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# Quantum State Reduction for Universal Measurement Based Computation

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Measurement based quantum computation, which requires only single particle measurements on a universal resource state to achieve the full power of quantum computing, has been recognized as one of the most promising models for the physical realization of quantum computers. Despite considerable progress in the past decade, it remains a great challenge to search for new universal resource states with naturally occurring Hamiltonians and to better understand the entanglement structure of these kinds of states. Here we show that most of the resource states currently known can be reduced to the cluster state, the first known universal resource state, via adaptive local measurements at a constant cost. This new quantum state reduction scheme provides simpler proofs of universality of resource states and opens up plenty of space to the search of new resource states.

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Measurement based quantum computation (MBQC) [1], an interesting computation model that incorporates peculiar aspects in quantum mechanics such as entanglement and measurement, achieves the full power of quantum computing by adaptive local measurements on a resource state. One-way quantum computing, the first MBQC scheme, employs the now well-known cluster state [2]. The highly entangled feature of the cluster state indicates that high entanglement is a key requirement for universality in MBQC. Although this is true in some sense [3,4], it is also clear now that too much entanglement could also undermine universality [5,6]. In other words, the entanglement should be manageable in a structured way.

As shown in the recent breakthrough made by Gross *et al.* in Refs. [7,8], the matrix product state (MPS) formalism [9,10] or, in higher spatial dimensions, the computational tensor networks [8,11], provides such an infrastructure for manipulating the entanglement and brings a new scheme of MBQC called the correlation space quantum computation. In this framework, a lot of new resource states beyond the cluster state are proposed. Most of the new resource states have different properties from the cluster state concerning, for example, local entropy, the two-point correlation function, and the locality of the Hamiltonians of which they are unique ground states. In particular, some of the new resources are unique ground states of more practical Hamiltonians [12,13], thereby overcoming the major flaw of the cluster state of not being a unique ground state of any two-body nearest-neighbor gapped Hamiltonian [14].

Here, we introduce the concept of quantum state reduction for MBQC, and the motivation is twofold. First of all, state reduction serves as a tool for revealing the entanglement structure of universal resource states [15,16]. Similar to the common technique to study entanglement by considering local transformations [17–19], quantum state reduction is also a type of local transformation tailored to meet the nature of MBQC. We find that almost all known resource states can be locally transformed to a cluster state via quantum state reduction, indicating that these resource states possess similar entanglement structure as the cluster state.

Secondly, although the application of the MPS formalism in the theory of MBQC is elegant and fruitful, the routine for analyzing the universality of a resource state remains a complicated procedure, including the initialization, embedding of universal rotations, the readout, and compensation for the randomness. The quantum state reduction approach largely simplifies the analysis. For example, the universality of the Affleck-Kennedy-Lieb-Tasaki (AKLT) state [20] is now cleanly summarized in Fig. 2. The simplicity also enables us to find new resource states, giving the universality of two deformations of AKLT state almost for free.

To be more precise, our state reduction is a transformation from one resource state to some other universal target state (usually the cluster state) using local measurement and adaptive classical control. This transformation is named reduction as it resembles the reduction in complexity theory—as long as  $|\Psi\rangle$  is reducible to  $|\Phi\rangle$ , it is in principle no harder to construct MBQC schemes for  $|\Psi\rangle$ .

than for  $|\Phi\rangle$ . Although the resulting state may consist of a much smaller number of particles than the state before reduction, the cost or efficiency of the reduction, measured by the diminution in the number of particles, is always expected to be a constant fraction.

*Matrix product states.*—Following the notion in Refs. [7,8], a matrix product state  $|\Psi_n\rangle$  of  $n$  particles has the following form:

$$|\Psi_n\rangle = \sum_{x_1, \dots, x_n=0}^{d-1} \langle R | A[x_n] \cdots A[x_1] | L \rangle |x_1 \cdots x_n\rangle. \quad (1)$$

The physical dimension of each site is  $d$ , while the bond dimension of the state defined by the size of the matrices is  $\delta$ . In general, one may consider a MPS where the defining matrices are site dependent. The matrix product form can be obtained by a series of Schmidt decompositions [4]. The defining matrices are usually far more important than the boundary conditions  $\langle R|$ ,  $|L\rangle$ , and sometimes we will use only the  $d$ -tuple  $(A[0], \dots, A[d-1])$  to specify a MPS. We will ignore the effect of a local change of basis on the physical space. For example, a state may be said to have some MPS representation even though this is only true up to some local unitary operations. Also for simplicity,  $A[x]$ 's will be given up to some normalization constant, which can be figured out from  $\sum_x A[x]^\dagger A[x] = I$  shown in Ref. [9].

