ORDERINGS AND BOOLEAN ALGEBRAS NOT ISOMORPHIC TO RECURSIVE ONES

by

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Abstract

Chapter I: With each linear ordering, $\mathcal{L}$, with first and last element we can associate a Boolean algebra $D_{\mathcal{L}}$. We discuss a Cantor-Bendixon like classification of order types due to Erdős and Hajnal [1], whereby with each order type $\mathcal{L}$ and each ordinal ν we associate the ν^{th} derived linear ordering, $\mathcal{L}^{(\nu)}$. $\partial(\mathcal{L})$ is defined to be the least ordinal, , for which $\mathcal{L}^{(\nu)} = \mathcal{L}^{(\nu+1)}$. We discuss a similar classification of Boolean algebra isomorphism types due to Mostowski and Tarski [2], whereby with each Boolean algebra isomorphism type, B, and each ordinal, ν, we associate the ν^{th} derived Boolean algebra isomorphism type, $B^{(\nu)}$. $\delta(B)$ is defined to be the least ordinal, ν, such that $B^{(\nu)} = B^{(\nu+1)}$. We show that, for any linear ordering with first and last element, $\mathcal{L}$, and any ordinal, ν, $(D_{\mathcal{L}})^{(\nu)}$ is isomorphic to $D_{\mathcal{L}}(\nu)$. We use this fact to give non-topological proofs of some standard properties of Boolean algebras.

Chapter II: We discuss countable Boolean algebras. We define strict Boolean algebras and we discuss Lindenbaum algebras of Theories. We define what is meant by a Π^1_1, (Σ^1_1, Δ^1_1, arithmetic, recursive) Boolean algebra.

Chapter III: We show that, for any subset X of the natural numbers, there is a partial ordering, $\mathcal{P}$, such that $\mathcal{P}$ is

r.e. in X but not isomorphic to a X-recursive partial ordering; there is a linear ordering $\mathcal{L}$, such that $\mathcal{L}$ is Π_1^0 in X, but $\mathcal{L}$ is not isomorphic to any X-recursive linear ordering; and there is an H_ω^X-recursive Boolean algebra, B, such that B is not isomorphic to any X-arithmetic Boolean algebra.

Chapter IV: We show that any Σ_1^1 strict Boolean algebra with a scattered base is isomorphic to a recursive Boolean algebra. We show that there is a Π_1^1 strict Boolean algebra, B, with a scattered base such that B is not isomorphic to any Σ_1^1 Boolean algebra. We show that if B is the Lindenbaum algebra of a Π_1^1 axiomatizable theory, then $\delta(B) \leq \omega_1^{\text{Kleene}}$.

However, there is a Lindenbaum algebra B of a Σ_1^1-axiomatizable theory such that $\delta(B) > \omega_1^{\text{Kleene}}$.

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NOTATION

N is the set of natural numbers. $\mathcal{N} = \{N, 0, ', +, .\}$ is the standard model for arithmetic. $J(x,y)$ is the standard pairing function which associates ordered pairs with natural numbers. $J^{-1}(x) = (K(x), L(x)) \epsilon N^2$. p_n is the $(n+1)$-th prime. s is a sequence number, denoted by seq (s), iff $(x)(p_x$ divides $s \longrightarrow (y)_{\leq x}$ $(p_y$ divides $s))$. If seq (s), then $\ell(s) = n$, iff n is the greatest number such that p_n divides s, and $y = s(i)$, iff $y + 1$ is the greatest number, r, such that p_i^r divides s.

$H_1 =$ the complete set $H_{n+1} = H_n'$, for $n \geq 1$. $H_\omega = \{i | K(i) \epsilon H_{L(i)}\}$. $H_1^X = X'$. $H_{n+1}^X = (H_n^X)'$, for $n \geq 1$. $H_\omega^X = \{i | K(i) \epsilon H_{L(i)}^X\}$. If $\mathcal{L}$ and $\mathcal{L}'$ are linear orderings (partial orderings), and $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, we write $\mathcal{L} \approx \mathcal{L}'$. We say "e is the Gödel number for a recursive linear ordering", or $e \epsilon Li$ iff there is a relation, $\mathfrak{R}(x,y)$, whose field is N, such that $\mathfrak{R}(x,y)$ is a linear ordering, and $(x)(y)$ $(\mathfrak{R}(x,y) \longleftrightarrow \{e\}(x,y) = 0)$. Since every r.e. linear ordering is isomorphic to a recursive linear ordering (See Crossley [i]), we consider only recursive linear orderings. Ω is the least uncountable ordinal, and ω_1 is the least non-recursive ordinal.

If B is a Boolean algebra, we say that B is <u>strict</u>, iff $(a)_{\epsilon B} (b)_{\epsilon B} (a \equiv b \longrightarrow a = b)$. If $f:B \longrightarrow B'$, we say that f is a <u>Boolean homomorphism</u>, iff for every a, b, c ϵ B:

(i) $a \cup b \equiv c \longrightarrow f(a) \cup f(b) \equiv f(c)$;

(ii) $a \cap b \equiv c \longrightarrow f(a) \cap f(b) \equiv f(c)$;

(iii) $a \equiv \overline{b} \longrightarrow f(a) \equiv \overline{f(b)}$;

(iv) $a \leq b \longrightarrow f(a) \leq f(b)$

Furthermore, if, in addition to (i)-(iv), we have $(a)_{\epsilon B}(b)_{\epsilon B} (f(a) \leq f(b) \longrightarrow a \leq b)$, and $(a)_{\epsilon B'} (\exists b)_{\epsilon B}(f(b) \equiv a)$, we say that f is a Boolean isomorphism, and we write $B \approx B'$. If B is a Boolean algebra and B has Π^1_1 (Σ^1_1, Δ^1_1, recursive) field, operations, and relation, we say that B is Π^1_1 (Σ^1_1, Δ^1_1, recursive).

$a^{(1)}$ or $^{(2)}$ written before a logical w.f.f. is not a reference to a footnote, but is a device used to refer to the formula later on.

INTRODUCTION

In this paper we study countable structures which are not isomorphic to a structure whose field, operations and relations are recursive. Similar known results include:

(i) the construction of a sentence with no r.e. models; and

(ii) the proof that no non-standard model of arithmetic is r.e.

Most of the constructions of a sentence with no r.e. models reduce to the fact that Von Neumann Bernays set theory has no r.e. models. A very nice proof of this fact can be found in Rabin [ii]. The proof that there are no r.e. non-standard models of arithmetic is due to Tenenbaum [iii].

However, in this paper we discuss partial orderings, linear orderings and Boolean algebras. The problem which we discuss is that of finding a partial ordering (linear ordering, Boolean algebra) at level Θ, say, of the Hensel-Putnam Hierarchy [iv] which is not isomorphic to any partial ordering (linear ordering, Boolean algebra) at any level $\Theta' < \Theta$.

In Chapter III we use a method of "Coding functions into the isomorphism type of an ordering" to show that for any subset of the natural numbers, there are:

(i) partial orderings which are r.e. in X, which are not isomorphic to any partial ordering recursive in X.

(ii) linear orderings which are Π^0_1 in X which are not isomorphic to any linear ordering which is recursive in X; and

(iii) Boolean algebras whose field operations and relations are recursive in H^X_ω which are not isomorphic to any Boolean algebra whose field, relation and operations are arithmetic in X.

We prove (i) and (ii) by constructing an X - r.e. partial ordering (X-Π^0_1 linear ordering) which is not elementarily equivalent to any X-recursive partial ordering (X-recursive linear ordering). We prove (iii) by constructing a Boolean algebra whose operations and relations are recursive in H^X_ω, but which is not elementarily equivalent in the weak second order theory (see pp. 54-55) to any Boolean algebra whose operations and relation are X-recursive.

For any ordinal $\theta < \omega$, if we let $X = H_\theta$, we observe that, by (i) and (ii), there is a partial ordering (linear ordering) at level $\theta + 1$ of the Hensel-Putnam hierarchy which is not isomorphic to any partial ordering (linear ordering) at level θ. We can modify the proof of (iii) to show that, if $\lambda < \omega_1$ and λ is a limit, then there is a Boolean algebra at level λ which is not isomorphic to any Boolean algebra at any level $\theta < \lambda$. This fact is not proved explicitly in this paper.

In Chapter I, we use results of Erdős and Hajnal [1] and Tarski and Mostowski [2] to give non-topological proofs of some classical properties of Boolean algebras. These properties when dualized via the Stone representation theorem, [v] become well known theorems of 0-dimensional topology. In particular, we assign a rank $\delta(B)$ to each Boolean algebra B. In chapter IV, we perform a constructive analysis of δ by means of the analytic hierarchy to obtain:

(i) If B is Σ_1^1 strict Boolean algebra with a scattered base (see Chapter I, p.11 of this paper.) then B is isomorphic to a recursive Boolean algebra; and

(ii) there is a Π_1^1 strict Boolean algebra with a scattered base which is not isomorphic to a Σ_1^1 Boolean algebra.

We also prove:

(i) If B is a Σ_1^1 strict Boolean algebra, then $\delta(B) \leq \omega_1$. However, there is a Π_1^1 strict Boolean algebra such that $\delta(B) > \omega_1$;

(ii) If B is the Lindenbaum algebra of a Π_1^1-axiomatizable theory, then $\delta(B) \leq \omega_1$. However, there is a Lindenbaum algebra, B, of a Σ_1^1-axiomatizable theory, such that $\delta(B) > \omega_1$.

BIBLIOGRAPHY FOR INTRODUCTION AND NOTATION

i. J. N. Crossley <u>Constructive Order Types I</u>, Formal Systems and Recursive Functions, Amsterdam North-Holland Publishing Company, 1965, c.p. 198.

ii. M. Rabin <u>On Recursively Enumerable and Arithmetic Models of Set Theory</u>, J. S. L., <u>23</u>, (1958) 408-416.

iii. S. Tenenbaum <u>Non-archimedean Models for Arithmetic</u> Amer. Math. Soc. Notices, <u>6</u> (1959) c.p. 270.

iv. G. Hensel and H. Putnam <u>On the Notational Independence of Various Hierarchies of Degrees of Unsolvability</u>, J. S. L., <u>30</u>, (1965) 69-87.

v. M. Stone <u>The Theory of Representations for Boolean Algebras</u> Trans. Amer. Math., <u>40</u>, (1936) 37-111.

CHAPTER I

The Erdős-Hajnal classification of denumerable order types, and the Tarski-Mostowski classification of Boolean Algebras with ordered bases

§1. ∂

Let $\mathcal{L} = \{L, \prec\}$ be a linear ordering. If S is a subset of L, we say S is dense in $\mathcal{L}$ iff S contains more than two members, and, for every $a, b \in S$, $a \prec b$ implies there is a $c \in S$ such that $a \prec c \prec b$. If L has no dense subsets, then $\mathcal{L}$ is said to be scattered. A subset $S \subset L$ is said to be a segment of $\mathcal{L}$ iff, for every $a, b, c \in L$, $(a \prec c \prec b \;\&\; a \in S \;\&\; b \in S) \longrightarrow c \in S$. If two linear orderings are isomorphic, they are said to be of the same order type. In Chapter I, we will usually make no distinction between linear orderings and order types. However, in Chapters II, III, and IV, the actual presentation of an order type will become important. η is the order type of any countable dense ordering with no first and no last element. A point $a \in L$ is said to be isolated iff, $\{x \in L \mid a \prec x\}$ has a $\prec$- least member. If $\mathcal{L}$ is denumerable and has no isolated points, then $\mathcal{L}$ is one of the types $1, 1 + \eta, 1 + \eta + 1, \eta + 1, \eta$.

The following definition is due to Erdős and Hajnal [1]

Definition 1.1: Let $\mathcal{L} = \{L, \prec\}$ be a linear ordering. For each ordinal τ, we define an equivalence relation, $\equiv^{\mathcal{L}}_{\tau}$ on L as follows: If $\tau = 0$, then $a \equiv^{\mathcal{L}}_{0} b$ iff $a = b$.

Suppose that $\tau > 0$, and that $\equiv^{\mathcal{L}}_{\nu}$ has been defined for every $\nu < \tau$. Furthermore, suppose that, for every $a, b \in L$, and every $\nu' < \nu < \tau$:

(i) $\equiv^{\mathcal{L}}_{\nu}$ is an equivalence relation on L;

(ii) $a \equiv^{\mathcal{L}}_{\nu'} b \longrightarrow a \equiv^{\mathcal{L}}_{\nu} b$;

(iii) $(a \prec c \prec b \ \& \ a \equiv^{\mathcal{L}}_{\nu} b) \longrightarrow a \equiv^{\mathcal{L}}_{\nu} c$.

Let ν be an ordinal $< \tau$. If $b \in L$, define $[b]_{\nu}$ to be the equivalence class of b under the relation $\equiv^{\mathcal{L}}_{\nu}$. Define $[a]_{\nu} \prec_{\nu} [b]_{\nu}$ to mean $a \prec b \ \& \ a \not\equiv^{\mathcal{L}}_{\nu} b$.

Observe that $\mathcal{L}^{(\nu)} = \{\{[b]_{\nu}\}_{b \in L}, \prec_{\nu}\}$ is a linear odering. $\mathcal{L}^{(\nu)}$ is called the ν^{th} derived linear ordering of $\mathcal{L}$. If τ is a limit define $a \equiv^{\mathcal{L}}_{\tau} b$ to mean $(\exists \nu)(\nu < \tau \ \& \ a \equiv^{\mathcal{L}}_{\nu} b)$. If, for some ν, $\tau = \nu + 1$,

define $a \equiv^{\mathcal{L}}_{\tau} b$ to mean

"$\{x \in \{[c]_{\nu}\}_{c \in L} \mid [a]_{\nu} \prec_{\nu} x \prec_{\nu} [b]_{\nu} \text{ or } [b]_{\nu} \prec_{\nu} x \prec_{\nu} [a]_{\nu}\}$

is finite"

In either case observe that (i), (ii) and (iii) hold for every $a, b, c \in L$, every $\nu' < \nu < \tau+1$.

Let $\partial(\mathcal{L})$ be the least ordinal τ such that, for every $a \in L$, $b \in L$, $a \equiv^{\mathcal{L}}_{\tau} b \longleftrightarrow a \equiv^{\mathcal{L}}_{\tau+1} b$. $\partial(\mathcal{L})$ is the least ordinal τ, such that $\mathcal{L}^{(\tau)}$ has no isolated points. $\partial(\mathcal{L})$ is an invariant of the order type of $\mathcal{L}$.

Example: If $\mathcal{L}$ = the reals, then $\partial(\mathcal{L}) = 0$. If $\mathcal{L} = \omega$, then $\partial(\mathcal{L}) = 1$. If $\mathcal{L} = \omega + 1$, then $\partial(\mathcal{L}) = 2$. $(\omega^2)^{(1)} = \omega$. $\partial(\omega^3) = 3$. ||

The following facts can be found in [1]. We shall list them here, and sketch the proof in some cases.

Lemma 1.2: If $\mathcal{L}$ is a denumerable linear ordering, then $\partial(\mathcal{L}) < \Omega$.

Lemma 1.3: If $\mathcal{L}$ is scattered, then $(\mathcal{L})^{\partial(\mathcal{L})} = 1$. If $\mathcal{L}$ has a dense subset, then $(\mathcal{L})^{\partial(\mathcal{L})}$ is dense.

Proof: Observe that, for any ν, $\mathcal{L}$ has a dense subset, iff $\mathcal{L}^{(\nu)}$ has a dense subset. If $\mathcal{L}^{(\nu)}$ has a dense subset, it is not difficult to see that $\mathcal{L}$ must also have a dense subset. Conversely, if S is a dense subset of $\mathcal{L}$, we can prove by induction on ν, that for any a, b, ϵ S, $a \not\equiv^{\mathcal{L}}_{\nu} b$. This shows that $\{[a]_\nu\}_{a \epsilon S}$ is a dense subset of $\mathcal{L}^{(\nu)}$. ||

The following facts are not stated explicitly in [1], but we shall need them later so we list them here.

Lemma 1.4: Let S be a segment of $\mathcal{L}$. Let $\mathcal{L} \upharpoonright S$ be $\mathcal{L}$ restricted to S. If a, b ϵ S then, for every ν, $a \equiv^{\mathcal{L}}_{\nu} b \longleftrightarrow a \equiv^{\mathcal{L}\upharpoonright S}_{\nu} b$. Therefore, $\partial(\mathcal{L} \upharpoonright S) \leq \partial(\mathcal{L})$.

Lemma 1.5: If $\mathcal{L}$ is scattered, then $\partial(\mathcal{L})$ is the least ν such that $\mathcal{L}^{(\nu)} = 1$. In other words, $\partial(\mathcal{L})$ is the least ν such that, for every $a, b \in \mathcal{L}$, $a \overset{\mathcal{L}}{\underset{\nu}{\equiv}} b$.

