonzero, the velocity vector changes magnitude,

or *stretches*. When  $a_n$  is nonzero, the velocity vector changes direction, or *swings*. The modulus of the total acceleration can be calculated as  $a = \sqrt{a_t^2 + a_n^2}$ .

**Note**

**Calculation of the radius of curvature for a trajectory**

In some situations the trajectory will be known as a curve of the form  $y = f(x)$ . The radius of curvature in this case can be computed according to the expression,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} . \quad (9)$$

This expression is not hard to derive. Try it!

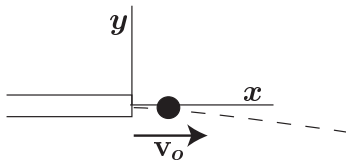
On the other hand, if the trajectory is known in parametric form as a curve of the form  $\mathbf{r}(t)$ , where  $t$  can be time, but also any other parameter, then the radius of curvature can be computed as

$$\rho = \frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{3/2}}{\sqrt{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}} ,$$

where  $\dot{\mathbf{r}} = d\mathbf{r}/dt$ , and  $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$ .

**Example**

A ball is ejected horizontally from the tube with a speed  $v_0$ . The only acceleration on the ball is due to gravity. We want to determine the radius of curvature of the trajectory just after the ball is released.



The simplest way to determine the radius of curvature is to note that, initially, the only nonzero component of the acceleration will be in the normal direction, i.e.  $a_n = g$ . Thus, from  $a_n = v_0^2/\rho$ , we have that,

$$\rho = \frac{v_0^2}{g} .$$

Alternatively, we can obtain an equation for the trajectory of the form  $y = f(x)$  and use expression (9) to calculate the curvature. The trajectory is given as,

$$\begin{aligned} x &= v_0 t \\ y &= -\frac{1}{2}gt^2 . \end{aligned}$$

Thus, eliminating  $t$ , we have

$$y = -\frac{g}{2v_0^2}x^2 .$$

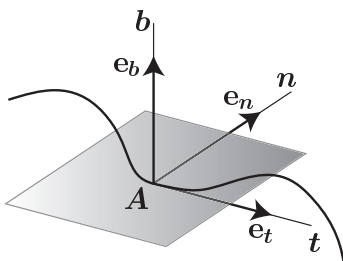
At  $x = 0$ ,  $dy/dx = 0$ ,  $d^2y/dx^2 = -g/v_0^2$ , and the above expression gives,  $\rho = v_0^2/g$ , as expected.

The quantities  $s$ ,  $v$  and  $a_t$  are related in the same manner as the quantities  $s$ ,  $v$  and  $a$  for rectilinear motion. In particular we have that  $v = \dot{s}$ ,  $a_t = \dot{v}$ , and  $a_t ds = v dv$ . This means that if we have a way of knowing  $a_t$ , we may be able to integrate the tangential component of the motion independently. We will be exploiting these relations in the future.

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The vectors  $\mathbf{e}_t$  and  $\mathbf{e}_n$ , and their respective coordinates  $t$  and  $n$ , define two orthogonal directions. The plane defined by these two directions, is called the *osculating* plane. This plane changes from point to point, and can be thought of as the plane that locally contains the trajectory (Note that the tangent is the current direction of the velocity, and the normal is the direction into which the velocity is changing).

In order to define a right handed set of axes we need to introduce an additional unit vector which is orthogonal to  $\mathbf{e}_t$  and  $\mathbf{e}_n$ . This vector is called the *binormal*, and is defined as  $\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n$ .



At any point in the trajectory, the position vector, the velocity and acceleration can be referred to these axes. In particular, the velocity and acceleration take very simple forms,

$$\begin{aligned} \mathbf{v} &= v\mathbf{e}_t \\ \mathbf{a} &= \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n . \end{aligned}$$

The difficulty of working with this reference frame stems from the fact that the orientation of the axis depends on the trajectory itself. The position vector,  $\mathbf{r}$ , needs to be found by integrating the relation  $d\mathbf{r}/dt = \mathbf{v}$  as follows,

$$\mathbf{r} = \mathbf{r}_0 + \int_0^t \mathbf{v} dt ,$$

where  $\mathbf{r}_0 = \mathbf{r}(0)$  is given by the initial condition.

We note that, by construction, the component of the acceleration along the binormal is always zero.

