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16.07 Dynamics
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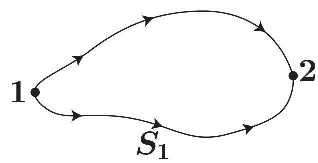
Lecture D8 - Conservative Forces and Potential Energy

We have seen that the work done by a force $\boldsymbol{F}$ on a particle is given by $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$. If the work done by $\boldsymbol{F}$, when the particle moves from any position $\boldsymbol{r}_1$ to any position $\boldsymbol{r}_2$, can be expressed as,

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -(V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)) = V_1 - V_2 \ , \tag{1}$$

then we say that the force is *conservative*. In the above expression, the scalar function $V(\boldsymbol{r})$ is called the *potential*. It is clear that the potential satisfies $dV = -\boldsymbol{F} \cdot d\boldsymbol{r}$ (the minus sign is included for convenience). There are two main consequences that follow from the existence of a potential: i) the work done by a conservative force between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is *independent of the path*. This follows from (1) since W_{12} only depends on the initial and final potentials V_1 and V_2 (and not on how we go from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$), and ii) the work done by potential forces is *recoverable*. Consider the work done in going from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$, W_{12}. If we go, now, from point $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$, we have that $W_{21} = -W_{12}$ since the total work $W_{12} + W_{21} = (V_1 - V_2) + (V_2 - V_1) = 0$.



In one dimension any force which is only a function of position is conservative. That is, if we have a force, $F(x)$, which is only a function of position, then $F(x)\, dx$ is always a perfect differential. This means that we can define a potential function as

$$V(x) = -\int_{x_0}^{x} F(x)\, dx \ ,$$

where x_0 is arbitrary.

In two and three dimensions, we would, in principle, expect that any force which depends only on position, $\boldsymbol{F}(\boldsymbol{r})$, to be conservative. However, it turns out that, in general, this is not sufficient. In multiple dimensions, the condition for a force field to be conservative is that it can be expressed as the gradient of a potential function. That is,

$$\boldsymbol{F} = -\boldsymbol{\nabla} V \ .$$

Note — ***The gradient operator, $\boldsymbol{\nabla}$***

The gradient operator, $\boldsymbol{\nabla}$ (called "del"), in cartesian coordinates is defined as

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial x}\boldsymbol{i} + \frac{\partial(\ )}{\partial y}\boldsymbol{j} + \frac{\partial(\ )}{\partial z}\boldsymbol{k} \ .$$

When operating on a scalar function $V(x,y,z)$, the result $\boldsymbol{\nabla} V$ is a vector, called the gradient of V. The components of $\boldsymbol{\nabla} V$ are the derivatives of V along each of the coordinate directions,

$$\boldsymbol{\nabla} V \equiv \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} \ .$$

If we consider a particle moving due to conservative forces with potential energy $V(x,y,z)$, as the particle moves from point $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ to point $\boldsymbol{r} + d\boldsymbol{r} = (x+dx)\boldsymbol{i} + (y+dy)\boldsymbol{j} + (z+dz)\boldsymbol{k}$, the potential energy changes by $dV = V(x+dx, y+dy, z+dz) - V(x,y,z)$. For small increments dx, dy, dz, and dV, can be expressed, using Taylor series expansions, as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \ ,$$

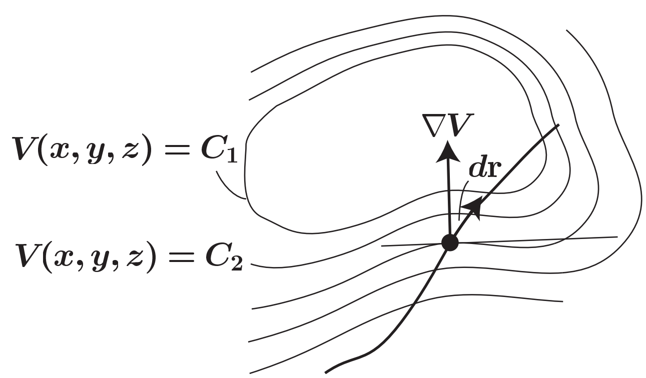
where $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$.

This equation expresses the fundamental property of the gradient. The gradient allows us to find the change in a function induced by a change in its variables.