Lemma 1.6: If $\mathcal{L}$ is scattered and has a greatest and a least element, then $\partial(\mathcal{L})$ isn't a limit.

Proof: Let a and b be the least and greatest elements of $\mathcal{L}$ respectively. $\mathcal{L}^{(\partial(\mathcal{L}))} = 1$. Therefore $a \overset{\mathcal{L}}{\underset{\partial(\mathcal{L})}{\equiv}} b$. If $\partial(\mathcal{L})$ is a limit, there is a $\nu < \partial(\mathcal{L})$, such that $a \overset{\mathcal{L}}{\underset{\nu}{\equiv}} b$. However, for any element c of $\mathcal{L}$, $a \preceq c \preceq b$. Therefore $a \overset{\mathcal{L}}{\underset{\nu}{\equiv}} c$, for any element c of $\mathcal{L}$. Thus, $\mathcal{L}^{(\nu)} = 1$, and $\partial(\mathcal{L}) \leq \nu$. Contradiction. ||

Theorem 1.7: If $\mathcal{L}$ is scattered, then $\partial((\mathcal{L}\cdot\omega) + 1) > \partial(\mathcal{L})$.

Proof: Let $\hat{\mathcal{L}} = (\mathcal{L}\cdot\omega) + 1$. Let $\mathcal{L}_i$ be the ith copy of $\mathcal{L}$ in $\hat{\mathcal{L}}$, and let p be the greatest element of $\hat{\mathcal{L}}$. We wish to prove that, for any $a \in \hat{\mathcal{L}}$, and for any ordinal $\tau \leq \partial(\mathcal{L})$, $p \neq a \longrightarrow p \overset{\hat{\mathcal{L}}}{\underset{\tau}{\not\equiv}} a$. The fact is obvious for $\tau = 0$. Suppose we have proved it for $\nu < \tau \leq \partial(\mathcal{L})$. If τ is a limit, then $p \overset{\hat{\mathcal{L}}}{\underset{\tau}{\equiv}} a$ implies $(\exists\nu)(\nu < \tau\ \&\ p \overset{\hat{\mathcal{L}}}{\underset{\nu}{\equiv}} a)$. Contradiction. Suppose τ is a successor. $\tau = \nu + 1$. Furthermore, suppose that $p \overset{\hat{\mathcal{L}}}{\underset{\tau}{\equiv}} a$, for some $a \neq p$. This means that $[[a]_\nu, [p]_\nu)$ contains a finite number of elements, $[q_1]_\nu \ldots [q_n]_\nu$, where $[q_n]_\nu$ is the greatest of these elements. $q_n \in \mathcal{L}_j$, for some j. Choose $q_{n+1} \in \mathcal{L}_{j+2}$. We claim that

$q_n \not\equiv^{\hat{\mathcal{L}}}_{\nu} q_{n+1} \not\equiv^{\hat{\mathcal{L}}}_{\nu} p$. If $q_n \equiv^{\hat{\mathcal{L}}}_{\nu} q_{n+1}$, then for any $x, y \in \mathcal{L}_{j+1}$, $x \equiv^{\hat{\mathcal{L}}}_{\nu} y$. Thus, by I-1.4, $\partial(\mathcal{L}_{j+1}) \leq \nu < \partial(\mathcal{L})$. Contradiction. By our inductive hypothesis, $q_{n+1} \not\equiv^{\hat{\mathcal{L}}}_{\nu} p$. Therefore, $[q_{n+1}]_\nu$ is greater than $[q_n]_\nu$, but less than $[p]_\nu$, which contradicts the fact that $[q_n]_\nu$ is the greatest element of $[[a]_\nu, [p]_\nu)$.

We have proved that $((\mathcal{L}\cdot\omega) + 1)^{\partial(\mathcal{L})}$ contains at least two elements. Since $\mathcal{L}$ is scattered, $(\mathcal{L}\cdot\omega) + 1$ is scattered, and by I-1.5, $\partial(\mathcal{L}) < \partial((\mathcal{L}\cdot\omega) + 1)$. ||

Definition 1.8: We define ω^τ as follows:

$\omega^0 = 1$;

if τ is a limit, $\omega^\tau = \lim\limits_{\nu<\tau} \omega^\nu$;

if $\tau = \nu + 1$, $\omega^\tau = \omega^\nu\cdot\omega$.

An ordinal is called a principal number for addition if it is not a finite sum of lesser ordinals.

Lemma 1.9: ω^α is the α^{th} principal number for addition.

Proof: Straightforward induction on α. ||

Theorem 1.10: For any τ:

(i) $\partial(\omega^\tau) = \tau$;

(ii) $\partial(\omega^\tau + 1) > \partial(\omega^\tau)$.

Proof: The theorem is trivial for $\tau = 0$. Suppose that for any $\nu < \tau$, $\partial(\omega^{\nu}) = \nu$ and $\partial(\omega^{\nu} + 1) > \partial(\omega^{\nu})$. We wish to prove that $\partial(\omega^{\tau}) = \tau$, and $\partial(\omega^{\tau} + 1) > \partial(\omega^{\tau})$.

Case (i): τ is a limit.

Choose any $\theta < \omega^{\tau}$. By I-1.8, there is a $\beta < \tau$ such that $\theta \leq \omega^{\beta} < \omega^{\tau}$, $\partial(\omega^{\beta}) = \beta < \alpha$. Therefore, $0 \equiv^{\omega}_{\beta} \theta$, which implies that $0 \equiv^{\omega}_{\tau} \theta$. This shows that $(\omega^{\tau})^{(\tau)} = 1$. Thus, $\tau \geq \partial(\omega^{\tau})$.

Now, suppose that $\nu < \tau$. $\omega^{\nu} < \omega^{\tau}$. Since τ is a limit, $\nu + 1 < \tau$, and, therefore $\omega^{\nu+1} < \omega^{\tau}$. $\partial(\omega^{\nu+1}) = \nu + 1 > \nu$. Because $\omega^{\nu+1}$ is a segment of ω^{τ}, $\partial(\omega^{\tau}) > \nu$, by I-1.4. Therefore $\partial(\omega^{\tau}) = \tau$. By I-1.6, $\partial(\omega^{\tau}+1)$ must be a successor. Since τ is a limit, $\partial(\omega^{\tau} + 1) > \partial(\omega^{\tau})$.

case (ii): $\tau = \nu + 1$

Consider a linear ordering $\mathcal{L}$ of order type $\omega^{\nu}\cdot\omega$. Let $a_i \in \mathcal{L}$ be the first element of the i^{th} copy of ω^{ν}. Let p be the greatest element of $\mathcal{L}$. By our inductive hypothesis, $\partial(\omega^{\nu} + 1) > \partial(\omega^{\nu})$ and $\partial(\omega^{\nu}) =$ and $(\omega^{\nu})^{(\nu)} = 1$. Using this fact one can easily prove that:

(i) for any $i \geq 0$, $a_i \not\equiv^{\mathcal{L}}_{\nu} a_{i+1}$;

(ii) for any element c of $\mathcal{L}$, if $c \neq p$, then, $c \equiv^{\mathcal{L}}_{\nu} a_i$, where i is the greatest integer such that a_i is less than c;

(iii) for any $c \neq p$, $p \not\equiv^{\mathcal{L}}_{\nu} c$.

This shows that $(\omega^{\nu+1})^{(\nu)} = \omega$ and $(\omega^{\nu+1} + 1) = \omega + 1$. Thus $(\omega^{\nu+1})^{(\nu+1)} = 1$ and $(\omega^{\nu+1} + 1)^{(\nu+1)} = 2$. Therefore $\partial(\omega^{\nu+1}) = \nu+1$ and $\partial(\omega^{\nu+1} + 1) > \nu + 1$. ||

<u>Theorem 1.11</u>: If $\mathcal{L}$ is a denumerable linear ordering, then:

(i) $\mathcal{L} = \sum_{r \in \hat{\eta}} \mathcal{L}_r$, where each $\mathcal{L}_r$ is scattered and $\hat{\eta}$ is one of the types 1, $1+\eta+1$, $\eta+1$, η, $1+\eta$;

(ii) $\partial(\mathcal{L}) = \ell\cdot u\cdot b.\ \{\partial(\mathcal{L}_r) \mid r \in \hat{\eta}\}$.

<u>proof</u>: (i) See [1, p.119].

(ii) Straightforward application of results stated previously.

§2.

Let B be a Boolean algebra. Let I be an ideal of B. There is a natural Boolean homomorphism $i:B \longrightarrow B/I$. If $a \in B$, we define $|a|_I$ to be $i(a)$. The set, $I_1^B \subseteq B$, of elements bounded by a finite union of atoms is easily seen to be an ideal of B. B/I_1^B is called the <u>first derived algebra of B</u>. If $f:B \longrightarrow B'$ is a Boolean homomorphism, then f is called a Boolean <u>monomorphism</u> iff $(a)_{\epsilon B}(f(a) \equiv 0 \longleftrightarrow a \equiv 0)$.

<u>Lemma 2.1</u>: If $f:B \longrightarrow B'$ is a Boolean monomorphism, then:

(i) $(a)_{\epsilon B}\,(b)_{\epsilon B}\,(a \leq b \longleftrightarrow f(a) \leq f(b))$;

(ii) $f^{-1}(I_1^{B'}) \subseteq I_1^B$.

<u>Lemma 2.2</u>: Let B be a Boolean algebra and let b be an element of B, then $b \notin I_1^B$, iff b bounds χ_0 disjoint elements of B.

<u>Definition 2.3</u>: (Tarski-Mortowski [3]). If B is a Boolean algebra, then for each ordinal τ we define an ideal, $I_\tau^B \subseteq B$ as follows: $I_0^B = \{a | a \epsilon B \ \& \ a \equiv 0\}$. If ν is a limit, then $I_\tau^B = \bigcup_{\nu<\tau} I_\nu^B$. If $\tau = \nu + 1$, then

$$I_\tau^B = \{a | a \epsilon B \ \& \ |a|_{I_\nu^B} \in I_1^{(B/I_\nu^B)}\}.$$

For all ν, let $|a|_\nu$ be $|a|_{I^B_\nu}$, and let $B^{(\nu)}$ be B/I^B_ν. Let $\delta(B)$ be the least ν such that $I^B_\nu = I^B_{\nu+1}$. $\delta(B)$ is the least ν such that $B^{(\nu)}$ is atomless.

Examples: If B is finite, then $\delta(B) = 1$. If B is the algebra of all finite-cofinite subsets of the integers, then $\partial(B) = 2$. If B is atomless, then $\delta(B) = 0$.

Lemma 2.4: If $f:B \longrightarrow B'$ is a monomorphism, then, for any ordinal ν, $f^{-1}(I^{B'}_\nu) \subseteq I^B_\nu$.

Proof: Do induction on ν, using I-2.1. ||

In the study of the elementary properties of Boolean Algebras, one very useful source of examples is the class of interval algebras. If $\mathcal{L} = \{L, \prec\}$ is a linear ordering with first and last element, then the interval algebra, $D_\mathcal{L}$, of $\mathcal{L}$ is the set

$\{S | S \subseteq L$ & $S = [a_1, b_1) \cup \ldots \cup [a_n, b_n)$,

for some finite sequence, $a_1 \prec b_1 \prec \ldots \prec a_n \prec b_n$, of elements of L$\}$. Let b be the greatest element of $\mathcal{L}$. It is easily seen that $D_\mathcal{L}$ is a subalgebra of the power set of $L - \{e\}$. If $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, then $D_\mathcal{L}$ is isomorphic to $D_{\mathcal{L}'}$. However, the converse if false. For example, $D_{\omega+1}$ is isomorphic to $D_{1+\omega^*+1}$.

Definition 2.5: A subset $S \subseteq B$ is called an ordered basis of B iff:

(i) the ordering of B restricted to S is total, that is for any s, t in S either $s \leq t$, or $t \leq s$;

(ii) S generates B, that is any element of B is a Boolean combination of elements of S.

A strict ordered basis, S, of B is an ordered basis of B such that for any $a, b \in S$, $a \not\equiv b$. For any ordered basis, S, of B, there is a strict ordered basis $\hat{S}$ of B such that $\hat{S} \subseteq S$. If $\mathcal{L}$ is a linear ordering with first and last element a and b, respectively, then $\{[a,x) | x \in \mathcal{L}\}$ is a strict ordered basis of $\mathcal{L}$. Conversely, if B has a strict ordered basis, $\hat{S}$, of order type $\mathcal{L}$, then B is isomorphic to $D_{\mathcal{L}}$. (See [2])

We say that a Boolean algebra has a scattered basis iff it has a strict ordered basis which is scattered.

Lemma 2.6: Let $I \subseteq B$ be an ideal of B and let $S \subseteq B$ be a subset of B. If S generates B, then $\{|a|_I \mid a \in S\}$ generates B/I. If S is an ordered basis of B, then $\{|a|_I \mid a \in S\}$ is an ordered basis of B/I.

We now turn to the main result of this Chapter. As far as the author can tell it has not appeared in the literature, though consequences of it can be found in [2] and Mayer and

Pierce [4].

Theorem 2.7: If $\mathfrak{L}$ is a linear ordering with first and last element, then, for any ordinal ν, $(D_{\mathfrak{L}})^{(\nu)}$ is isomorphic to $D_{\mathfrak{L}}(\nu)$.

This theorem will be an immediate consequence of the following lemma.

Lemma 2.8: Let $B = D_{\mathfrak{L}}$, where $\mathfrak{L}$ has first and last element. For any $a, b \in \mathfrak{L}$, any ordinal ν, $a \equiv^{\mathfrak{L}}_{\nu} b \longleftrightarrow [a,b) \in I^{B}_{\nu}$.

Proof: If $\nu = 0$, then $a \equiv^{\mathfrak{L}}_{0} b \longleftrightarrow [a,b) = \emptyset$. Suppose the lemma is true for $\nu < \tau$. We prove it for $\nu = \tau$.

case (i): τ is a limit.

$$a \equiv^{\mathfrak{L}}_{\tau} b \longleftrightarrow (\exists \nu)(\nu < \tau \ \& \ a \equiv^{\mathfrak{L}}_{\nu} b) \longleftrightarrow$$

$$(\exists \nu)(\nu < \tau \ \& \ [a,b) \in I^{B}_{\nu}) \longleftrightarrow [a,b) \in I_{\tau} .$$

case (ii): $\tau = \nu + 1$.

Suppose $a \not\equiv^{\mathfrak{L}}_{\nu} b$. By I-1.1, there is an infinite sequence $\{c_i\}$ of elements of $\mathfrak{L}$, such that:

(i) for every i, $c_i \not\equiv^{\mathfrak{L}}_{\nu} c_{i+1}$;

(ii) either $(i)(c_i \prec c_{i+1})$ or $(i)(c_{i+1} \prec c_i)$;

(iii) $\{c_i\} \subseteq [a,b)$.

By our induction hypothesis, for every i, $[c_i, c_{i+1}) \cup [c_{i+1}, c_i) \notin I^{B}_{\nu}$. Therefore, $|[a,b)|_{\nu}$ bounds $\aleph_0$ disjoint non-zero elements of B/I_ν^B, and, by I-2.3, $|[a,b)|_\nu$ is not a member of I_1^{B/I_ν^B}.

Conversely, suppose $a \equiv_\tau^{\mathfrak{L}} b$. That is, there is a finite sequence $a \prec c_1 \prec c_2 \prec c_3 \prec \ldots \prec c_n \prec b$, such that $a = c_1 \not\equiv_\nu^{\mathfrak{L}} c_2 \not\equiv_\nu^{\mathfrak{L}} c_3 \not\equiv_\nu^{\mathfrak{L}} \ldots \not\equiv_\nu^{\mathfrak{L}} c_n = b$, and for any element z of $\mathfrak{L}$, $a \leq z \leq b$ implies $z \equiv_\nu^{\mathfrak{L}} c_i$, for some $1 \leq i \leq n$. $[a,b) = [c_1, c_2) \cup \ldots \cup [c_{n-1}, c_n)$. $|[a,b)|_\nu = |[c_1,c_2)|_\nu \cup \ldots \cup |[c_{n-1},c_n)|_\nu$. We now use the induction hypothesis to show that $|[c_i,c_{i+1})|_\nu$ is an atom, for every $1 \leq i < n$. Let $[z_1,z_2)$ be an interval such that $[z_1,z_2) \subseteq [c_i,c_{i+1})$. We say that $[z_1,z_2)$ "fills" $[c_i,c_{i+1})$ iff, $z_1 \equiv_\nu^{\mathfrak{L}} c_i$ and $z_2 \equiv_\nu^{\mathfrak{L}} c_{i+1}$. By our induction hypothesis, if $[z_1,z_2)$ fills $[c_i,c_{i+1})$ then $[c_i,c_{i+1}) - [z_1,z_2) \in I_\nu^B$, and if $[z_1,z_2)$ does not fill $[c_i,c_{i+1})$, then $[z_1,z_2) \in I_\nu^B$. Let ξ be a member of $D_{\mathfrak{L}}$ such that $\xi \subseteq [c_i,c_{i+1})$. By our induction hypothesis, $[c_i,c_{i+1}) - \xi \in I_\nu^B$ or $\xi \in I_\nu^B$, depending on whether or not there exists an interval of ξ which fills $[c_i,c_{i+1})$. Thus $|[c_i,c_{i+1})|_\nu$ is an atom. ||

To conclude the proof of II-2.7, we note that, by II-2.6, $\{|[0,a)|_\nu\}_{a \in \mathfrak{L}}$ is an ordered basis for $B^{(\nu)}$. Thus II-2.8 and II-2.5, $B^{(\nu)}$ is isomorphic to $D_{\mathfrak{L}(\nu)}$. ||

From now on when we write $D_{\mathfrak{L}}$, we shall assume that

$\mathcal{L}$ has first and last elements.