When the trajectory is planar, the binormal stays constant (orthogonal to the plane of motion). However, when the trajectory is a space curve, the binormal changes with  $s$ . It can be shown (see note below) that the derivative of the binormal is always along the direction of the normal. The rate of change of the binormal

with  $s$  is called the *torsion*,  $\tau$ . Thus,

$$\frac{d\mathbf{e}_b}{ds} = -\tau \mathbf{e}_n \quad \text{or,} \quad \frac{d\mathbf{e}_b}{dt} = -\tau v \mathbf{e}_n.$$

We see that whenever the torsion is zero, the trajectory is planar, and whenever the curvature is zero, the trajectory is linear.

## Equations of Motion

Newton's second law is a vector equation,  $\mathbf{F} = m\mathbf{a}$ , which can now be written in intrinsic coordinates.

In tangent, normal and binormal components,  $tnb$ , we write  $\mathbf{F} = F_t\mathbf{e}_t + F_n\mathbf{e}_n$  and  $\mathbf{a} = a_t\mathbf{e}_t + a_n\mathbf{e}_n$ . We observe that the positive direction of the normal coordinate is that pointing towards the center of curvature.

Thus, in component form, we have

$$\begin{aligned} F_t &= m a_t = m \dot{v} = m \ddot{s} \\ F_n &= m a_n = m \frac{v^2}{\rho} \end{aligned}$$

Note that, by definition, the component of the acceleration along the binormal direction,  $\mathbf{e}_b$ , is always zero, and consequently the binormal component of the force must also be zero. This may seem surprising, at first, but recall that the tangent and normal directions are determined by the motion, and, hence, we can say that the motion "chooses" the binormal direction to be always orthogonal to the applied force. In other words, if we apply a force to a particle, the particle will experience an acceleration which is parallel to the force. The normal direction is chosen so that the acceleration vector is always contained in the plane defined by the tangent and the normal. Thus, the binormal is always orthogonal to the external force.

Intrinsic coordinates are sometimes useful when we are dealing with problems in which the motion is constrained, such as a car on a roller coaster. The geometry of the trajectory is known, and, therefore, the directions of the tangent, normal and binormal vectors are also known. In these cases it may be possible to integrate the component of the equation of motion along the tangential direction (especially if there is no friction), and then calculate, a-posteriori, the reaction force using the normal component of the equation of motion.

### **Note (optional)**

### **Frenet formulae**

The Frenet formulae give us the variations of the unit vectors  $\mathbf{e}_t$ ,  $\mathbf{e}_n$  and  $\mathbf{e}_b$  with respect to the path coordinate  $s$ . The first formula

$$\frac{d\mathbf{e}_t}{ds} = \frac{1}{\rho} \mathbf{e}_n,$$

has already been defined. Now, since  $\mathbf{e}_b$  is a unit vector,  $d\mathbf{e}_b/ds$  will be orthogonal to  $\mathbf{e}_b$ . Hence, it will be of the form,

$$\frac{d\mathbf{e}_b}{ds} = b_t\mathbf{e}_t + b_n\mathbf{e}_n.$$

If we perform the dot product of this expression with  $\mathbf{e}_t$ , we obtain

$$b_t = \frac{d\mathbf{e}_b}{ds} \cdot \mathbf{e}_t = -\frac{d\mathbf{e}_t}{ds} \cdot \mathbf{e}_b = -\frac{1}{\rho} \mathbf{e}_n \cdot \mathbf{e}_b = 0 .$$

The second equality follows from the fact that the derivative of  $\mathbf{e}_t \cdot \mathbf{e}_b$  is zero, i.e.  $d\mathbf{e}_t \cdot \mathbf{e}_b + \mathbf{e}_t \cdot d\mathbf{e}_b = 0$ . Therefore, only  $b_n$  is nonzero. Defining  $b_n = -\tau$ , we obtain the third Frenet formula

$$\frac{d\mathbf{e}_b}{ds} = -\tau \mathbf{e}_n .$$

Finally, the second formula can be obtained in a similar manner (we leave the details as an exercise) and gives,

$$\frac{d\mathbf{e}_n}{ds} = -\frac{1}{\rho} \mathbf{e}_t + \tau \mathbf{e}_b ,$$

or, multiplying by  $v$ ,

$$\frac{d\mathbf{e}_n}{dt} = -\frac{v}{\rho} \mathbf{e}_t + \tau v \mathbf{e}_b . \quad (10)$$

As we move along  $s$ , the osculating plane (and hence  $\mathbf{e}_b$ ) may rotate, making the curve non planar. As an example, an aeroplane may be rolling as it flies along  $\mathbf{e}_t$ . The derivative of  $\mathbf{e}_b$  is in the direction opposite to  $\mathbf{e}_n$  if the rotation is in the direction of a right hand screw, and this is taken as the positive direction for the torsion.