A lot of states of particular interest in quantum information are indeed matrix product states of small bond dimension. The 1D cluster state, for example, is a MPS with defining matrices  $(H, HZ)$ , where  $H, X, Y, Z$  are used to denote the Hadamard and Pauli matrices, respectively. Another intriguing example is the AKLT state [20] first studied in condensed matter theory. It will be shown in the next section that it has a simple MPS representation  $(I, X, Z)$ .

The correlation space quantum computation employs the structure of MPS as in Eq. (1). It starts with the initial state  $|L\rangle$  of the so-called correlation space, measures the physical spaces sequentially, and thereby processes the correlation space, and finally reads out the information stored in the correlation space. Several new universal resources for MBQC were introduced in Refs. [7,8] including a modified AKLT state with defining matrices  $(H, X, Y)$ . Later, the original AKLT state is also shown to be universal for MBQC [12]. Recently, the concept of quantum wires is defined and fully characterized in Ref. [21], which essentially gives the explicit condition for a MPS with  $d = \delta = 2$  to be a universal resource. There are two normal forms of the matrices for quantum wires. One is the “by-product normal form,”

$$A[0] = W/\sqrt{2}, \quad A[1] = WS(\phi)/\sqrt{2}, \quad (2)$$

and the other is the “biased normal form” [22],

$$A[0] = \sin\gamma W', \quad A[1] = \cos\gamma W'Z, \quad (3)$$

where  $W, W'$  are rotations along axes in the  $X$ - $Z$  plane of the Bloch sphere and  $S(\phi) = \exp(-i\phi Z/2)$ .

There is a higher spatial dimensional generalization of MPS, known by the names of the computational tensor networks [8], or the projected entangled pair state [11]. Because of limitations of space, we refer the readers to the references above for details.

*Tabular form.*—For the convenience of later discussions, we introduce a tabular form of MPS. In the tabular form, one writes the defining matrices of a block of sites explicitly in a table, where each column consists of the  $d$  matrices of a corresponding site. The physical indexes determine a selection of one matrix from each column, whose product gives the correct amplitude together with the boundary conditions  $\langle R|$  and  $|L\rangle$ . A  $k$ -column tabular form therefore corresponds to  $k$  consecutive particles in a MPS. From the definition in Eq. (1) and the properties of MPS [9], we have (1) for any two neighboring columns, multiplication of  $M$  to the right of all matrices in the left column and  $M^{-1}$  to the left of all matrices in the right column simultaneously does not change the state, (2) a unitary transformation in the physical space corresponds to linear combinations of entries in the column with coefficients of the unitary, (3) measurement in the computational basis corresponds to the deletion of column entries not consistent with the measured outcome, and (4) columns of single entry can be removed by absorbing them to a neighboring column. We will use binary relations  $=$ ,  $\simeq$ , and  $<$  to represent equality, local unitary equivalence, and quantum state reduction, respectively.

As an example, Table 1 of Fig. 1 consists of a block of two sites of the AKLT states which will be discussed in the next section and it equals Table 2 by property (1) of the tabular form.

*Reduction of the AKLT state.*—The AKLT state [20] has become one of the prototypical states of spin systems. It also gives an excellent example for quantum state reduction.

As the origin of matrix product states, the AKLT state bears a simple MPS representation with

$$A[0] = Z, \quad A[1] = \sqrt{2}|0\rangle\langle 1|, \quad A[2] = \sqrt{2}|1\rangle\langle 0|. \quad (4)$$

Up to a local unitary operation, the matrices of the AKLT state can also be chosen as  $(X, Y, Z)$ . In fact, any three different matrices of the identity and the Pauli matrices will work. For example, Fig. 1 presents the proof that  $(I, X, Z)$  also stands for the AKLT state. In this figure, Table 2 is obtained by adding the  $Y$ 's in between, and hence represents the same state as Table 1 by property (1) of the tabular form; Tables 2 and 3 describe two states that are equivalent under local unitary operations by property (2).

We now show the reduction from the AKLT state to the 1D cluster state. It is convenient to start with the  $(I, X, Z)$  form. Two different measurements,  $\mathcal{N}_1$  and  $\mathcal{N}_2$ , will be used alternatively, where  $\mathcal{N}_1$  measures  $\{|0\rangle, |1\rangle\}$  versus  $|2\rangle$  and  $\mathcal{N}_2$  measures  $\{|0\rangle, |2\rangle\}$  versus  $|1\rangle$ . The measurements are called success (failure) if the outcome corresponds to the two- (one-)dimensional subspaces. We measure the two

$$\begin{array}{|c|c|} \hline 1 \\ \hline X & X \\ Y & Y \\ Z & Z \\ \hline \end{array} = \begin{array}{|c|c|} \hline 2 \\ \hline XY & YX \\ YY & YY \\ ZY & YZ \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline 3 \\ \hline Z & Z \\ I & I \\ X & X \\ \hline \end{array}$$

FIG. 1 (color online). AKLT as  $(I, X, Z)$ .

measurements sequentially along the AKLT chain and switch the measurement we use only when the previous one succeeds. This simple procedure is called the alternating measurement scheme.