Lemma 2.9: $D_{\mathcal{L}}$ is atomless, iff $\mathcal{L}$ has no isolated points.

Corollary 2.10: $\partial(\mathcal{L}) = \delta(D_{\mathcal{L}})$.

proof: See I-1.11. ||

Corollary 2.11: There are $\aleph_1$ isomorphism types of countable Boolean Algebras.

Proof: If $B \approx B'$, then $\delta(B) = \delta(B')$. If $\tau < \tau'$, then $\partial(\omega^{\tau}+1) < \partial(\omega^{\tau'}+1)$, and, by I-2.10, $D_{\omega^{\tau}+1} \not\approx D_{\omega^{\tau'}+1}$. ||

Remark: See [4, pp. 937-938] for a topological proof of this fact.

The following facts will be useful in Chapter III.

Lemma 2.12: $\mathcal{L}$ is scattered, iff $(D_{\mathcal{L}})^{\partial(\mathcal{L})} \approx D_1$. Thus, if B has a scattered ordered basis, then every basis of B is scattered.

Proof: Follows immediately from II-2.7 and I-1.3.

Lemma 2.13: If $f: D_{\mathcal{L}} \to D_{\mathcal{L}'}$ is a monomorphism and $\mathcal{L}'$ is scattered, then so is $\mathcal{L}$.

Let B be $D_{\mathcal{L}}$ and B' be $D_{\mathcal{L}'}$. Let i be the natural homomorphism from B' into $B'/I^{B'}_{\partial(\mathcal{L}')}$. $i \circ f$ induces a monomorphism from $B/f^{-1}(I^{B'}_{\partial(\mathcal{L}')})$ into $B'/I^{B'}_{\partial(\mathcal{L}')}$. However, $(B')^{\partial(\mathcal{L}')} = D_1$. Therefore $B/f^{-1}(I^{B'}_{\partial(\mathcal{L}')}) \approx D_1$. By I-2.1, $f^{-1}(I^{B'}_{\partial(\mathcal{L}')}) \subseteq I^{B}_{\partial(\mathcal{L}')}$, $B/I^{B}_{\partial(\mathcal{L}')} \approx D_1$. Thus, $\partial(\mathcal{L}) \leq \partial(\mathcal{L}')$, and $(B)^{\partial(\mathfrak{L})} \approx D_1$. By I-2.12, $\mathfrak{L}$ must be scattered.

Lemma 2.14: If Θ and Θ' are ordinals and $\Theta < \Theta'$, then there is a monomorphism from $D_{\Theta+1}$ into $D_{\Theta'+1}$.

Proof: Let $\hat{f}$ be an order preserving map from Θ onto an initial segment of Θ'. Let f be an order preserving map from $\Theta + 1$ into $\Theta' + 1$ such that $f \restriction \Theta = \hat{f}$ and $f(\Theta+1) = \Theta'+1$. f induces a monomorphism from $D_{\Theta+1}$ into $D_{\Theta'+1}$. ||

Theorem 2.15: If $\mathfrak{L}$ is denumerable and scattered then $D_{\mathfrak{L}} \approx D_{\Theta}$ for some ordinal Θ.

Proof: See Mazurkievicz and Sierpinski [3, pp. 2-21.].||

Remark: In Erdös and Hajnal [1], we find the following classification of denumerable scattered linear orderings. Let $O_o = \{0,1\}$. Let O_{σ} = all ω-sums and ω^*-sums of members of $\bigcup_{\sigma'<\sigma} O_{\sigma'}$. Then $O = \bigcup_{\sigma<\Omega} O_{\sigma'}$ is the collection of all denumerable scattered linear order types. The authors express their belief that the classification is so natural that it must have appeared somewhere before in the literature. This author has not been able to find it if it has. However if we look at the Stone spaces of the interval algebras of the order types, it turns out that Erdös and Hajnal's classification theorem is equivalent to Sierpinski-Mazurkievicz's theorem that every countable compact space is homomorphic to an ordinal. We can do induction on σ to show that for every denumerable scattered linear ordering $\mathfrak{L}$, with first and last element, $D_{\mathfrak{L}}$ is isomorphic to D_{θ}, for some $\theta < \Omega$. Now the Stone Representation theorem tells us that every countable compact Space (a countable compact Space must be Boolean) is homeomorphic to an ordinal. In this non-topological proof of a classifical topological theorem the classical fact that every countable ordinal is an ω-sum of smaller ordinals takes the place of separability and the well ordering of the ordinals takes the place of compactness. Conversely, for any denumerable scattered linear ordering $\mathfrak{L}$ with first and last element, $\mathfrak{L}$ is isomorphic to $D_{\omega^{\alpha} n+1}$.

We can do induction on (α, n) to show that, for any denumerable scattered $\mathfrak{L}$ with first and last element, $\mathfrak{L} \in O$. The classification theorem follows easily from this. We can also use the Classification theorem to prove that, if B is a countable Boolean algebra and B/(atomless elements) has a scattered basis, then B is isomorphic to $D_{\sum_{\tau \in Q} (\theta_{\tau} + \eta)}$, where $Q \subseteq \hat{\gamma}(\gamma < \Omega)$.

CHAPTER II

Countable Boolean Algebras and Boolean Algebras whose elements are integers.

§1. Countable Boolean Algebras

The following theorem is part of the "folklore" of the subject of Boolean algebras. We include it here for completeness.

Theorem 1.1: Every denumerable Boolean Algebra has an ordered basis.

Proof: Let $a_1, a_2, \ldots$ be an enumeration of the members of B. We define inductively an increasing sequence of finite sets as follows

$$A_1 = \{a_1\}$$

Suppose $A_n = \{b_1, \ldots, b_n\}$ has the following properties:

(i) $0 \leq b_1 \leq \ldots \leq b_n \leq 1.$

(ii) Each a_i, $1 \leq i \leq n$, is a Boolean combination of the members of A_n.

Let $c_1 = b_1 \cup \overline{a_{n+1}}$, $c_i = b_i \cup \overline{c_i}$, for $1 < i \leq n$. Now $b_i \leq c_i$, for $1 \leq i \leq n$, and hence:

-- $0 \leq b_1 \cap a_{n+1} \leq b_1$;

-- $b_{i-1} \leq b_i \cap c_{i-1} \leq b_i$ for $1 \leq i < n$; and

-- $c_{n-1} \cap b_n \leq b_n \leq c_n$.

Thus, $A_n \cup \{b_1 \cap a_{n+1}\} \cup \bigcup_{1 \leq i < n} \{b_i \cap c_{i-1}\}$ is totally ordered by $\leq$.

Let $A_{n+1} = A_n \cup \{b_1 \cap a_{n+1}\} \cup \{c_n\} \cup \bigcup_{1 \leq i \leq n} \{b_i \cap c_{i-1}\}$

We now claim that a_{n+1} is a Boolean combination of $a_{n+1} \cap b_1$ and c_1. To see this, we perform the following computation.

$$\overline{a_{n+1}} = (b_1 \cup \overline{a_{n+1}}) \cap (\overline{b_1} \cup \overline{a_{n+1}})$$

$$\overline{a_{n+1}} = (b_1 \cup \overline{a_{n+1}}) \cap (\overline{b_1 \cap a_{n+1}})$$

$$a_{n+1} = \overline{(b_1 \cup \overline{a_{n+1}})} \cup (b_1 \cap a_{n+1})$$

$$a_{n+1} = \overline{c_1} \cup (b_1 \cap a_{n+1})$$

Similarly, c_i is a Boolean combination of $c_i \cap b_{i+1}$ and c_{i+1}, for $1 \leq i < n$. Thus, a_{n+1} is a Boolean combination of $\{a_{n+1} \cap b_1, b_2 \cap c_1, b_3 \cap c_2, \ldots, b_n \cap c_{n-1}, c_n\}$, and, hence, a_{n+1} is a Boolean combination of members of A_{n+1}.

Let $S = \bigcup_{n \geq 1} A_n$. S is totally ordered by $\leq$, and S generates B. ||

We will now discuss Boolean algebras whose elements are integers, and whose relations and operations are relations and function of integers. In doing this, we can take two points of view. We can regard the equivalence relation "$\equiv$" as an equality relation, and say that two integers represent the same elementof the Boolean algebra, iff they are the same integer. In this case, our Boolean algebra will be a <u>strict</u> Boolean algebra, that is, one in which any two equivalent elements are, in fact, identical. On the other hand, we can take the point of view that "$\leq$" is not a strict partial ordering, and that two distinct integers may be equivalent in the algebra. As an example of our first point of view, we will discuss interval algebras of linear orderings of <u>integers</u>. It can be seen by our definition of an interval algebra (I-p.11 line 12) that an interval algebra is strict. As an example of our second point of view, we will discuss Lindenbaum algebras of theories. We will regard the elements of the Lindenbaum algebra to be Gödel numbers of sentences. Certainly, two Gödel numbers can represent equivalent sentences or even the same sentence.

We will now begin our discussion of interval algebras with the following lemmas, which will be useful in Chapter IV.

<u>Lemma 1.2</u>: If $\mathcal{L}$ is a Σ_1^1 (π_1^1, recursive) linear ordering $\{L, \prec\}$, then $D_{\mathcal{L}}$ can be presented as a Boolean Algebra whose

operations and relations are Σ_1^1 (π_1^1, recursive).

Proof: Let a and b be the first and last elements of $\mathcal{L}$, respectively. We will present $D_{\mathcal{L}}$ as the union of the set $\{0\}$ with the set of all strictly ascending, finite sequences, s, of elements of $\mathcal{L}$, such that the cardinality of s is even. The integer 0 will represent the 0 element of $D_{\mathcal{L}}$. To prove the lemma we shall write down the definitions of $\leq$, $\cap$, $\cup$, $^-$ and observe that they are Σ_1^1 (π_1^1, recursive) if $\mathcal{L}$ is Σ_1^1 (π_1^1, recursive.).

(i) $x \in D_{\mathcal{L}}$:

$$x \in D_{\mathcal{L}} \longleftrightarrow x = 0 \vee (\text{seq}\ (x)\ \ \&\ \ (i)_{\leq \ell(x)}(K(x(i) \prec L(x(i))$$
$$\&\ \ (i)\big(0 \leq i < \ell(x) \longrightarrow L(x(i)) \prec K(x(i+1))\big)$$

(ii) $x \in \hat{D}_{\mathcal{L}}$:

$$x \in \hat{D}_{\mathcal{L}} \longleftrightarrow x = 0 \vee (\text{seq}\ (x)\ \ \&\ \ (i)_{\leq \ell(x)}(K(x(i) \preceq L(x(i))$$
$$\&\ \ (i)\big(0 \leq i < \ell(x) \longrightarrow L(x(i)) \preceq K(x(i+1)\big)\ .$$

(iii) $x \leq^{\mathcal{L}} y$:

$$x \leq^{\mathcal{L}} y \longleftrightarrow x \in D_{\mathcal{L}}\ \ \&\ \ y \in D_{\mathcal{L}}\ \ \&\ \ (i)_{\leq \ell(x)}(\exists j)_{\leq \ell(y)}$$
$$\big(K(y(j)) \preceq K(x(i))\ \ \&\ \ L(x(i) \preceq L(y(j)))\big) \vee\ \ x = 0\ .$$

(iv) $\underline{x \sim y}$:

$$x \sim y \longleftrightarrow x \in \hat{D}_{\mathfrak{L}} \ \& \ y \in \hat{D}_{\mathfrak{L}} \ \&$$

$$(i)_{\leq \ell(x)}\Big(K(x(i)) \neq L(x(i)) \longrightarrow (\exists j)_{\leq \ell(y)} \ (x(i) = y(j))\Big)$$

$$(j)_{\leq \ell(y)}\Big(K(y(j)) \neq L(y(j)) \longrightarrow (\exists i)_{\leq \ell(x)} \ (x(i) = y(j))\Big) .$$

$x \sim y$ says that x and y are equal modulo empty intervals.

(v) $\underline{x = \overline{y}^{-\mathfrak{L}}}$:

$$x = \overline{y}^{-\mathfrak{L}} \longleftrightarrow x \in D_{\mathfrak{L}} \ \& \ y \in D_{\mathfrak{L}} \ \& \ \big((x=0 \ \& \ y = 2^{1+J(a,b)})$$

$$\vee \ (y = 0 \ \& \ x = 2^{1+J(a,b)}) \ \vee \ (x \sim z \ \& \ \ell(y) + 1 \ \&$$

$$K(z(0) = a \ \& \ L(z(\ell(z))) = b \ \& \ (i)\Big(0 \leq i < \ell(z) \longrightarrow$$

$$L(z(i)) = K(y(i))\Big) \ \& \ (i)\Big(0 < i < \ell(z) \longrightarrow K(z(i))$$

$$= L(y(i-1))\Big) .$$

(vi) $\underline{x U^{\mathfrak{L}} y = z}$: If x, y and z are sequence numbers let "$z \Subset x \uplus y$" be the recursive predicate which asserts that every term of z is either a term of x or a term of y.

$$x \cup^{\mathfrak{L}} y = z \longleftrightarrow x \in D_{\mathfrak{L}} \ \& \ y \in D_{\mathfrak{L}} \ \& \ z \in D_{\mathfrak{L}} \ \& \ x \leq^{\mathfrak{L}} z$$

$$\& \ y \leq^{\mathfrak{L}} z \ \& \ \Big((y = 0 \ \& \ x = z) \ \vee \ (x = 0 \ \& \ z = y) \ \vee$$

$(i)_{\leq \ell(z)}\ (\exists s)(seq(s)\ \&\ s \subseteq x \uplus y\ \&\ K(s(0)) = K(z(0))$

$\&\ L(z(\ell(z))) = K(s(\ell(z))\ \&\ (j)_{\leq \ell(s)}\ (2^{(1+s(j))} \leq^{\mathcal{L}} 2^{1+z(i)})).$

(vii) $x \cap^{\mathcal{L}} y = z \longleftrightarrow \quad \overline{x}^{\mathcal{L}} \cup^{\mathcal{L}} \overline{y}^{\mathcal{L}} = \overline{z}^{\mathcal{L}}$

We observe that, in (i) - (vii), "$\leq$" occurs only in a positive fashion. Therefore, we bring the function quantifiers of (i) - (vii) to the front (as in Kleene [5, p. 315]) and observe that, if $\mathcal{L}$ is Σ_1^1 (π_1^1, recursive), then so are the predicates (i) - (vii). ||

Lemma 1.2': Let $x \subseteq N$. If $\mathcal{L}$ is an x-recursive linear ordering, then $D_{\mathcal{L}}$ can be presented as a Boolean algebra whose operations and relations are x-recursive.

Lemma 1.3: There exist recursive functions g_1, g_2, g_3, g_4, g_5 such that if e is the Gödel number which defines a recursive linear ordering, the for every x, y, z :

(i) $x = 0$ iff x is the zero element of $D_{1+\mathcal{L}+1}$;

(ii) $\{g_1(e)\}(x)$ iff $x \in D_{1+\mathcal{L}+1}$;

(iii) $\{g_2(e)\}(x,y)$ iff $x \leq^{1+\mathcal{L}+1} y$;

(iv) $\{g_3(e)\}(x,y,z) = 0$ iff $x \cup^{1+\mathcal{L}+1} y = z$;

(v) $\{g_4(e)\}(x,y,z) = 0$, iff $x \cap^{1+\mathcal{L}+1} y = z$;

(vi) $\{g_5(e)\}(x,y) = 0$, iff $x = \overline{y}^{1+\mathcal{L}+1}$.

Proof: Similar to the proof of II-1.2. ||

Lemma 1.4: If B is a denumerable strict Boolean algebra with recursive field, and $\cup$, $\cap$, $^{-}$ are recursive, then there is a recursive linear ordering, $\mathcal{L}$, such that $B \approx D_{\mathcal{L}}$.

Proof: Use the proof of II-1.1 to show that B has a recursively enumerable strict ordered basis S. Let $\mathcal{L}$ be a recursive linear ordering of the same order type as S. $B \approx D_{\mathcal{L}}$.

