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16.07 Dynamics
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Lecture D6 - Equations of Motion: Application Examples

In this lecture we will look at some applications of Newton's second law, expressed in the different coordinate systems that were introduced in lectures D3-D5. Recall that Newton's second law

$$\boldsymbol{F} = m\boldsymbol{a} \ , \tag{1}$$

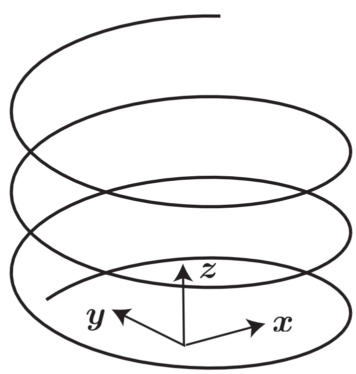
is a vector equation which is valid for inertial observers.

In general, we will be interested in determining the motion of a particle given that we know the external forces. Equation (1), written in terms of either velocity or position, is a differential equation. In order to calculate the velocity and position as a function of time we will need to integrate this equation either analytically or numerically. On the other hand, the reverse problem of computing the forces given motion is much easier and only requires direct evaluation of (1). Is is also common to have mixed type problems, in which we know some components of the force and some components of the acceleration. The goal is then to determine the remaining unknown terms.

While no general rules can be given regarding the appropriate choice of a coordinate system, we note that intrinsic coordinates are particularly useful in constrained problems, where the trajectory is known beforehand.

Example ***Aircraft flying on a helix***

A $10,000$ lb aircraft is descending on a cylindrical helix. The rate of descent is $\dot{z} = -10$ft/s, the speed is $v = 211$ ft/s, and $\dot{\theta} = 3^o \approx 0.05$rad/s. This is standard for gas turbine powered aircraft. We want to know the force on the aircraft and the radius of curvature of the path.



We have,

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z = v\boldsymbol{e}_t$$

Since, $r = R$, $\dot{r} = 0$. Therefore, $211 = \sqrt{(0.05R)^2 + 10^2}$, or $R = 4,215$ ft. For the acceleration,

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z = \dot{v}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n \ ,$$

and, considering only the non-zero terms,

$$\boldsymbol{a} = -R\dot{\theta}^2\boldsymbol{e}_r = \frac{v^2}{\rho}\boldsymbol{e}_n \ .$$

We see that $\boldsymbol{e}_n = -\boldsymbol{e}_r$, and that,

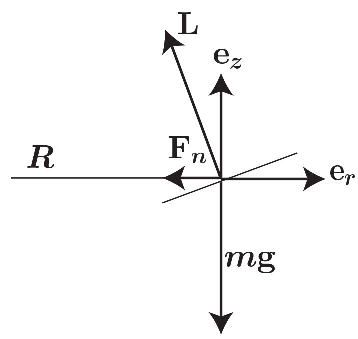
$$a = (0.05)^2 4,215 = 10.54\,\text{ft/s}^2 = \frac{v^2}{\rho}, \qquad \rho = \frac{211}{10.54} = 4,225\,\text{ft} \ .$$

The normal force on the aircraft is

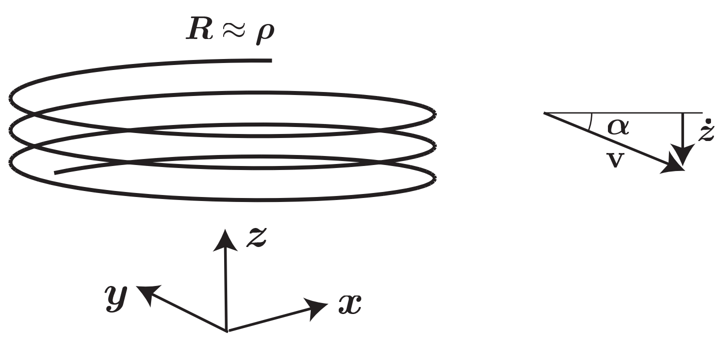
$$F_n = ma_n = \frac{10,000}{32}10.54 = 3,273\,\text{lb} \ ,$$

and finally, the lift, $\mathbf{L}$, is

$$\mathbf{L} = -3,273\,\boldsymbol{e}_r + 10,000\,\boldsymbol{e}_z\ \text{lb} \ .$$



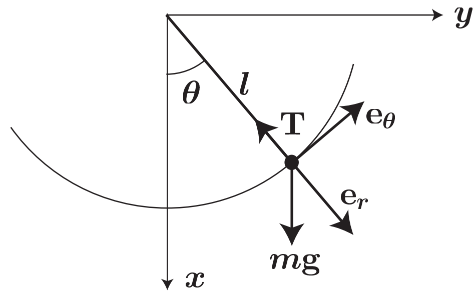
Here we see that $\rho \approx r$ which means that the helix is very tight.



The angle of descent α is calculated as $\sin\alpha = -\dot{z}/v$, or, $\alpha = -2.72^o$. This angle is sometimes called the *pitch* of the helix.

Example ***Pendulum***

Now, we consider a simple pendulum consisting of a mass, m, suspended from a string of length l and negligible mass.