Table 1 in Fig. 2 denotes a possible result after the alternating measurements on the AKLT state. More specifically, one first measures  $\mathcal{N}_1$  and succeeds. Next, the measurement  $\mathcal{N}_2$  is used. It results in the single dimensional space  $|1\rangle$  once, and succeeds subsequently, and so on. After renaming the physical indexes and absorbing the  $X$  and  $Z$  in the second and fourth columns to their previous columns, we have Table 2 in Fig. 2 by properties (4) and (2). This is actually already a 1D cluster state by the second line of reasoning in Fig. 2. Although we have only analyzed a single case of possible measurement outcomes, the general analysis follows similarly.

*Families of universal states of AKLT type.*—The simplicity of the above analysis enables us to generalize the same approach to a larger family of AKLT type of states. Notice that the key property that validates the first line of Fig. 2 is simply  $X^2 = Z^2 = I$ , and the key to the second line is that  $H^2 = I$  and  $XH = HZ$ . We now choose two unitary matrices  $A$  and  $B$  such that  $A^2 = B^2 = I$ , where  $A, B$  correspond to  $\pi$  rotations along  $\mathbf{n}_a$  and  $\mathbf{n}_b$  on the Bloch sphere, respectively. Let  $C \propto A + B$  be the  $\pi$  rotation along  $\mathbf{n}_a + \mathbf{n}_b$ . We will have  $C^2 = I$  and  $AC = CB$ . Therefore, we can prove the following reduction similarly:

$$(I, A, B) \prec (C, CB). \quad (5)$$

Employing the gauge freedom of the representation of MPS, one can always choose  $B$  to be  $Z$  and  $C$  to be  $\sin\theta X + \cos\theta Z$ , making  $(C, CB)$  a quantum wire in the normal form of Eq. (2). Note that the 1D AKLT state is a special case where  $\theta = \pi/4$  and that the error group  $\langle C, B \rangle$  is isomorphic to the dihedral group for infinitely many  $\theta$ 's. With the techniques of Ref. [9], one can check that the new AKLT-type resource is always a unique ground state of a nearest-neighbor, frustration-free Hamiltonian.

$$\begin{array}{|c|c|c|c|} \hline 1 \\ \hline I & I & I & I \\ X & \color{red}{X} & & X \\ & & Z & \color{red}{Z} \\ & & & Z \\ \hline \end{array} \simeq \begin{array}{|c|c|c|c|} \hline 2 \\ \hline I & I & I & I \\ X & Z & X & Z \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 3 \\ \hline I & I \\ X & Z \\ \hline \end{array} = \begin{array}{|c|c|} \hline 4 \\ \hline IH & HI \\ XH & HZ \\ \hline \end{array} = \begin{array}{|c|c|} \hline 5 \\ \hline H & H \\ HZ & HZ \\ \hline \end{array}$$

FIG. 2 (color online). AKLT reduced to 1D cluster.

In Ref. [10], Fannes *et al.* considered another one-parameter deformation of the AKLT model whose ground state is a MPS with

$$A[0] = \sin\theta Z, \quad A[1] = \cos\theta|0\rangle\langle 1|, \quad A[2] = \cos\theta|1\rangle\langle 0|.$$

We will show the universality of these states also by reduction. First, the defining matrices can be chosen as

$$(\sin\theta Z, \cos\theta X/\sqrt{2}, \cos\theta Y/\sqrt{2}),$$

up to local unitary transformation. Let  $\hat{\theta}$  be the angle that satisfies  $\tan\hat{\theta} = \sqrt{2}\tan\theta$ . The defining matrices can be simplified to  $(\sin\hat{\theta}I, \cos\hat{\theta}X, \cos\hat{\theta}Z)$ , in the same way as in Fig. 1. Using a similar alternating measurement scheme in Fig. 2, this is further reducible to  $(\sin\hat{\theta}H, \cos\hat{\theta}HZ)$ , a universal state in the biased normal form of Eq. (3). One caveat is that, in this case, we cannot simply absorb the single  $X$  and  $Z$  columns in Table 1 of Fig. 2 to neighboring sites because of the bias. But one can always measure the computational basis in several neighboring sites and cancel their effects by a random walk on the Pauli group.