Lemma 1.4': If B is a strict Boolean algebra with X-recursive field, operations and relations, then $B \approx D_{\mathcal{L}}$ for some X-recursive linear ordering $\mathcal{L}$.

Corollary 1.5: If B is a strict Boolean algebra with hyperarithmetic field, relation and operations, then $B \approx D_{\mathcal{L}}$, where $\mathcal{L}$ is hyperarithmetic.

We conclude this chapter with a discussion of our second kind of Boolean algebra. From this second point of view, we regard the Boolean algebra as being specified by its natural ordering, $\leq$. So when we say that a Boolean algebra of the second kind is π^1_1 (Σ^1_1, Δ^1_1, arithmetic, recursive), we mean that $\leq$ is π^1_1 (Σ^1_1, Δ^1_1, arithmetic, recursive). For example, consider the Lindenbaum algebra of a theory, T. In this case, we regard, "$\leq$" as being the derivability relation "$\alpha \vdash_T \beta$" where α and β are sentences of the language, $\mathfrak{S}$, of T.

If B is a Lindenbaum algebra, then B has the following nice properties:

(i) The field of B is recursive. (It is just the sentences of $\mathfrak{G}$).

(ii) There are recursive functions f_1, f_2, f_3 such that for every $x, y, z \in B$, $f_1(x,y) \equiv x \cup y$. $f_2(x,y) \equiv x \cap y$, $f_3(x) \equiv \overline{x}$.

(The functions f_1, f_2, f_3 are just the propositional connectives $\vee, \wedge, \sim$.)

In general, however, the predicates $x \cap y \equiv z$, $x \cup y \equiv z$, $x \equiv \overline{z}$ are arithmetically definable in terms of $\leq$. This gives us the following lemma.

Lemma 1.6: If B is a Boolean algebra whose natural ordering, $\leq$, is hyperarithmetic (arithmetic) then B is isomorphic to a strict Boolean algebra $\hat{B}$ whose operations, relation and field are hyperarithmetic (arithmetic).

Proof: If $a \in B$, let $[a]$ be $\{x | x \in B \ \& \ x \equiv a\}$. Let f be a hyperarithmetic (arithmetic) choice function which chooses a member from each class in the collection $\{[a]\}_{a \in B}$. Let $\hat{B}$ be the range of f. Since the relations, $x \cap y \equiv z$, $x \cup y \equiv z$, $x \equiv \overline{z}$ are arithmetically definable in terms of $\leq$, these relations are hyperarithmetic (arithmetic). Since f is hyperarithmetic (arithmetic) so is $\hat{B}$. The relations $x \cup y = z$, $x \cap y = z$, $x = \overline{z}$, when restricted to $\hat{B}$, are hyperarithmetic (arithmetic) operations. The relation " $\equiv$ ", when restricted to $\hat{B}$ becomes the equality relation "= ". B is isomorphic to $\hat{B}$. ||

CHAPTER III

Coding Functions into the Isomorphism Type of an Ordering

§1. Preliminaries

The following lemma and definitions will be useful in what follows.

Definition 1.1: Let $A(x_1,\ldots,x_n,y)$ be an arithmetic predicate. We say that y is E.A.N. in $A(x_1,\ldots,x_n,y)$ iff,

$$\mathcal{N} \models (x_1)\ldots(x_m)(r)\big(\sim A(x_1,\ldots,x_m,r) \longrightarrow (y)_{\geq r} \sim A(x_1,\ldots,x_m,y)\big).$$

Remark: E.A.N. stands for "everywhere or almost nowhere."

Definition 1.2: An arithmetic predicate is said to be in predicate form iff it is in the form $Q_1z_1, Q_2z_2,\ldots,Q_nz_n$ $R(\vec{X},z_1,\ldots,z_n)$, where $R(\vec{X},z_1,\ldots,z_n)$ is recursive, and $Q_1, \ldots, Q_n$ are alternating unbounded quantifiers. Every arithmetic predicate is equivalent to a predicate in predicate form. (See Rogers [7, p.126].)

Definition 1.3: If $Q_1z_1,\ldots,Q_nz_n\ R(\vec{X},z_1,\ldots,z_n)$ is in predicate form where R is recursive, then $Q_1z_1,\ldots,Q_nz_n$ $R(\vec{X},z_1,\ldots,z_n)$ is said to be in E.A.N. form iff, for every $1 \leq k \leq n$, $Q_k = \forall$ implies z_k is E.A.N. in

$Q_{k+1}z_{k+1}\ldots Q_nz_n\ R(\vec{X}, z_1,\ldots,z_k,\ldots,z_n)$.

Lemma 1.4: Every arithmetic predicate, A, is equivalent to a predicate in E.A.N. form.

Proof: By III-1.2, we may assume A is in predicate form We do an induction on the number, N, of universal quantifiers preceding the recursive predicate to show that A is equivalent to a predicate in E.A.N. form. If N = 1, then A is in one of the following forms where R is recursive:

(i) $(y)\,R\,(\vec{X},y)$; or

(ii) $(y)(\exists z)R(\vec{X},y,z)$; or

(iii) $(\exists t)(y)(\exists z)R(\vec{X},y,z)$.

In the first case A is equivalent to $(y)(u)_{\leq y}\,R(\vec{X},u)$, which is in E.A.N. form. In case (ii),

$$A \longleftrightarrow (y)(u)_{\leq y}\,(\exists z)R(\vec{X},u,z)$$

$$\stackrel{(1)}{\longleftrightarrow} (y)(\exists z)(u)_{\leq y}\,R(\vec{X},u,(z)_u),$$

where (1) is in E.A.N. form. The third case reduces to the second by observing that, in general, if $A(\vec{X},z)$ is in E.A.N. form, then so is $(\exists z)A(\vec{X},z)$. Before proceeding to the induction step, we prove the following claim.

Claim: If y is E.A.N. in $A(\vec{X},u,y)$, then it E.A.N. in $(u)_{\leq v}\,A(\vec{X},u,y)$.

Proof: Suppose $\sim(u)_{\leq v}\,A(\vec{X},u,r)$ holds for some $\vec{X},r$. This implies $(\exists u)_{\leq v} \sim A(\vec{X},u,r)$. Thus, for some $\tilde{u} \leq v, \sim A(\vec{X},\tilde{u},r)$. Because y is E.A.N., $(y)_{\geq r} \sim A(\vec{X},\tilde{u},y)$ holds. This implies

$(y)_{\geq r}\ (\exists u)_{\leq V} \sim A(\vec{X},u,y)$ or equivalently $(y)_{\geq r} \sim (u)_{\leq V}\ A(\vec{X},u,y)$. |

Using the claim, the induction on N may be completed. Suppose the theorem is true for $N = n$. Let A be $(y_1)(\exists z_1)(y_2)(\exists z_2)\ \ldots\ R(\vec{X},y_1,z_1,y_2,z_2,\ldots)$, where A has $n + 1$ universal quantifiers preceding the recursive predicate. By our induction hypothesis, we can assume that $(y_2)(\exists z_2)\ldots R(\vec{X},y_1,z_1,\ldots)$ is in E.A.N. form. $A \longleftrightarrow$ (2) $(y_1)(u)_{\leq y_1}(\exists z_1)(y_2)\ \ldots\ R(\vec{X},u,z_1,y_2,\ldots)$ where y_1 is E.A.N. in (2). We now drive "$(u)_{\leq y_1}$" inwards (as in Kleene [6,p.]) and observe that, every y_k is E.A.N. in $(\exists z_k)\ldots R(\vec{X},u,(z_1)_u,\ y_1,(z_2)_u,\ \ldots,\ (z_{k-1})_u,\ y_k,\ldots)$ and, therefore, by our claim every y_k is E.A.N. in

$(u)_{\leq y_1}\ (\exists z_k)\ \ldots\ R(\vec{X},u,(z_1)_u,\ \ldots,\ y_k,\ \ldots)$

Therefore after "$(u)_{\leq y_1}$" is driven all the way inward, the resulting predicate is in E.A.N. form. ||

§2. Partial Orderings

The proof that every recursively enumerable linear ordering whose field is total is a recursive linear ordering depends on the fact that a linear ordering $\mathcal{L}$ is connected, that is, for any a, b $\in \mathcal{L}$, either $a \prec b$ or $b \prec a$. If $\mathcal{P}$ is an r.e. partial ordering and $\mathcal{P}$ is not connected, then, as we will prove below, $\mathcal{P}$ need not be isomorphic to a recursive partial ordering.

Theorem 2.1: There is a recursively enumerable strict partial ordering which is not isomorphic to a recursive strict partial ordering.

Proof: Let $\mathcal{G}$ be the language of the elementary theory, , of partial orderings, and let $\prec$ be its relation symbol. Let $\mathcal{P} = \{P, \prec\}$ be a model of T. $\mathcal{P}$ is a strict partial ordering iff, for any two elements a, b $\in$ P, a = b implies $a \not\prec b$. A set S of elements of P is called an antichain iff any two elements of S are $\prec$- incomparable. It is clear that any anti-chain can be extended to a maximal anti-chain. It is also clear that, for every integer n, there is a sentence Φ^n of T such that Φ^n asserts the existence of a maximal anti-chain of cardinality n.

Lemma 2.2: (i) If $\mathcal{P}$ is a recursive partial ordering, then

$\hat{n}(\mathcal{P} \models \Phi^n) \in \Sigma_2$.

(ii) If $\mathcal{P}$ is a recursively enumerable partial ordering then $\hat{n}(\mathcal{P} \models \Phi^n) \in \Sigma_3$.

Proof: (i) "$\mathcal{P} \models \Phi^n$" $\longleftrightarrow$

$$\mathcal{N} \models^{(1)} (\exists s)(seq(s) \ \& \ \ell(s) = n \ \&$$

$$(i)_{\leq n} (j)_{\leq n} (i) \neq j \longrightarrow \sim (s(i) \preceq s(j)) \ \&$$

$$(z)(\exists j)_{\leq n} (s(j) \preceq z \vee z \preceq s(j)),$$

where $a \preceq b$ means $a \prec b$ or $a = b$. "$\preceq$" is a recursive relation. We apply the Tarski-Kuratowski algorithm (see [7, pp. 131-133]) to (1) to see that (1) is in Σ_2. (ii) is proved similarly. ||

Thus, to prove the theorem, it suffices to find a r.e. partial ordering $\mathcal{P}$ such that $\hat{n}(\mathcal{P} \models \Phi^n) \in \Sigma_3 - \Sigma_2$. Let $R(m,y_1,y_2,y_3,n)$ be a recursive predicate such that:

(i) "$n \geq 2 \ \& \ (\exists m)(y_1)(\exists y_2) \ R(m,y_1,y_2,n)$" $\in \Sigma_3 - \Sigma_2$;

and

(ii) y_1 is E.A.N. in $(\exists y_2)R(m,y_1,y_2,n)$.

We seek a r.e. partial ordering $\mathcal{P}$ such that

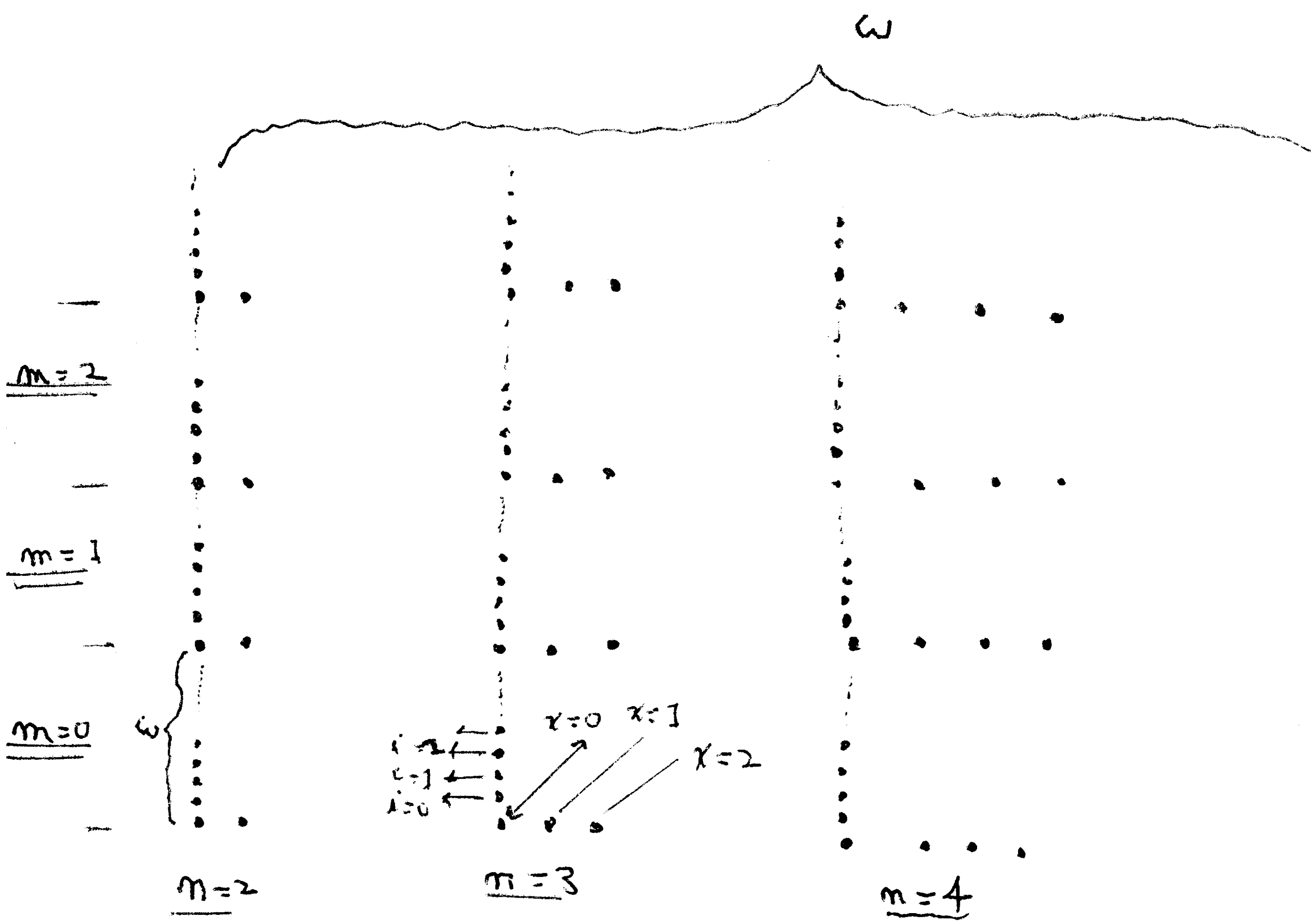
$$(n)_{\geq 2} \ (\mathcal{P} \models \Phi^n \longleftrightarrow \mathcal{N} \models (\exists m)(y_1)(\exists y_2)R(m,y_1,y_2,n)).$$

For notational convenience, we will deal with a symbol * in addition to the integers. That is, the field of the relation we will construct will be a subset of the fourth cartesian power of $\{*,0,1,2,3,4,\ldots\}$.

Let P be the following set of 4-tuples.

$$\left\{ (n,m,x,i) \;\middle|\; \begin{array}{l} n \geq 2 \;\; \& \;\; 0 \leq x \leq n \;\; \& \;\; (x < n \longleftrightarrow i = *) \;\; \& \\ x \neq * \;\; \& \;\; n \neq * \;\; \& \;\; m \neq * \end{array} \right\}$$

P can be visualized as follows:



We define the following relation on P:

$$(n,m,x,i) \prec (n',m',x',i') \longleftrightarrow$$

$$(n,m) <_{\ell ex} (n',m') \vee \Big((n,m) = (n',m') \;\&$$

$$i < i' \;\&\; x = x' = n \;\&\; (\exists y_2)\, R(m,i,y_2,n) \;\&$$

$$(\exists y_2)\, R(m,i',y_2,n)\Big) \vee \Big((n,m) = (n',m') \;\&$$

$$x < n \;\&\; x' = n \;\&\; (\exists y_2)\, R(m,i',y_2,n)\Big).$$

Let $\mathcal{P} = \{P, \prec\}$. "$\prec$" is an r.e. relation.

Claim (i): $\mathcal{P}$ is a strict partial ordering.

Proof: Follows from an examination of the definition of $\mathcal{P}$. The reader will get some idea of what $\mathcal{P}$ looks like by checking that this claim is true.

Claim (ii): If S is an antichain of $\mathcal{P}$, and both (n,m,x,i) and (n',m',x',i') are members of S, then $(n,m) = (n',m')$.