We can formulate the problem in polar coordinates, and noting that $r = l$ (constant), write for the r and θ components,

$$\begin{aligned} mg\cos\theta - T &= -ml\dot{\theta}^2 \\ -mg\sin\theta &= ml\ddot{\theta}\ , \end{aligned} \tag{2}$$

where T is the tension on the string. If we restrict the motion to small oscillations, we can approximate $\sin\theta \approx \theta$, and the θ-equation becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0\ .$$

Integrating we obtain the general solution,

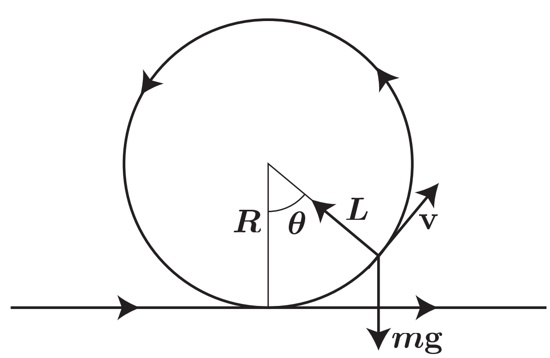
$$\theta(t) = C_1 \cos(\sqrt{\frac{g}{l}}\, t) + C_2 \sin(\sqrt{\frac{g}{l}}\, t)\ ,$$

where the constants C_1 and C_2 are determined by the initial conditions. Thus, if $\theta(0) = \theta_{\max}$,

$$\theta(t) = \theta_{\max}\ \cos(\sqrt{\frac{g}{l}}\, t)\ .$$

Example ***Aircraft flying a perfect loop*** (Hollister)

Consider an aircraft flying a perfect loop, i.e. a circle in the vertical plane. Assume that the engine thrust exactly cancels the aerodynamic drag so that the lift and gravity are the only unbalanced forces on the aircraft. This assumption makes the problem into the same dynamical model that we have used in the previous example.



Since the lift, **L**, is perpendicular to the flight path, we have that the force on the aircraft, in normal and tangential components, is

$$\boldsymbol{F} = -mg\sin\theta\,\boldsymbol{e}_t + (L - mg\cos\theta)\,\boldsymbol{e}_n \ .$$

Thus,

$$\begin{aligned} a_t &= \dot{v} = r\ddot{\theta} = -g\sin\theta \\ a_n &= \frac{v^2}{R} = \frac{L}{m} - g\cos\theta \ . \end{aligned} \tag{3}$$

Since, $v\,dv = a_t\,ds = a_t R\,d\theta = -Rg\sin\theta\,d\theta$. Thus, integrating,

$$v^2 = v_0^2 + 2Rg(\cos\theta - 1) \ , \tag{4}$$

where v_0 is the velocity at the bottom of the loop when $\theta = 0$. To be able to go over the top we need $v > 0$ when $\theta = \pi$. This means that we need $v_0 > 2\sqrt{Rg}$.

Note that for $v_0 < 2\sqrt{Rg}$, we can calculate the maximum angle the aircraft can reach, θ_{max}. If we set $v = 0$ when $\theta = \theta_{max}$, we have,

$$\theta_{max} = \cos^{-1}(1 - \frac{v_0^2}{2Rg}) \ .$$

The necessary lift, L, can be calculated as a function of θ. From (3) and (4), we have

$$\frac{L}{m} = \frac{v^2}{R} + g\cos\theta = \frac{v_0^2}{R} + 3g\cos\theta - 2g \ .$$

We have that, in order for θ to go from 0 to π, the aircraft has to have a range of lift capability that extends over $5g$.

It turns out that most aircraft do not have this capability and consequently do not fly perfect loops.

Example ***Aircraft flying an "imperfect" loop*** (Hollister)

It is more common to fly a loop keeping the normal acceleration, a_n, approximately constant at, say, ng ($n \sim 3 - 4$).

Let β be the flight path angle which the velocity vector (or tangent) makes with the horizontal, and let ρ be the radius of curvature of the path. Then,

$$\begin{aligned} a_n &= \frac{v^2}{\rho} = v\dot{\beta} = ng \\ a_t &= -g\sin\beta \ . \end{aligned}$$

From, $v\,dv = a_t\,ds$, with $ds = \rho d\beta$, we have,

$$v\,dv = a_t\rho\,d\beta = (-g\sin\beta)(\frac{v^2}{ng})d\beta \ ,$$

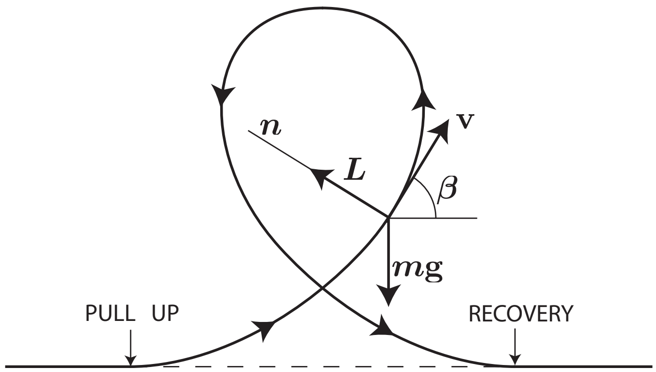
or, integrating,

$$\frac{n\,dv}{v} = -\sin\beta d\beta \ , \qquad n\ln v|_{v_0}^{v} = \cos\beta|_0^\beta \ , \qquad v = v_0 e^{-\frac{1}{n}(1-\cos\beta)} \ ,$$

and

$$\rho = \frac{v_0^2}{ng} e^{-\frac{2}{n}(1-\cos\beta)} \ . \tag{5}$$

A sketch of this path is shown in the figure below.



In going over the top of the imperfect loop, the aircraft does not go as high or loose as much velocity as it does going over the perfect loop. Unlike the perfect loop case however, the aircraft does need to pull up before, and recover after, the point of maximum altitude (see diagram). Note that the solution of this example and the previous one would have been rather difficult using rectangular coordinates. Note also that the form of the solution given by equation (5) is rather unusual, e.g. the radius of curvature is given as a function of the attitude angle. A possible way to plot the trajectory starting from an initial position and velocity would be to first determine β, and then draw a small circle segment with the appropriate ρ. After calculating the new position, the process would be repeated to draw the entire trajectory.

References

[1] W.H. Hollister, *Unified Engineering Notes*, Course 93-94.