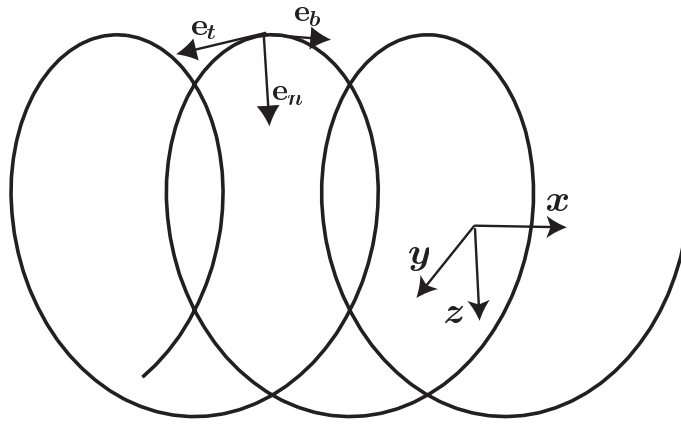
**Example**

**“Simplified” Aircraft Kinematics (W. M. Hollister)**

The flight of an aircraft through the sky is an example of curvilinear motion. Think of  $\mathbf{e}_t$  as the roll axis aligned with the velocity vector of the aircraft. Think of  $\mathbf{e}_b$  as being the pitch axis. The lift is then directed along  $\mathbf{e}_n$ . The roll rate of the aircraft can be interpreted as  $\tau v$ , and the pitch rate as  $v/\rho$ . In order to turn the aircraft out of the vertical plane, it is necessary to rotate the direction of the lift  $\mathbf{e}_n$  so that there is a component of acceleration out of the vertical plane. Neglecting gravity, the velocity vector along  $\mathbf{e}_t$  determines where the aircraft is going, and the lift along  $\mathbf{e}_n$  determines where the velocity vector is going. The roll rate determines how the lift vector will be rotated out of the osculating plane. As shown by equation (10), the direction of the lift vector is changed by rolling  $\tau v$ , as well as pitching  $v/\rho$ .

Consider the following example. An aircraft follows a spiral path in the sky while doing a barrel roll. The coordinates are given below, where  $v_0 = 194$  ft/s,  $\omega = 0.4$  rad/s, and  $h = 125$  ft are constants.

$$\begin{aligned} x &= v_0 t \\ y &= h \cos \omega t \\ z &= h \sin \omega t \end{aligned}$$



We have,

$$\begin{aligned}
 \mathbf{v} &= v_0 \mathbf{i} - h\omega \sin \omega t \mathbf{j} + h\omega \cos \omega t \mathbf{k} \\
 v &= \sqrt{v_0^2 + h^2 \omega^2} = 200 \text{ft/s} \approx v_0 \\
 \mathbf{a} &= -h\omega^2 \cos \omega t \mathbf{j} - h\omega^2 \sin \omega t \mathbf{k} \\
 a &= h\omega^2 = 20 \text{ft/s}^2
 \end{aligned}$$

Since  $\dot{v} = 0$ ,  $\mathbf{a} = (v^2/\rho)\mathbf{e}_n$ , or,

$$h\omega^2 = \frac{v^2}{\rho}, \quad \rho = \frac{v^2}{h\omega^2} = 2000 \text{ft}$$

,

$$\begin{aligned}
 \mathbf{e}_t &= \frac{v_0}{v} \mathbf{i} - \frac{h\omega}{v} \sin \omega t \mathbf{j} + \frac{h\omega}{v} \cos \omega t \mathbf{k} \\
 \mathbf{e}_n &= -\cos \omega t \mathbf{j} - \sin \omega t \mathbf{k} \\
 \mathbf{e}_b &= \frac{h\omega}{v} \mathbf{i} + \frac{v_0}{v} \sin \omega t \mathbf{j} - \frac{v_0}{v} \cos \omega t \mathbf{k}
 \end{aligned}$$

Finally,

$$\frac{d\mathbf{e}_b}{dt} = \frac{v_0\omega}{v} \cos \omega t \mathbf{j} + \frac{v_0\omega}{v} \sin \omega t \mathbf{k}$$

which corresponds to a roll rate of

$$\tau v = \frac{v_0\omega}{v} \approx \omega = 0.4 \text{rad/s}$$

#### ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition  
2/5, 3/5 (normal and tangential coordinates only)