Thus if S is an anti-chain we define $S_1 = n$ and $S_2 = m$, where (n,m,x,i) is an element of S. Let

$$A_m^n = \left\{(n',m',x,i) \,\middle|\, \begin{array}{l}(n',m',x,i) \in P \;\&\; 0 \le x < n \\ \&\; (n,m) = (n',m')\end{array}\right\}:$$

$$B_m^n = \left\{(n',m',x,i) \,\middle|\, \begin{array}{l}(n',m',x,i) \in P \;\&\; x = n \;\& \\ (\exists y_2)\; R(m,i,y_2,n) \;\&\; (n',m') = (n,m)\end{array}\right\}$$

$$C^n_m = \left\{ (n',m',x,i) \,\middle|\, \begin{array}{l} (n',m',x,i) \in P \ \& \ x = n \\ \& \ \sim(\exists y_2)\ R(m,i,y_2,n) \ \& \ (n',m') = (n,m) \end{array} \right\}$$

Claim (iii): Suppose that S is a maximal anti-chain of P and that $S_1 = n$, $S_2 = m$, and the cardinality of S (card S) is ≥ 2, then:

(i) $(y_1)(\exists y_2)R(m,y_1,y_2,n) \longrightarrow$ card $S = n$, and

(ii) $\sim(y_1)(\exists y_2)R(m,y_1,y_2,m) \longrightarrow$ card $S = \aleph_o$.

Proof: An examination of the definition of $\mathcal{P}$ will be the justification for the assertions made in this proof.

(i) If $(y_1)(\exists y_2)R(m,y_1,y_2,n)$, then S must be A^n_m. Card $A^n_m = n$.

(ii) Since y_1 is E.A.N. in $(\exists y_2)R(m,y_1,y_2,n)$, there is an π such that $(y_1)_{\geq r} \sim(\exists y_2)R(m,y_1,y_2,n)$. Thus, card $C^n_m = \aleph_o$. Since S is maximal, $C^n_m \subseteq S$, so card $S = \aleph_o$. ||

We now conclude the proof of Theorem 2.1. If $(\exists m)(y_1)(\exists y_2)R(m,y_1,y_2,n)$, then for some $\tilde{m}$, $(y_1)(\exists y_2)R(\tilde{m},y_1,y_2,n)$. By Claim (ii), $A^n_{\tilde{m}}$ is a maximal anti-chain of cardinality n, and, thus, $\mathcal{P} \models \Phi^n$. Conversely, suppose $(m)\sim(y_1)(\exists y_2)R(m,y_1,y_2,n)$. In this case, if S is a maximal anti-chain, and $S_1 = n$, then card $S = \aleph_o$. On the other hand, if $S_1 = \tilde{n} \neq n$, then either card $S = \aleph_o$ or card $S = \tilde{n} \neq n$.

Therefore, $\mathcal{P} \not\models \Phi^n$. ||

<u>Theorem 2.1</u>: Let $X \subseteq N$. There is a X-r.e. partial ordering which is not isomorphic to an χ-recursive partial ordering.

§3. Linear Orderings

The proof that every r.e. linear ordering is isomorphic to a recursive linear ordering depends upon the fact that every r.e. relation is isomorphic to an r.e. relation whose field is total. (This latter fact will be proved in chapter III, §1). In this section we will construct a Π^0_1 linear ordering, $\mathcal{L}$, which is not isomorphic to a recursive linear ordering. In particular, $\mathcal{L}$ is not isomorphic to any Π^0_1 relation whose field is total. For if $\mathcal{L}$ is Π^0_1 and the field of $\mathcal{L}$ is total, then for every $x, y \in \mathcal{L}$, $x \prec y \longleftrightarrow \sim(y \prec x)$, and, thus, $\mathcal{L}$ is r.e. which means that $\mathcal{L}$ is recursive.

<u>Theorem 3.1</u>: There is a Π^0_1 scattered linear ordering not isomorphic to a recursive linear ordering.

<u>Proof</u>: Let $\prec$ be the relation symbol in the language $\mathfrak{G}$ of the elementary theory, T, of linear order. Let $\mathcal{L} = \{L, \prec\}$ be a model of T. A set $S \subseteq L$ is said to be a <u>successor-chain</u> iff for any $a \in S$, $b \in S$, $[a,b)$ is finite. It is clear that any successor-chain can be extended to a maximal successor chain. It is also clear that for every integer n, there is a sentence Φ^n of $\mathfrak{G}$ such that Φ^n asserts the existence of a maximal successor-chain of cardinality n.

Lemma 3.2: (i) If $\mathcal{L}$ is a recursive linear ordering then $\hat{n}(\mathcal{L} \models \Phi^n) \in \Sigma_3$.

(ii) If $\mathcal{L}$ is a Π_1^0 linear ordering then $\hat{n}(\mathcal{L} \models \Phi^n) \in \Sigma_4$.

Proof: (i) $\mathcal{L} \models \Phi^n$ is equivalent to $\mathcal{N} \models A$ where A is

$$(\exists s)(seq(s) \ \& \ \ell(s) = n \ \& \ (i)_{<n} (s(i) < s(i+1)) \ \&$$

$$(z)(s(0) < z < s(n) \longrightarrow (\exists i)_{\leq n}(z = s(i))) \ \&$$

$$(z)(z < s(0) \longrightarrow (\exists y)(z < y < s(0)) \ \&$$

$$(z)(s(n) < z \longrightarrow (\exists y)(s(n) < y < z).$$

We apply the Tarski-Kuratowski algorithm to (A) to show that (A) $\in \Sigma_3$. The proof of (ii) is similar. ||

Thus to prove the theorem it will suffice to construct a Π_1^0 linear ordering $\mathcal{L}$ such that

$\hat{n}(\mathcal{L} \models \Phi^n) \in \Sigma_4 - \Sigma_3$.

III-1.4 will be useful in constructing such an $\mathcal{L}$. Let $R(m,y_1,y_2,y_3,n)$ be a recursive predicate such that;

(i) "$n \geq 2 \ \& \ (\exists m)(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n)$" $\in \Sigma_4 -$

(ii) "$(\exists m)(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n)$" is in E.A.N. form.

As in the proof of Theorem 2.1, we deal with a symbol $*$ in addition to the integers. Let L be the following set of 5-tuples

$$\{(n,m,x,i,j) \mid n \geq 2 \ \& \ 0 \leq x \leq n \ \& \ (x < n \longrightarrow (i,j) = (*,*)) \ \& \ (x = n \longrightarrow (i \neq * \ \& \ j \neq *)) \ \& \ x \neq * \ \& \ m \neq *\}.$$

We define the following linear ordering on L.

$$(n,m,x,i,j) \prec (n',m',x',i',j') \longleftrightarrow ((n,m) <_{\ell ex.} (n',m')) \vee ((n,m) = (n',m') \ \& \ x < x') \vee ((n,m) = (n',m') \ \& \ x = x' = n \ \& \ i > i') \vee ((n,m) = (n',m') \ \& \ x = x' = n \ \& \ i = i' \ \& \ j < j').$$

It is not too difficult to see that the order type of $\{L, \prec\}$ is

$$\sum_{n \geq 2} (n + \omega \cdot \omega^*) \cdot \omega.$$

We now define a Π^0_1 subset $\hat{L}$ of L.

$$(n,m.x.i.j) \in \hat{L} \longleftrightarrow (n,m,x,i,j) \in L \ \& \ (x < n \vee (x = n \ \& \ i = 0) \vee (x = n \ \& \ i > 0 \ \& \ (\exists y_2)_{\leq j} (y_3) R(m,i-1,y_2,y_3,n)).$$

If $A \subset L$, let $\mathcal{L} \restriction A$ be the restriction of $\mathcal{L}$ to A. Let A^n_m be the set

$\{(n',m',x,i,j) \mid (n',m') = (n,m) \ \& \ x = n\}$.

We wish to show that,

(2) $(n)_{\geq 2}(\mathcal{L} \restriction \hat{L} \models \Phi^n \longleftrightarrow (\exists m)(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n))$.

The following observations, which will be useful in establishing (2), are direct consequences of the definitions of $\mathcal{L}$, L, and $\hat{L}$.

(i) For every n and m,

$(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n) \longrightarrow \mathcal{L} \restriction \hat{L} \cap A^n_m \approx \omega \cdot \omega^*$.

(ii) For every n and m,

$\sim(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n) \longrightarrow \mathcal{L} \restriction \hat{L} \cap A^n_m \approx \omega \cdot q$

where q is the least integer such that

$\sim(\exists y_2)(y_3)R(m,r,y_2,y_3,n)$.

(iii) If S is a maximal successor-chain of $\mathcal{L} \restriction \hat{L}$, then S must have a least element. Furthermore, if S is to be finite, it is necessary (though not sufficient) that the least element of S be of the form (n,m,x,i,j) where $x = 0$.

We now conclude our proof of III-3.1. An examination of the definitions of L, $\hat{L}$ and $\mathcal{L}$ will be the justification for the assertions made below.

Suppose that

$(\exists m)(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n)$ holds for some n.

Therefore, for some $\tilde{m}$, $(y_1)(\exists y_2)(y_3)R(\tilde{m},y_1,y_2,y_3,n)$ holds. By (i), $\left\{(n',m',x,i,j) \,\middle|\, \begin{array}{l}(n',m') = (n,\tilde{m}) \\ \& \;\; x < n\end{array}\right\}$ is a maximal successor chain of $\mathfrak{L}\ \hat{L}$. ($(n,\tilde{m},n-1,*,*)$ is a lower limit point. $(n,\tilde{m},0,*,*)$ is an initial point if $n = 2$, and an upper limit point if $n > 2$.)

Conversely, suppose that, for some n, $\sim(\exists m)(y_1)(\exists y_2)(y_3)R(m,y_1,y_2,y_3,n)$. Furthermore, suppose that S is a maximal successor-chain of $\mathfrak{L} \restriction \hat{L}$ and that S has least element $(\tilde{n},\tilde{m},0,*,*)$. Let $B^{\tilde{n}}_{\tilde{m}}$ be the set

$\{(n',m',x,i,j) \mid (n',m',x,i,j) \in L \;\&\; (n',m') = (\tilde{n},\tilde{m}) \;\&\; x < n'\}$.

If $\mathfrak{L} \restriction A^{\tilde{n}}_{\tilde{m}} \cap \hat{L} \approx \omega\cdot\omega^*$, then $S = B^{\tilde{n}}_{\tilde{m}}$, and, therefore the cardinalit of S is $\tilde{n}$. On the other hand, if $\mathfrak{L} \restriction A^{\tilde{n}}_{\tilde{m}} \cap \hat{L} \approx \omega\cdot q$, for some integer q, then one easily checks that the cardinality of S is infinite. In fact, in this case, $\mathfrak{L} \restriction S \approx \omega$. Thus, it follows from Lemma 2.6, that either card $s = \tilde{n}$ or card $S = \aleph_0$, depending upon whether or not $(y_1)(\exists y_2)(y_3)R(\tilde{m},y_1,y_2,y_3,\tilde{n})$ holds. However, it follows from our assumption that if $n = \tilde{n}$, then, for any $\tilde{m}$, $(y_1)(\exists y_2)(y_3)\tilde{R}(m, y_1, y_2, y_3, \tilde{n})$ does not hold. Thus, either card $S = \tilde{n} \neq n$ or card $S = \aleph_0$. ||

<u>Theorem 3.1'</u>: Let $X \subseteq N$. There is a scattered linear ordering which is Π^0_1 in X and which is not isomorphic to any X-recursive linear ordering.

§ 4 Coding H_ω-recursive functions into Recursive Orderings

The following theorem will give us a method for constructing a recursive structure whose isomorphism type has a non-arithmetic collection of elementary properties. In particular, at the end of this section we will construct a recursive linear ordering, $\mathcal{L}$, such that the set of all elementary statements true of $\mathcal{L}$ is not arithmetic.

Definition 4.1: Let e be a number, and $\mathcal{L}$ be an order type. We say $|e| = \mathcal{L}$, iff for every n, $\{e\}(n)$ is defined and $\mathfrak{J}^{-1}(W_e)$ is a linear ordering of order type $\mathcal{L}$.

Theorem 4.2: There is a recursive function $\epsilon(e,a,m,n)$ such that if e defines a recursive function of $n + 1$ arguments (see Kleene [8, pp. 288-289]), then for every integer a:

(i) $(\exists z_1)(z_2)(\exists z_3)(z_4)\ldots(\{e\}(z_1,z_2,z_3,\ldots,a) = 0)$

$\longrightarrow |\epsilon(e,a,m,n)| = \omega^{m+n+1}$;

(ii) $\sim(\exists z_1)(z_2)(\exists z_3)(z_4)\ldots(\{e\}(z_1,z_2,z_3,\ldots,a) = \ )$

$\longrightarrow |\epsilon(e,a,m,n)| = \omega^{m+n}$.

The proof of this theorem will require several lemmas.

Lemma 4.3: There is a recursive function $\pi(e,a,m,n)$ such that, if e defines a recursive function of $n + 1$ arguments, then, for every $z_1,\ldots,z_n,a$:

(i) $(\{e\}(z_1,\ldots,z_n,a) = 0) \longrightarrow |\pi(e,a,m,n)| = \omega^{m+1}$;

(ii) $(\{e\}(z_1,\ldots,z_n,a) \neq 0) \longrightarrow |\pi(e,a,m,n)| = \omega^{m}$.

Proof: Straightforward. ||

The function, π, will correspond to the recursive matrix of an arithmetic predicate in predicate form. We will now define two recursive functions, Σ and $\tilde{\tilde{\Sigma}}$, which will correspond to $\forall$ and $\exists$, respectively.

Lemma 4.4: There is a recursive function Σ such that, if e is a number such that, for every n, $\{e\}(n)$ is defined and $|\{e\}(n)| = \mathcal{L}_n$, then $|\Sigma(e)| = \sum_i \mathcal{L}_i$.

Proof: Suppose we are given a number e. Let $\Sigma(e)$ be the Gödel number of the following partial recursive function: Given n, to find $\{\Sigma(e)\}(n)$, we:

(i) let $Q_n \subseteq \{0,\ldots,n\} \times \{0,\ldots,n\}$ be the set of all pairs (i,j) $\in\{0,\ldots,n\}^2$ such that the computation of $\{e\}(i)$ terminates in at least n steps, and, furthermore the computation of $\{\{e\}(i)\}(j)$ terminates in at least n steps;

(ii) let $\mathbb{P}_{0ij} = K(\{\{e\}(i)\}(j))$, and let

$\mathbb{P}_{1ij} = L(\{\{e\}(i)\}(j))$;

(iii) let

$$A_n = \left\{ J(J(\mathbb{P}_{kij}, i), J(\mathbb{P}_{k'i'j'}, i')) \;\middle|\; \begin{array}{l} (i,j)\in Q_n \;\;\&\;\; (i',j')\in Q_n \\ \&\;\; k \le 1 \;\;\&\; k' \le 1 \\ \&\;\; (i<i' \vee (i,j)=(i',j')\;\; \& k<k \end{array} \right.$$

(iv) Let **r** be the least m such that $A_m \neq \emptyset$ and let $\{\Sigma(e)\}(n)$ be the least element of

$C_n = A_{r+n} - \{\{\Sigma(e)\}(0),\ldots,\{\Sigma(e)\}(n-1)\}$ if $C_n \neq \emptyset$, and let $\{\Sigma(e)\}(n) = \{\Sigma(e)\}(n-1)$, otherwise. ||

Σ is recursive and has the desired properties. ||

<u>Lemma 4.5</u>: There is a recursive function, $\widetilde{\Sigma}$, such that, if e is a number such that, for every n, $\{e\}(n)$ is defined and $|\{e\}(n)| = \mathcal{L}_n$, then $|\widetilde{\Sigma}(e)| = \sum\limits_{i} \sum\limits_{j \le i} \mathcal{L}_j$

<u>Proof</u>: Similar to that of 4.4. ||

<u>Lemma 4.6</u>: Let $A(\vec{z},y)$ be an arithmetic predicate. Let e define $f(\vec{z},y)$ recursively. If for every $\vec{z},y$;

(i) $A(\vec{z},y) \longrightarrow (|f(\vec{z},y)| = \omega^{m+1})$; and

(ii) $\sim A(\vec{z},y) \longrightarrow (|f(\vec{z},y)| = \omega^{m}\cdot q)$, where q is an integer whose value depends on $\vec{z},y$; then, for every $\vec{z}$,

(1) $(\exists y)A(\vec{z},y) \longrightarrow (|\tilde{\Sigma}(S_1^n(e,\vec{z}))| = \omega^{m+2}$; and

(2) $\sim(\exists y)A(\vec{z},y) \longrightarrow (|\tilde{\Sigma}(S_1^n(e,\vec{z}))| = \omega^{m+1}$.