If we write $V(x,y,z) = C$, for some constant C, this is the implicit equation of a surface, which is called a constant energy surface. This surface is made up by all the points in the x,y,z space for which the function $V(x,y,z)$ is equal to C. It is clear that if a particle moves on a constant energy surface, $dV = 0$, since V is constant on that surface. Therefore, when a particle moves on a constant energy surface, $d\boldsymbol{r}$ will be tangent to that surface, and since

$$0 = dV = \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \ ,$$

we have that $\boldsymbol{\nabla} V$ is perpendicular to any tangent to the surface. This situation is illustrated in the picture below for the two dimensional case. Here, the constant energy surfaces are contour curves, and we can see that the gradient vector is always normal to the contour curves.



Note ***Gradient operator in cylindrical coordinates***

The gradient operator can be expressed in cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, and

$r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. Thus, applying the chain rule for differentiation, we have

$$\begin{aligned}\frac{\partial(\ )}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial \theta} = \cos\theta\frac{\partial(\ )}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(\ )}{\partial r} \\ \frac{\partial(\ )}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial \theta} = \sin\theta\frac{\partial(\ )}{\partial r} + \frac{\cos\theta}{r}\frac{\partial(\ )}{\partial r}\ .\end{aligned}$$

If we note that $\boldsymbol{i} = \cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta$ and $\boldsymbol{j} = \sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta$, we have that

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial(\ )}{\partial z}\ .$$

An expression for spherical coordinates can be derived in a similar manner.

Conservation of Energy

When all the forces doing work are conservative, the work is given by (1), and the principle of work and energy derived in the last lecture,

$$T_1 + W_{12} = T_2\ ,$$

reduces to,

$$T_1 + V_1 = T_2 + V_2$$

or more generally, since the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are arbitrary,

$$E = T + V = \text{constant}\ . \tag{2}$$

Whenever applicable, this equation states that the total energy stays constant, and that during the motion only exchanges between kinetic and potential energy occur.
In the general case, however, we will have a combination of conservative, $\boldsymbol{F}^C$, and non-conservative, $\boldsymbol{F}^{NC}$, forces. In this case, the work done by the conservative forces will be calculated using the corresponding potential function, i.e., $W_{12}^C = V_1 - V_2$, and the work done by the non-conservative forces will be path dependent and will need to be be calculated using the work integral. Thus, in the general case, we will have,

$$T_1 + V_1 + \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}^{NC} \cdot d\boldsymbol{r} = T_2 + V_2\ .$$

Examples of Conservative Forces

Gravity near the earth's surface

On a "flat earth", the specific gravity $\boldsymbol{g}$ points down (along the -z axis), so $\boldsymbol{F} = -mg\boldsymbol{k}$. Call $V = 0$ on the surface $z = 0$, and then

$$V(z) = -\int_0^z (-mg)\,dz, \qquad V(z) = mgz\ .$$

For the motion of a projectile, the total energy is then

$$E = \frac{1}{2}mv^2 + mgz = \text{constant} \ .$$

Since v_x and v_y remain constant, we also have $\frac{1}{2}mv_z^2 + mgz = \text{constant}$.

Gravity

In a central gravity field

$$\boldsymbol{F} = -G\frac{Mm}{r^2}\boldsymbol{e}_r = -\boldsymbol{\nabla}(-G\frac{Mm}{r}) \ ,$$

and so, taking $V(r \to \infty) = 0$,

$$V = -G\frac{Mm}{r} \ .$$

Spring Force

The force supported by a spring is $F = -kx$. The elastic potential energy of the spring is the work done on it to deform it an amount x. Thus, we have

$$V = -\int_0^x -kx \, dx = \frac{1}{2}kx^2 \ .$$

If the deformation, either tensile or compressive, increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is the difference between its final and initial values, or,

$$\Delta V = \frac{1}{2}k(x_2^2 - x_1^2) \ .$$

Exercise ***Non-conservative Forces***

Consider a block sliding on a horizontal surface. Is the friction force conservative? Explain why? Give examples of non-conservative forces.

Note ***Path independent forces for which there is no potential (Optional)***

The condition $\boldsymbol{F} = -\boldsymbol{\nabla}V$ is *sufficient*, but not *necessary* for the work to be path independent. An important example is the magnetostatic force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, where the work $\boldsymbol{F} \cdot d\boldsymbol{r} = \boldsymbol{F} \cdot \boldsymbol{v}dt$ is zero (the force is always perpendicular to $\boldsymbol{v}$). Thus, for a general velocity-dependent electro-magnetic force, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$, where only $\boldsymbol{B}$ is irrotational, the work is still path independent, and the total energy $(mv^2/2 + q\phi)$ is conserved.

References

[1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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