*Reduction of quantum wires to cluster states.*—We now discuss the reduction of universal quantum wires to the cluster state. The special case of  $(W, WZ)$  is much easier to deal with. To transform it into  $(H, HZ)$ , one can simply implement  $HW^\dagger$  in the correlation space using the sites beforehand. In the general case of  $(W, WS(\phi))$ , however, one cannot succeed using projective measurement only—the local entropy determined by  $\phi$  [21] can never be increased. Yet, if the more general quantum measurement is employed, this is again possible. It will be easier to work with the biased normal form in this case. Suppose we want to transform  $(\sin\gamma W', \cos\gamma W'Z)$  to  $(H, HZ)$ . Assume that  $\gamma \in (0, \pi/4]$  with out loss of generality and apply on the site a general measurement with operators

$$M_0 = |0\rangle\langle 0| + \tan\gamma|1\rangle\langle 1|, \quad M_1 = \sqrt{1 - \tan^2\gamma}|1\rangle\langle 1|,$$

known as the filtering operation. When the outcome happens to be 0, we have changed the matrices to  $(W', W'Z)$  and can proceed as in the easy case; otherwise, we need to undo the action of  $W'Z$  on the correlation space and start all over again.

*Higher spatial dimensional cases.*—This section investigates the idea of quantum state reduction in the case of higher-dimensional resource states, which are necessary for the full power of universal quantum computing.

The tricluster state, an interesting variant of the cluster state, is proposed in Ref. [13] as a universal resource state. It is not difficult to see that there is a reduction to cluster state on exactly the same lattice of the tricluster state, and we leave the details to Appendix A in Ref. [23].

For most of the known 2D resource state, a general coupling scheme has been used to make 2D resource states out of 1D chains as in constructing the 2D AKLT resource, analyzing 2D weighted graph state [7,8], and weaving quantum wires into quantum webs [21]. In this scheme,



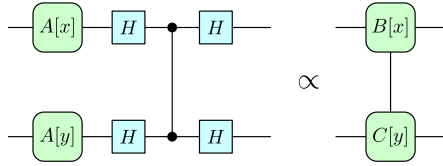


FIG. 3 (color online). Reduction of 2D resources of the coupling scheme.

one can always (a) isolate several usually horizontal, 1D, universal chains from the 2D state and (b) couple the correlation space of two neighboring 1D chains whenever necessary. Resources of this type can be transformed to 2D cluster state. To see this, one first isolates 1D chain states from it, use the reductions we already have for the 1D case to obtain 1D cluster states, and then employ an appropriate coupling to link the 1D cluster states into a two-dimensional cluster state. The first two steps are obvious, while the third step is possible as shown in Fig. 3. In this figure,  $A[x]$ 's are defining matrices of cluster state chosen to be  $A[0] = |+\rangle\langle 0|$ ,  $A[1] = |-\rangle\langle 1|$ , and  $B, C$  are tensors in the notion of Refs. [7,8]

$$\begin{aligned} B[0] &= |+\rangle_r\langle 0|_l \otimes |+\rangle_d, & B[1] &= |-\rangle_r\langle 1|_l \otimes |-\rangle_d, \\ C[0] &= |+\rangle_r\langle 0|_l \otimes |0\rangle_u, & C[1] &= |-\rangle_r\langle 1|_l \otimes |1\rangle_u. \end{aligned}$$

The Hadamard gates and the CZ gate  $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$  are implemented on the correlation spaces. The right-hand side of Fig. 3 represents two nodes of degree 3 of a cluster state.

In the weighted graph state, for example, the isolated wire has the form  $HZ^x S^z$ , where  $z$  depends on the outcomes of neighboring sites [7,8]. We can measure all sites with odd  $z$  in the 0-1 basis and the resulting state is a 1D cluster state up to random Clifford by-product operations. The CZ gate can be applied in the same way as in Ref. [8]. Other coupling based schemes mentioned above can be analyzed in a similar way.

**Discussions.**—The method of reduction for proving universality of the MBQC resource state is applicable to other examples that have not been covered in the previous sections. These include, for example, the modified AKLT state  $(H, X, Y)$  proposed in Ref. [7] and the second toric code state example in Ref. [8].

It is worth comparing the idea of reduction and that of the “universal state preparator” result of Ref. [24]. A reduction to cluster state would imply the universal preparator property of the resource. On the other hand, although any universal preparator could in principle be transformed to a cluster state, the transformation does not respect the underlying lattice and may be less efficient than quantum state reduction.

The tabular form we propose is well hinged to the structure and properties of MPS. It simplifies the analysis by hiding the unwanted details and provides an intuitive way of manipulating the matrices. We have mainly investigated reductions of MPS and projected entangled pair

state resources, but the idea seems to be able to generalize to a potentially new MBQC scheme not known yet. It is also reasonable to believe that investigations of the reduction method will improve our understanding of both the MBQC itself and the structure of universal resource states.

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