Proof: By III-4.5, $|\tilde{\Sigma}(S_1^n(e,\vec{z}))|$ is an ω-sum of ordinals. That is, $|\tilde{\Sigma}(S_1^n(e,\vec{z}))| = \Sigma\Theta_i$, where, for every i, $\omega^m \leq \Theta_i \leq \omega^{m+1}$. If $(\exists y)A(\vec{z},y)$ holds, then ω^{m+1} occurs $\aleph_o$ times in the ω-sum, and, therefore, the sum will be ω^{m+2}. If $(y) \sim A(\vec{z},y)$, then, for every i, $\Theta_i < \omega^{m+1}$. Therefore the ω-sum will be ω^{m+1}. ||

Lemma 4.7: Let $A(\vec{z},y)$ be an arithmetic predicate. Let e define $f(\vec{z},y)$ recursively. If y is E.A.N. in $A(\vec{z},y)$, and, if for every $\vec{z},y$;

(i) $A(\vec{z},y) \longrightarrow (|f(\vec{z},y)| = \omega^{m+1})$; and

(ii) $\sim A(\vec{z},y) \longrightarrow (|f(\vec{z},y)| = \omega^{m})$;

then, for every $\vec{z}$;

(1) $(y)A(\vec{z},y) \longrightarrow (|\Sigma(S_1^n(e,\vec{z}))| = \omega^{m+2})$; and

(2) $\sim(y)A(\vec{z},y) \longrightarrow (|\Sigma(S_1^n(e,\vec{z})| = \omega^{m}\cdot q$, where q is an integer whose values depends on $\vec{z}$.

Proof: By III-4.4, $|\Sigma(S_1^n(e,\vec{z}))| = \sum_{i\in N} \Theta_i$, where, for every i, either $\Theta_i = \omega^m$ or $\Theta_i = \omega^{m+1}$. If $(y)A(\vec{z},y)$, then $(i)(\Theta_i = \omega^m)$, and $\Sigma\Theta_i = \omega^{m+1}\cdot\omega = \omega^{m+2}$. y is E.A.N. in $A(\vec{z},y)$. Therefore, if $\sim(y)A(\vec{z},y)$, then there is an r such that $(y)_{>r} \sim A(\vec{z},y)$. In this case, $(i)_{\geq r}(\Theta_i = \omega^m)$, and $\Sigma\Theta_i = \omega^{m+1}\cdot q$ where $q \leq r$. ||

Lemma 4.8: There is a recursive function λ such that, if e defines a recursive function of $n + 1$ arguments, then:

(i) $\lambda(e)$ defines a recursive function of $n + 1$ arguments;

(ii) for every a, $(\exists z_1)(z_2)(\exists z_3)(z_4)\dots(\{e\}(z_1,\dots,z_n,a) = 0$ $\longleftrightarrow (\exists z_1)(z_2)(\exists z_3)\dots(\{\lambda(e)\}(z_1,\dots,z_n,a) = 0)$;

(iii) $(\exists z_1)(z_2)(\exists z_3)\dots(\{\lambda(e)\}(z_1,\dots,z_m,a) = 0)$ is in E.A.N. form.

Proof: Look at the proof of I-1.4. This proof gives an effective procedure which given the Gödel number defining the recursive matrix of an arithmetic predicate, A, yields a Gödel number of the recursive matrix of an arithmetic predicate $\hat{A}$, where $\hat{A}$ is in E.A.N. form and $\hat{A}$ is equivalent to A. ||

We now complete the proof of IV-4.2. We will define $\epsilon(e,a,m,n)$ for the case where n is odd. (The case where n is even is similar and will not be discussed further). Let $\{e\}(z_1,\dots,z_n,a)$ be a recursive function. We define $\epsilon_1(e,a,m,n)\dots,\epsilon_n(e,a,m,n)$ inductively as follows:

(i) $\epsilon_1(e,a,m,n)$ defines $\tilde{\Sigma}(S_1^{n-1}(\pi(\lambda(e),a,m,n),z_1,\ldots,z_{n-1}))$ as a n-1 place function;

(ii) For $0 < 2i+1 \leq n$, $\epsilon_{2i+1}(e,a,m,n)$ defines $\tilde{\Sigma}(S_1^{n-(2i+1)}(\epsilon_{2i},z_1,\ldots,z_{n-(2i+1)})$ as a $n-(2i+1)$ place function. For $0 < 2(i+1) \leq n$, $\epsilon_{2(i+1)}$ defines $\Sigma(S_1^{n-2(i+1)}(\epsilon_{2i+1},z_1,\ldots,z_{n-2(i+1)}))$ as a $n-2(i+1)$ place function.

Let $\epsilon(e,a,m,n) = \epsilon_n(e,a,m,n)$. It is a straightforward inductive proof, using III-4.6 and III-4.7, to show that $\epsilon(e,a,m,n)$ has the desired properties. Observe also that $\epsilon(e,a,m,n)$ is recursive. ||

<u>Definition 4.1'</u>: Let e be a number, and $\mathcal{L}$ be an order type. We say $|e^X| = \mathcal{L}$, iff for every n, $\{e\}(n)$ is defined and $J^{-1}(W_e^X)$ is a linear ordering of order type $\mathcal{L}$.

<u>Theorem 4.2'</u>: There is a recursive function $\pi(e,a,m,n)$, such that, if $X \subseteq N$ and if e defines an $n+1$ place function recursively in X (see Kleene [8, pp. 266-281]), then for every $z_1,\ldots,z_n,a$:

(i) $(\exists z_1)(z_2(\exists z_3)\ldots(\{e\}^X(z_1,\ldots,z_n,a) = 0) \longrightarrow (|\pi(e,a,m,n)^X|=\omega^{m+n}$

(ii) $\sim(\exists z_1)(z_2)(\exists z_3)\ldots(\{e\}^x(z_1,\ldots,z,a) = 0)$

$\longrightarrow (|\pi(e,a,m,n)^x| = \omega^{m+n})$.

Proof: Alter the proof of III-4.2 by making the following replacements: __an expression of the form "$\{t\}.$" is replaced by "$\{t^x\}$".

__an expression of the form "$|t| = \mathfrak{x}$" is replaced by "$|t^x| = \mathfrak{x}$".

__an expression of the form "t defines a recursive function..." is replaced by "t defines, relative to x, an x-recursive function ..."

Observe that this altered proof is a proof of 4-2'. The assertions in the altered proof are shown to be correct by an argument very similar to the proof of the relativization of the Kleene S^m_n theorem (Kleene [10, pp. 150-155].) ||

The following corollaries will give us III-4.2 in the form we need it.

Lemma 4.9: There is a recursive function Θ such that for every n, $\Theta(n)$ defines an n + 1 place recursive function, and

$$(a)(n)(a\epsilon H_n \longleftrightarrow (\exists z_1)(z_2)(\exists z_3)(z_4)\ldots(\{\Theta(n)\}(z_1,\ldots,z_n,a)=0))$$

Proof: Look at the proof of the Post representation theorem. (See Davis [9 , pp. 158-161]) Observe that this proof gives us an effective procedure which given a number, n, yields the Gödel number which defines the recursive matrix of H_n. ||

Corollary 4.10: Let $\beta(a,m,n) = \pi(\theta(n),a,m,n)$. For every a, n:

(i) $a \in H_n \longrightarrow (|\beta(a,m,n)| = \omega^{m+n+1})$;

(ii) $a \notin H_n \longrightarrow (|\beta(a,m,n)| = \omega^{m+n})$.

If in III-4.9 and III-4.10, we make the replacements listed in the proof of III-4.2', we obtain the relativized versions. III-4.9' and III-4.10'.

Corollary 4.10': There is a recursive function $\beta(a,m,n)$ such that if $X \subseteq N$, then for every a, n:

(i) $a \in H_n^X \longrightarrow (|\beta(a,m,n)^X| = \omega^{m+n+1})$;

(ii) $a \notin H_n^X \longrightarrow (|\beta(a,m,n)^X| = \omega^{m+n})$.

We will now use IV-4.10 to construct a recursive linear ordering, $\mathfrak{L}$, such that the set of elementary statements true in $\mathfrak{L}$ is Turing equivalent to H_ω. First, we will need several preliminaries.

Definition 4.11: Let $\mathfrak{A}$ be a structure and let $\mathfrak{S}$ be a language of the same similarity type as $\mathfrak{L}$. The truth set of is the set $\{\Phi | \Phi$ is a sentence of $\mathfrak{S}$ & $\mathfrak{R} \models \Phi\}$.

Lemma 4.12: If $\mathfrak{R}$ is a denumerable structure of finite similarity type, whose field is an arithmetic subset of N, and whose relations and operations are arithmetic, then the truth set of $\mathfrak{R}$ is Turing reducible to H_ω.

Proof: Look at the model-theoretic definition of "$\models$", and keep in mind that every member of $\Pi_n^o \cup \Sigma_n^o$ is Turing reducible to H_n. ||

Lemma 4.13: Let $\mathfrak{S}$ be the language of the elementary theory of linear order. The predicate "There exists an n^{th} inductive upper limit which is also a lower limit point." is expressible in $\mathfrak{S}$.

Proof: Let $\nu_1, \nu_2, \nu_3 \ldots$ be the variables and $\prec$ be the relation symbol of $\mathfrak{S}$. We define the w.f.f.'s $L_n^-(\nu_1)$ inductively:

$$L_o^-(\nu_1) \longleftrightarrow (\exists \nu_2)(\nu_2 \prec \nu_1) \quad \& \quad (\nu_2)(\nu_2 \prec \nu_1 \longrightarrow (\exists \nu_3)(\nu_2 \prec \nu_3 \prec \nu_1).$$

$$L_{n+1}^-(\nu_1) \longleftrightarrow L_o^-(\nu_1) \text{ relativized to } L_n^-(\nu_1).$$

Now we define the w.f.f.'s $L_o^+(\nu_1)$ and $\Phi^n(\nu_1)$.

$$L_o^+(\nu_1) \longleftrightarrow (\exists \nu_2)(\nu_1 \prec \nu_2 \quad \& \quad (\nu_2)(\nu_1 \prec \nu_2 \longrightarrow (\exists \nu_3)(\nu_1 \prec \nu_3 \prec \nu_2)$$

$$\Phi^n \longleftrightarrow (\exists \nu_1)(L_n^-(\nu_1) \quad \& \quad L_o^+(\nu_1))$$

Φ^n asserts the existence of an n^{th} inductive upperlimit point which is also a lower limit point. ||

<u>Lemma 4.14</u>: The exists a 1-1 recursive function φ such that:

(i) $0 \notin$ range φ;

(ii) for every integer n, $\{\varphi(n) + 1, \ldots, \varphi(n) + L(n) + 1\}$ is disjoint from the range of φ.

We are now in a position to construct a recursive linear ordering whose truth set is Turing equivalent to H_ω. Consider an ordering of the form, $\mathcal{L} = \sum_{m \in Q} (\omega^m+1+\omega^*)$, where $Q \subseteq N$. It is not difficult to see that $\mathcal{L} \models \Phi^m$, iff $m \in Q$. We are looking for a recursive linear ordering of the form $\sum_{m \in Q} (\omega^m+1+\omega^*)$, such that, for every a,n:

(i) $\varphi(J(a,n)) + n + 1 \in Q \longleftrightarrow a \in H_n$;

(ii) $\varphi(J(a,n))+ n \in Q \longleftrightarrow a \notin H_n$

If we can find such a recursive linear ordering $\mathcal{L}$, then $(a)(n)(a \in H_n \longleftrightarrow \mathcal{L} \models \Phi^{\varphi(J(a,n)+n+1})$, and we see that H_ω is 1-1 reducible to the truth set of $\mathcal{L}$. Let $\Lambda_i = |\beta(K(i)), \varphi(i), L(i)|$ where β is the function defined in III-4.10. It is not difficult to prove the existence of a number $\tilde{e}$ such that $(i)(|\{\tilde{e}\}(i)| = \Lambda_i+1+\omega^*)$. Let $\mathcal{L} = |\Sigma(\tilde{e})|$.

By III-4.14, for every i, either $\wedge_i = \omega^{(\varphi(i)+L(i)+1)}$ or $\wedge_i = \omega^{(\varphi(i)+L(i))}$ depending on whether or not $K(i) \epsilon H_{L(i)}$. Therefore, by our construction of φ, $K(i) \epsilon H_{L(i)} \longleftrightarrow (\exists i)(\wedge_i = \omega^{(\varphi(i)+L(i)+1)})$. Hence, we see that $(i)(K(i) \epsilon H_{L(i)} \longleftrightarrow \mathfrak{L} \models \Phi^{(\varphi(i),L(i)+1)})$. ||

§5. Boolean Algebra

In this section we apply the methods of III-§2 and III-§3 to the problem of constructing a Boolean algebra which is not isomorphic to a recursive one.

In attempting to code a non-recursive function into the isomorphism type of a Boolean algebra, we immediately run up against that fact that every Boolean algebra has a recursive truth set in the elementary theory of Boolean algebra [See Tarski [11, pp. 62-64]. Therefore we look at a variant of the weak second order theory described in Ehrenfeucht [12]. Let A be a set of relation and operation symbols. Let "Indiv" be a one variable relation symbol such that "Indiv" ϵ A. We will denote by $\mathfrak{L}(A)$ the set of all formulas of the lower predicate calculus with identity, "=", which contains the predicates ϵ and predicates from A only. As models of $\mathfrak{L}(A)$ we will admit those models for the set formulas in which:

(i) $\hat{x}(\text{Indiv}(x))$ is a set of individuals;

(ii) $|M|$ (the set of elements of the model M) is the smallest set X such that $\hat{x}(\text{Indiv}(x)) \subset X$ and, if $x_1 \epsilon X, \ldots, x_k \epsilon X$ and, for every $1 \leq i \leq k$, Indiv (x_i), then $\{x_1, \ldots, x_k\} \epsilon X$;

(iii) the members of A are interpreted as relations and operations on $\hat{x}(\text{Indiv}(x))$;

(iv) ϵ is the set-theoretical ϵ-relation in M;

(v) "=" means "equality."

Let $\mathfrak{N} = \{|\mathfrak{N}|, R_1, R_2, \ldots\}$ be a structure (where $|\mathfrak{N}|$ is the universe of $\mathfrak{N}$). We designate by $\mathfrak{N}^{\#}$, the structure

$$\{\hat{S}(S \epsilon |\mathfrak{N}| \vee \{S \subseteq |\mathfrak{N}| \ \& \ \text{card } S < \aleph_0)\}, \epsilon, =, \text{Indiv}, R_0, R_1, \ldots\}.$$

<u>Lemma 5.1</u>: If $\mathfrak{N}$ is an arithmetic structure of finite similarity type, then the truth set of $\mathfrak{N}^{\#}$ in $\mathfrak{L}($ Indiv, $R_1, \ldots, R_n)$ is $\leq_T H_\omega$.

<u>Proof</u>: Since $\mathfrak{N}$ is an arithmetic structure, there is a sequence of integers, $m_1, \ldots, m_{n+1}$, such that $|\mathfrak{N}| \leq_T H_{m_1}$, $R_1 \leq_T H_{m_2}$, ..., $R_n \leq_T H_{m_{n+1}}$. Choose $m > \max\{m_1, \ldots, m_{n+1}\}$. Using H_m as an oracle, we can Gödel number the members of $|\mathfrak{N}^{\#}|$ in such a way that:

(i) "$x \epsilon |\mathfrak{N}^{\#}|$" is H_m-recursive;

(ii) "Indiv (x)" is H_m-recursive;

(iii) "ϵ" is H_m-recursive;

(iv) $R_1, \ldots, R_n$ are H_m-recursive.

(For every $1 \leq i \leq n$, R_i will fail to hold unless each of its arguments is in $\hat{x}$(Indiv (x).)).

Now that $\mathfrak{N}^{\#}$ is presented in this way, we use III-4.12. ||

Let $A = \{\text{Indiv.}, \leq, \cap, \cup, -\}$. If we can find an

H_ω-recursive Boolean algebra B such that H_ω is 1-1 reducible to the truth set of $B^{\#}$ in $\mathcal{R}$(Indiv., $\leq$, $\cap$, $\cup$, $-$), then B can't be isomorphic to a Boolean algebra whose relations and operations are arithmetic. III-4.10' will be useful towards this end.

First we will list some properties of Boolean algebras which can be expressed in the weak second order theory. Let $\nu_1, \nu_2, \ldots$ stand for individuals and $V_1, V_2, \ldots$ stand for finite sets of individuals.

(i) "ν_1 is bounded by the union of the members of V_1", or as we shall write, "$\mathbb{V}(\nu_1, V_1)$":

$$(\nu_1, V_1) \longleftrightarrow ((\nu_2)((\nu_3)(\nu_3 \in V_1 \longrightarrow \nu_3 \leq \nu_2) \longrightarrow \nu_1 \leq \nu_2))$$

(ii) "$|\nu_1|_n$ is an atom", or $A^n(\nu_1)$ (See I-2.3):

$$A^o(\nu_1) \longleftrightarrow ((\nu_1 \not\equiv 0 \quad \& \quad (\nu_2)(\nu_1 \cap \nu_2 \equiv 0 \vee \nu_1 \leq \nu_2)).$$

Suppose $A^o(\nu_1), \ldots, A^n(\nu_1)$ have been defined. Let "$\nu_1 \in I_n$" (See I-2.3) be the w.f.f., $(\exists V_1)((\nu_3)(\nu_3 \in V_1 \longrightarrow (\nu_3 \equiv 0 \vee A^o(\nu_3) \ldots \vee A^n(\nu_3))$ & $\mathbb{V}(\nu_1, V_1)$. We now define $A^{n+1}(\nu_1)$ as follows:

$$A^{n+1}(\nu_1) \longleftrightarrow (\sim(\nu_1 \in I_n) \quad \& \quad (\nu_2)(\nu_1 \cap \nu_2 \in I_n \vee \nu_2 - \nu_1 \in I_n))$$

(iii) "$|\nu_1|_n \equiv |\nu_2|_n$" .

(iv) "$|\nu_1|_n \leq |\nu_2|_n$",

(v) "$|\nu_1|^n$ bounds $\aleph_0$ atoms".

$(\exists\nu_2)(A^n(\nu_2)$ & $|\nu_2|_n \leq |\nu_1|_n)$ & $(V_1)((\nu_2)(\nu_2 \epsilon V_1 \longrightarrow (A^n(\nu_2)$ & $|\nu_2|_n \leq |\nu_1|_n)) \longrightarrow (\exists\nu_3)((\nu_4)(\nu_4 \epsilon V_1 \longrightarrow |\nu_4|_n \not\equiv |\nu_3|_n$ & $|\nu_3|_n \leq |\nu_1|_n$ & $A^n(\nu_3))$.

(vi) "$|\nu_1|^n$ is completely atomic"

$(\nu_2)(|\nu_2|_n \leq |\nu_1|_n \longrightarrow (\exists\nu_3)(A^n(\nu_3)$ & $|\nu_3|_n \leq |\nu_2|_n)$.

(vii) "$|\nu_1|_n$ bounds $\aleph_0$ atoms, but no completely atomic element bounded by $|\nu_1|_n$ bounds $\aleph_0$ atoms", $C^n(\nu_1)$.

(viii) Let Φ^n be $(\exists\nu_1)C^n(\nu_1)$.

We now consider Boolean algebras of the form $B = D_{\mathfrak{L}}$, where $\mathfrak{L} = (\sum_{m \epsilon Q} (\omega^m + \eta)) + 1$ and $Q \subseteq N$. We wish to prove that $(m)(m \epsilon Q$ iff $B^{\#} \models \Phi^m)$. We shall do this in several steps.

Claim (i): $\mathfrak{L}^{(k)} = (\sum_{m \epsilon Q} (\omega^{m \dot{-} k} + \eta)) + 1$. (See I-1.1)

Claim (ii): Let $\xi \epsilon B$. If for every m, ξ does not bound an interval of order type $\omega^m + \eta$, then $\xi = \xi_1 \cup \xi_2$, where ξ_1 is atomic and ξ_2 is atomless.

Proof: $\xi = [a_1,b_1) \cup \ldots \cup [a_n,b_n)$, where $a_1 \prec b_1 \prec a_2 \prec \ldots \prec a_n \prec b$

We claim that for every $1 \leq i \leq n$, $[a_i,b_i) = \xi_1^i \cup \xi_2^i$ where ξ_2^i is atomless and ξ_1^i is completely atomic.

case I: Suppose b_i lies in some summand η. If a_i lies below η, then $[a_i,b_i)$ bounds a segment of type ω^m for some m. Therefore a_i also lies in η and $[a_i,b_i)$ is atomless.

case II: Suppose b_i lies in some summand ω^p. ω^p is preceded by a summand of the form $\omega^q + \eta$. If a_i lies in or below ω^q, then $[a_i,b_i)$ bounds a segment of type ω^q. Therefore a_i must either lie in η or in ω^p. If a_i lies in ω^p then $[a_i,b_i)$ is completely atomic. Let * be the least element of ω^p. If a_i lies in η then $[a_i,b_i)$ $= [a_i,*) \cup [*,b_i)$ where $[a_i,*)$ is atomless and $[*,b_i)$ is atomic.

To complete the proof, we let $\xi_1 = \bigcup_{1\leq i\leq n} \xi_1^i$, $\xi_2 = \bigcup_{1\leq i\leq n} \xi_2^i$. ||

Claim (iii): $B \models \Phi^o$ iff $1 \in Q$.

Proof: If $1 \in Q$, then choose a point a in the first rational interval. $[0,a)$ bounds $\aleph_o$ atoms, but no completely atomic subinterval of $[0,a)$ bounds $\aleph_o$ atoms. Suppose $1 \notin Q$. Let ξ be a member of B. If ξ bounds a segment of type $\omega^m + \eta$ for some m, then $m > 1$, and, therefore, a completely atomic subinterval of some interval of ξ will bound $\aleph_0$ atoms. If not, then $\xi = \xi_1 \cup \xi_2$ where ξ_1 is completely atomic and ξ_2 is atomless. If ξ bounds $\aleph_0$ atoms, then ξ_1, which is completely atomic, must bound $\aleph_0$ atoms, since ξ_2 is atomless. ||

Claim (iv): $B^{\#} \models \Phi^n$, iff $n + 1 \in Q$.

Proof: Observe $B^{(n)} = D_{\left[\sum_{m \in Q} (\omega^{m \dot{-} n} + \eta) + 1\right]}$.

Since $B^{\#} \models \Phi^n$, iff $(B^{(n)})^{\#} \models \Phi^0$, we see $B^{\#} \models \Phi^n$ iff $1 \in \hat{x}(x = m \dot{-} n \;\&\; m \in Q)$. Hence $B^{\#} \models \Phi^n$ iff $n + 1 \in Q$.

Corollary 5.2: The weak second order theory of Boolean Algebras has $2^{\aleph_0}$ completions.

We now turn to the main theorem of § 5.

Theorem 5.3: There is a strict Boolean algebra B (See II) whose field, relation and operations are H_ω-recursive such that B isn't isomorphic to any Boolean algebra whose field, relation and operations are arithmetic.

Proof: Recall 4.10', for the case where $X = H_\omega$. Let $\wedge_i = |\beta(k(i), \varphi(i), L(i))^{H_\omega}|$. It is not difficult to see that there is a number $\tilde{e}$ such that $|(\{\tilde{e}\}(n))^{H_\omega}| = \wedge_i + \eta$. Let $\hat{\mathfrak{L}} = |(\Sigma(\tilde{e}))^{H_\omega}|$, and let $\mathfrak{L} = \hat{\mathfrak{L}} + 1$. $\mathfrak{L}$ is isomorphic to an H_ω-recursive linear ordering. Let $B = D_{\mathfrak{L}}$. By an argument very similar to that given in the example at the end of §4, we can show that

$$(\exists i)(\wedge_i = \omega^{[\varphi(i)+L(i)+1]}) \longleftrightarrow K(i) \in H_{L(i)}^{H_\omega}.$$

Therefore by claim (iv),

$$(i)\left(K(i) \in H_{L(i)}^{H_\omega} \longleftrightarrow B \models \Phi^{(\varphi(i)+L(i))}\right).$$

Thus $H_\omega^{H_\omega}$ is 1-1 reducible to the truth set of $B^{\#}$. Thus, by III-5.1, B is not isomorphic to a Boolean algebra whose field, relation and operations are arithmetic. By II-1.2', the field, relation and operations of B are H_ω-recursive. ||

<u>Theorem 5.3'</u>: Let $X \subseteq N$. There is a strict Boolean algebra, B, whose relation, field and operations are H_ω^X-recursive such that B is not isomorphic to a strict Boolean algebra whose operations, field, and relation are arithmetic in X.

CHAPTER IV

Analysis of ∂ by Means of the Analytic Hierarchy

§1. Preliminaries

The following facts will be needed in §2 wherein the main results of this Chapter are discussed. We give the preliminaries all at once in order to facilitate the exposition in §2.

Lemma 1.1: Any infinite recursively enumerable relation $\mathfrak{R}(x,y)$ is isomorphic to an r.e. relation whose field is total.

Proof: Let f be the recursive function such that

$$(x)(y)(\mathfrak{R}(x,y) \longleftrightarrow J(x,y) \in \text{Range } f).$$

We define $\hat{f}$ as follows. $\hat{f}(0) = J(0,1)$ if $K(f(0)) \neq L(f(0))$. $\hat{f}(0) = J(0,0)$, if $K(f(0)) = L(f(0))$. Suppose that $f(0),\ldots,f(n)$ have been defined. If $K(f(n+1)) = K(f(i))$, for some $0 \leq i \leq n$, then let $K(\hat{f}(n+1)) = K(\hat{f}(i))$. If $K(f(n+1)) = L(f(i))$, for some $0 \leq i \leq n$, then let $K(\hat{f}(n+1)) = L(\hat{f}(i))$. Otherwise, let $K(f(n+1))$ be the least number x such that, for any $0 \leq i \leq n$, $x \neq K(\hat{f}(i))$, $x \neq L(\hat{f}(i))$. If $L(f(n+1)) = K(f(n+1))$, then let $L(\hat{f}(n+1)) = K(\hat{f}(n+1))$. If, for some i, $K(f(n+1))$ is equal to $K(f(i))$ or $L(f(i))$, then let $L(\hat{f}(i))$ be $K(\hat{f}(i))$ or $L(\hat{f}(i))$ respectively. Otherwise, let $L(f(n+1))$ be the least number x such that, for any $1 \leq i \leq n$, $x \neq k(\hat{f}(i))$, $x \neq L(\hat{f}(i))$, $x \neq k(\hat{f}(n+1))$.

$\tilde{f}$ is recursive. Range $\tilde{f} = N$, and the relation "$J(x,y) \in$ Range $\tilde{f}$" is isomorphic to $\mathcal{R}$.

Lemma 1.1': Let $X \subseteq N$. Any X-r.e. relation isomorphic to an X-r.e. relation whose field is total.

Corollary 1.2: Any hyperarithmetic relation is isomorphic to a hyperarithmetic relation whose field is total.

Lemma 1.3: Let $\{\beta_i\}_{i \in N}$ be a sequence of ordinals. If $B_i \uparrow \alpha$, then:

(i) $\omega^{\beta_i} \uparrow \omega^{\alpha}$;

(ii) $\omega^{\alpha} = \sum_{i \in N} \omega^{\beta_i}$.

Proof: (i) Follows from the definition of ω^{α}.

(ii) By I-1.9, $\omega^{\alpha} > \sum_{i \leq n} \omega^{\beta_i} \geq \omega^{\beta_n}$, for every n. ||

Lemma 1.4: Let O be the standard Π_1^1 set of notation for the recursive ordinals (See Kleene [13, pp. 51-52]. There is a partial recursive function, f, such that if $e \in O$ and $[e]$ is the ordinal named by e, then $|f(e)| = \omega^{[e]}$. (See III-4.1).

Proof: We use effective transfinite induction. We seek a partial recursive function such that:

$f(1) = d_1$, where $|d_1| = \omega$;

$f(2^e) = \Sigma(d_2(b))$, where $d_2(b)$ is the Gödel number of the constant function which assigns $f(b)$ to every integer. (See III-4.4);

$f(3.5^y) = \Sigma(d_3(b))$, where $d_3(b)$ is the Gödel number of the recursive function $f(\{y\}(n))$.

We see by the Rogers' recursion lemma that such a partial recursive function exists (See Rogers [14, p.849]. Using IV-1.3 we prove by induction on $<_0$ that f has the desired properties. ||

Corollary 1.5:

(i) If α is a recursive ordinal, then so is ω^α.

(ii) $\omega_1 = \omega^{\omega_1}$.

(iii) $\theta < \omega_1 \longleftrightarrow \partial(\theta) < \omega_1$.

Proof: (i) Immediate from 1.4

(ii) Any finite sum of recursive ordinals must clearly be recursive. Thus, by I-2.9, $\omega_1 = \omega^\alpha$, for some α. If $\alpha < \omega_1$, then ω_1 is recursive. Contradiction. Thus $\alpha = \omega_1$.

(iii) If $\theta < \omega$, then by I-1.8, there is a $\beta < \omega_1$ such that $\theta < \omega^\beta < \omega_1$. Thus, $\partial(\theta) \leq \beta < \omega_1$. Conversely, if $\theta \geq \omega_1$, then $\partial(\theta) \geq \omega_1$. ||

Lemma 1.6: If $\mathcal{L}$ is isomorphic to a $\Sigma_1^1(\Pi_1^1)$ linear ordering, then so is $1 + \mathcal{L} + 1$.

Proof: Let $\mathcal{L} = \{L, \prec\}$ where L and $\prec$ are $\Sigma_1^1(\Pi_1^1)$. Define $x \ll y$ as follows: $x \ll y \longleftrightarrow x = 0 \vee y = 1 \vee x + 2 \prec y + 2$.

$\hat{\mathcal{L}} = \{\hat{x}(x=0 \vee x=1 \vee x=z + 2 \text{ where } z \in L), \ll\}$

$\hat{\mathcal{L}}$ is $\Sigma_1^1(\Pi_1^1)$ and $\hat{\mathcal{L}}$ is isomorphic to $1 + \mathcal{L} + 1$. ||

§2. Analysis of ∂ by means of the analytic hierarchy

Definition 2.1: (Spector [15]) A set S of natural numbers) is said to be "inductively defined with respect to a predicate Q" iff, for each ordinal τ, $S_\tau = \hat{x}((\exists \nu)(\nu < \tau \ \& \ Q(x, S_\nu)))$ where $(T_1 \subseteq T_2, \ \& \ Q(x,T_1)) \longrightarrow Q(x,T_2)$, and $S = S_c$, c being the least ordinal such that $S_c = S_{c+1}$.

Theorem 2.1: (Spector [15]) The ordinal c of a set inductively defined with respect to a Π_1^1 predicate is $\leq \omega_1$.

Theorem 2.3: If $\mathcal{L} = \{L, \prec\}$ is a Δ_1^1 linear ordering, then $\partial(\mathcal{L}) \leq \omega_1$.

Proof: By IV-1.1, we may assume that the field of $\mathcal{L}$ is total Let $S_\nu = \{J(x,y) \mid x \equiv_\nu^{\mathcal{L}} y\}$. By IV-2.2, it will suffice, for the proof of the theorem, to show that $S_{\partial(\mathcal{L})}$ is inductively defined with respect to a Δ_1^1 predicate. (See I-1.1). Let $Q(x,T)$ be the following predicate:

$$Q(x,T) \longleftrightarrow x = J(y,y) \lor x \in T \lor (\exists s)(seq(s) \ \&$$

$$(z)(K(x) \prec z \prec L(x) \lor L(x) \prec z \prec K(x)) \longrightarrow (\exists i)(0 \leq i \leq \ell(s) \ \&$$

$$J(z, s(i)) \in T).$$

$Q(x,T)$ is Δ_1^1, and, by I-1.1, $S_{\partial(\mathcal{L})}$ is defined inductively with respect to Q. Therefore $\partial(\mathcal{L}) = c \leq \omega_1$. ||

Corollary 2.4: If $\mathcal{L}$ is a Δ_1^1 scattered linear ordering, then $\partial(\mathcal{L}) < \omega_1$.

Proof: Since $\mathcal{L}$ is a Δ_1^1 scattered linear ordering, so is $(\mathcal{L}\cdot\omega) + 1$. If $\partial(\mathcal{L}) = \omega_1$, then, by I-1.7, $\partial((\mathcal{L}\cdot\omega) + 1) > \partial(\mathcal{L}) = \omega_1$. Contradiction. ||

Lemma 2.5: If B is a strict Boolean algebra with a scattered base and the operations and relations of B are hyperarithmetic, then;

(i) $\delta(B) < \omega_1$;

(ii) $B \approx D_\Theta$ where $\Theta < \omega_1$;

(iii) $B \approx B'$ where B' is strict and the operations and relations of B' are recursive.

Proof: By II-1.5, $B \approx D_{\mathcal{L}}$, where $\mathcal{L}$ is hyperarithmetic. By I-2.12, $\mathcal{L}$ is scattered. By I-2.10, and IV-2.4, $\delta(B) < \omega_1$. By L-2.14 and IV-1.5, $B \approx D_\Theta$, where $\Theta < \omega_1$. (iii) holds by II-1.2. ||

We now wish to prove that if $\mathcal{L}$ is Σ_1^1, then $\partial(\mathcal{L}) \leq \omega_1$. If we try to apply IV-2.2 directly, we run into the following difficulty: If $\mathcal{L}$ is a Σ_1^1 linear ordering, then the set $S_{\partial(\mathcal{L})}$, defined in the proof of IV-2.3, need not necessarily be defined with respect to a Π_1^1 predicate. This is so because the equivalence relations, $\overset{\mathcal{L}}{\underset{\nu}{\equiv}}$ range over the field of $\mathcal{L}$, and, if $\mathcal{L}$ is Σ_1^1, then its field need not be Π_1^1. (In fact, if $\mathcal{L}$ is Σ_1^1 and not Δ_1^1, then its field can't be Π_1^1.) Therefore, we take the following indirect approach. Let "Scat" be

$$\left\{ e \,\middle|\, \begin{array}{l} e \text{ is the Gödel number of a scattered} \\ \text{recursive linear ordering} \end{array} \right\}$$

<u>Lemma 2.6</u>: (i) Scat ϵ Π^1_1.

(ii) Scat $\not\epsilon$ Σ^1_1.

<u>Proof</u>:

(i) "e ϵ Scat" $\longleftrightarrow$ e ϵ Li & $(f)[(\exists x)(\exists y)(\{e\}(f(x),f(y)) = 0$ & $(z) \sim(\{e\}(f(x),z) = 0$ & $\{e\}(z,f(y)) = 0)]$.

Since Li is arithmetic, "Scat" is Π^1_1.

(ii) Suppose "Scat" ϵ Σ^1_1. Scat ϵ $\Sigma^1_1 \cap \Pi^1_1 = \Delta^1_1$. We are going to use the hyperarithmetic predicate, "e ϵ Scat", to take a hyperarithmetic sum of all scattered recursive linear orderings.

Let p_n be the $(n + 1)^{th}$ prime. Let $g(m)$ be the m^{th} member of Scat. If Scat is Δ^1_1, then g is a hyperarithmetic function. Let $L = \{p^m_n\}_{(m,n)\epsilon N^2}$. We define a linear ordering, $\prec$, on $\mathcal{L}$ as follows:

$$x \prec y \longleftrightarrow x = p^m_n \;\&\; y = p^{m'}_{n'} \;\&\; (n < n' \vee$$

$$(n = n' \;\&\; \{g(n)\}(m,m') = 0).$$

Let $\mathcal{L} = \{L, \prec\}$. $\mathcal{L}$ is a scattered, Δ^1_1 linear ordering, and every scattered recursive linear ordering is isomorphic to a segment of $\mathcal{L}$. Therefore, for any $\alpha < \omega_1$,

ω^{α} is isomorphic to a segment of $\mathfrak{L}$. Thus, $\partial(\mathfrak{L}) \geq \alpha$, for any $\alpha \leq \omega_1$, and, therefore, $\partial(\mathfrak{L}) \geq \omega_1$. This contradicts Corollary 2.4. ||

Lemma 2.7: If B is a strict Boolean algebra with a scattered basis and the field, relations, and operations of B are Σ_1^1, then:

(i) $\delta(B) < \omega_1$:,

(ii) $B \approx D_{\Theta}$, for some ordinal $\Theta < \omega_1$.

Proof: (i) Suppose that B satisfies the hypothesis of IV-2.7, and $\delta(B) \geq \omega_1$. If $e \in Li$, let $||e|| = \{N, \prec\}$, where $(x)(y)(x \prec y \longleftrightarrow \{e\}(x,y) = 0)$. We give the following Σ_1^1 definition of Scat:

"$e \in$ Scat" $\longleftrightarrow$ $^{(1)}(e \in L \quad \& \quad (\exists f)(f:D_{1+||e||+1} \longrightarrow B \quad \&$

f is a Boolean monomorphism.)

First of all, we claim that (1) is, in fact, a definition of "scat.". If such a monomorphism f, exists, then, by I-2.13, $1 + ||e|| + 1$ is scattered, and, hence, $||e||$ is scattered. Conversely, suppose $||e||$ is scattered.

Then $1 + ||e|| + 1$ is scattered and recursive. By IV-2.4, $\partial(1 + ||e|| + 1) < \omega_1$. By IV-2.5, $D_{1+||e||+1} \approx D_{\Theta'}$, where $\Theta' < \omega_1$. By I-2.14, I-2.10, and IV-1.5, $B \approx D_{\Theta}$, where $\Theta \geq \omega_1$. By I-2.13, there is a monomorphism, $\hat{f}:D_{\Theta'} \rightarrow D_{\Theta}$.

We define f by means of the following diagram:

$$\begin{array}{ccc} D_{\Theta'} & \xrightarrow{\hat{f}} & D_{\Theta} \\ \approx \uparrow & & \downarrow \approx \\ D_{1+||e||+1} & \overset{f}{\dashrightarrow} & B \end{array}$$

Thus, it suffices, for the proof of IV-2.7, to show that (1) is Σ^1_1. To do this, we just write it out. (See II-1.3)

Let $\hat{0}$ be the zero element of B.

$$\text{"}e \in \text{Scat"} \longleftrightarrow {}^{(1)}\ (e \in \text{Li} \ \& \ (\exists f)(x)(y)(z)\big((\{g_1(e)\}(x) = 0$$

$$\longrightarrow x \in B) \ \& \ (\{g_2(e)\}(x,y) = 0 \longrightarrow x \leq y) \ \&$$

$$(\{g_3(e)\}(x,y,z) = 0 \longrightarrow x \cup y = z) \ \&$$

$$(\{g_4(e)\}(x,y,z) = 0 \longrightarrow x \cap y = z) \ \&$$

$$(\{g_5(e)\}(x,y) = 0 \longrightarrow x = \overline{y}) \ \&$$

$$((\{g_1(e)\}(x) = 0 \ \& \ f(x) = \hat{0}) \longrightarrow x = 0\big)$$

Noting that "$x \in B$", "$x \leq y$", "$x \cap y = z$", "$x \cup y = z$", "$x = \overline{y}$" are Σ^1_1, and bringing the function quantifiers of (1) to the front of (1), we see that (1) is in Σ^1_1. ||

(ii) This follows directly from (i). ||

Lemma 2.8: If $\mathcal{L}$ is a scattered, Σ^1_1 linear ordering, then $\partial(\mathcal{L}) < \omega_1$.

Proof: Follows directly from II-1.2, and IV-2.7. ||

Theorem 2.9: If $\mathcal{L} = \{L, \prec\}$ is a Σ^1_1 linear ordering then $\partial(\mathcal{L}) \leq \omega_1$.

Proof: By I-1.11, $\mathcal{L} = \sum_{r \in \hat{\eta}} \mathcal{L}_r$ where each $\mathcal{L}_r$ is scattered, and where $\partial(\mathcal{L}) = \ell.u.b.\ \{\partial(\mathcal{L}_r) | r \in \hat{\eta}\}$. If $\partial(\mathcal{L}) > \omega_1$, then there is some $\mathcal{L}_{\tilde{r}}$ such that $\partial(\mathcal{L}_{\tilde{r}}) > \omega_1$. There must be two elements a, $b \in \mathcal{L}_{\tilde{r}}$ such that $a \not\equiv^{\mathcal{L}}_{\omega_1} b$. Let $\mathcal{L} \restriction [a,b) = \{\hat{L}, \ll\}$. By I-1.1, $\partial(\mathcal{L} \restriction [a,b)) \geq \omega_1$. However, $\mathcal{L} \restriction [a,b)$ is scattered and Σ^1_1; i.e. $x \ll y \longleftrightarrow a \prec x \prec b \ \&\ a \prec y \prec b \ \&\ x \prec y$. Contradiction. ||

Theorem 2.10: If B is a strict Boolean algebra whose field, relation and operations are Σ^1_1, then $\delta(B) \leq \omega_1$.

Proof: Suppose B satisfies the hypothesis of IV-2.10, and $\delta(B) > \omega_1$. By II-1.1, and I-2.7, there is an isomorphism

$f: B \longrightarrow D_{\mathcal{L}}$ where $\partial(\mathcal{L}) > \omega_1$. As in the proof of IV-2.9, there are two elements $a, b \in \mathcal{L}$ such that $\mathcal{L}\restriction[a,b)$ is scattered and $\partial(\mathcal{L}\restriction[a,b)) \geq \omega_1$. Let $\xi = f^{-1}([a,b))$. We define the following Boolean algebra $\hat{B}$.

(i) The field of $\hat{B}$ is $\hat{x}(x \in B \ \& \ x \leq \xi)$.

(ii) For every $x, y, z \in \hat{B}$, $(x \mathbin{\hat{\cup}} y = z \longleftrightarrow x \cup y = z)$ and $(x \mathbin{\hat{\cap}} y = z \longleftrightarrow x \cap y = z)$ and $(x = \hat{\bar{z}}$ iff $x = \xi \cap \bar{z})$.

$\hat{B} \approx D_{\mathcal{L}\restriction[a,b]}$. $\hat{B}$ is strict and scattered and its field, operations and relations are easily seen to be Σ_1^1. Since $\partial(\mathcal{L}\restriction[a,b)) \geq \omega_1$, $\delta(\hat{B}) \geq \omega_1$. This contradicts IV-2.8. ||

<u>Example</u>: Let $\mathcal{L} = \{L, \prec\}$ be the Gandy ordering [16] with Π_1^1 initial segment O_1 of order type ω_1. $\mathcal{L}$ is recursive and $\partial(\mathcal{L}) = \omega_1$. Let $\hat{L} = \hat{x}(x \in L \ \& \ x \in O_1)$. Let $x \ll y$ mean $x \in \hat{L}$ and $y \in \hat{L}$ and $x \prec y$, and let $\hat{\mathcal{L}} = \{\hat{L}, \ll\}$. $\hat{\mathcal{L}}$ is a Π_1^1 linear ordering of order type ω_1. $\hat{\mathcal{L}}.2$ is a Π_1^1 linear ordering of order type $\omega_1 \cdot 2$. $\partial(\hat{\mathcal{L}} \cdot 2) > \omega_1$. $\delta(D_{\hat{\mathcal{L}} \cdot 2}) > \omega_1$. By II-1.2, $D_{\hat{\mathcal{L}} \cdot 2}$ is a strict Boolean algebra whose relation and operations are Π_1^1. By III-2.5, $D_{\hat{\mathcal{L}} \cdot 2}$ can't be isomorphic to any strict Boolean algebra whose operations, field and relations are Σ_1^1.

<u>Corollary 2.11</u>: There is a strict Boolean algebra B with a

scattered basis such that the operations, field and relation of B are Π^1_1 and B isn't isomorphic to any strict Boolean algebra whose field, operations and relations are Σ^1_1.

Now we turn to an interesting parallel between strict Boolean algebras whose field, operations, and relations are Σ^1_1 and Lindenbaum algebras of Π^1_1 - axiomatizable theories. This parallel is expressed in the following theorem. (See Chapter II, pp. 24-25).

Theorem 2.12: If B is the Lindenbaum algebra of a Π^1_1 - axiomatizable theory, T, then $\delta(B) \leq \omega_1$.

Proof: Let $\mathcal{G}$ be the language of T. If α is a sentence of $\mathcal{G}$, and $I \subseteq B$ is an ideal, let $|\alpha|_I$ be the equivalence class of α in B/I. If $C \subseteq B$, let I(C) be the ideal generated by C. Let $A \subseteq B$ be the Π^1_1 set of axioms.

Claim: The following predicates are Π^1_1 in α, β and C:

(i) $\alpha \leq \beta$;

(ii) $\alpha \leq \alpha \wedge \sim \alpha$;

(iii) $|\alpha|_{I(c)} \leq |\beta|_{I(c)}$;

(iv) $|\alpha|_{I(c)} \leq 0$;

(v) $|\alpha|_{I(c)}$ is an atom;

(vi) $|\alpha|_{I(c)}$ is a finite union of atoms.

Proof of claim: (i) $\alpha \leq \beta$ iff $\alpha \vdash_A \beta$. The only occurrence of the predicate "$\tau \in A$" in the proof theoretic definition of "$\alpha \vdash_A \beta$" is positive. (ii) Special case of (i). (iii) $|\alpha|_{I(c)} \leq |\beta|_{I(c)}$ $\longleftrightarrow$ "There is a sequence $\tau_1, \ldots, \tau_n$ of elements of C such that $\beta \cap \sim \alpha \leq \tau_1 \vee \ldots \vee \tau_n$". (v) $|\alpha|_{I(c)}$ is an atom $\longleftrightarrow$ "$(\tau)(\tau$ is a sentence of $\mathfrak{S}$) $\longrightarrow (|\tau \wedge \alpha|_{I(c)} = 0 \vee |\alpha|_{I(c)} \leq |\tau|_{I(c)}))$." (vi) Follows directly from (v). ||

We define $Q(\alpha, C)$ as follows:

$$Q(\alpha, C) \longleftrightarrow |\alpha|_{I(c)} = 0 \vee \text{"} |\alpha|_{I(c)} \text{ is a finite union of atoms "} \vee |\alpha| = 0.$$

$I^B_{\delta(B)}$ (see I-2.3) is inductively defined with respect to Q and Q is Π^1_1. Therefore, by IV-2.2, $\delta(B) \leq \omega_1$.

Example: Let $\mathfrak{L} = \{L, \prec\}$ be the Gandy ordering. We can assume without loss of generality that L = N. Let $P_1, P_2, P_3, \ldots$ be a countable set of propositional letters. Let $\mathfrak{S}$ be the set of all Boolean combinations of $\{P_n\}$. Let $A \subset \mathfrak{S}$ be the following set of axioms.

$$\{\sim p_n \quad p_{n'} | n < n' \vee (n' < n \ \& \ n' \not\in O_1)\}.$$

A is a Σ^1_1 set. Let T be the theory whose axioms are A, and let B be the Lindenbaum algebra of T. $\{p_n\}_{n \in N}$ is an ordered basis for B. The order type of $\{p_n\}_{n \in N}$ is $\omega_1 + 1$. Thus $\delta(B) > \omega_1$. ||

Remark: To justify IV-2.12 which might, at first glance seem somewhat artificial, we make the following observations. In order for the proof of IV-2.11 to go through it suffices that B be a Boolean algebra such that "$\leq$" is Π_1^1, and that there exist hyp. functions $f_1(x,y)$, $f_2(x,y)$, $f_3(x)$ such that for every $x, y \in B$:

(i) $f_1(x,y) \equiv x \cup y$;

(ii) $f_2(x,y) \equiv x \cap y$;

(iii) $f_3(x) = \bar{x}$.

However B is isomorphic to an algebra which satisfies these conditions iff B is isomorphic to the Lindenbaum algebra of a Π_1^1 axiomatizable theory. If B is the Lindenbaum algebra of some Π_1^1-axiomatizable theory then $\leq$ is Π_1^1 and f_1, f_2, f_3 are the propositional connectives " $,\wedge,\sim$". On the other hand, if B satisfies the conditions mentioned above, then we can use the standard proof that every Boolean algebra is isomorphic to the quotient of a free algebra and a filter, to show that B is isomorphic to the Lindenbaum algebra of a Π_1^1-axiomatizable theory. ||

BIBLIOGRAPHY

1. P. Erdös and A. Hajnal, A classification of the denumerable order types, Fund. Math., Vol. 51 (1962), 117-129.

2. A Mostowski and A. Tarski, Boolesche Ringe mit geordneter Basis, Fund. Math., 32 (1939), 69-86.

3. S. Mazurkievicz and W. Sierpinski, Contribution a' la Topologie des ensembles denumbrables Fund. Math., 1, 1920, 17-27.

4. R. D. Mayer and R. S. Pierce, Boolean Algebra with Ordered Bases, Pac. J. Math., 10 (1960) 925-942.

5. S. C. Kleene, Arithmetical Predicates and Function Quantifiers, Trans. of the Amer. Math. Soc. (1955), 312-339.

6. S. C. Kleene, Recursive Predicate and Quantifiers, Trans. of the Amer. Math. Soc., 53 (1943), 71-43.

7. H. Rogers, Computing Degrees of Unsolvability, Math. Annalen, 138 (1959) 129-140.

8. S. C. Kleene, Introduction to Metamathematics, New York and Toronto (van Nostrand). 1952.

9. M. Davis, Computalibility and Unsolvability, McGraw-Hill, 1958, x+210 pp.

10. S. C. Kleene, An Notation for Ordinal Numbers, J. S. L. 3 (1938) 150-155.

11. R. Tarski, Arithmetical Classes and Types of Boolean Algebras, Bull Amer. Math. Soc., 55 (1949)

12. A. Ehrenfeucht, Application of the Theory of Games to the Completeness Problem for Formalized Theories, Fund-Math., 50, (1961), 129-141.

13. S. C. Kleene, <u>On the Forms of Predicates in the Theory of Constructive Ordinals</u>, Amer. Jour. of Math. <u>66</u>, (1944) 41-58.

14. H. Rogers, <u>Recursive Functions Over Partial Well Orderings</u>, Proc. of Amer. Math. Soc., <u>10</u> (1959), 847-853.

15. C. Spector, <u>Inductively Defined Sets of Natural Numbers</u>. Warsaw Symposium on Infinitistic Methods, 1959, 98-102.

16. R. Gandy, <u>Proof of Mostowski's Conjecture</u>, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. Vol. 9, (1960), 577-582.

BIOGRAPHY

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