Phase transitions in dissipative quantum transport and mesoscopic nuclear spin pumping

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Topological phase transitions can occur in the dissipative dynamics of a quantum system when the ratio of matrix elements for competing transport channels is varied. Here we establish a relation between such behavior in a class of non-Hermitian quantum walk problems [M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. 102, 065703 (2009)] and nuclear spin pumping in double quantum dots, which is mediated by the decay of a spin-blockaded electron triplet state in the presence of spin-orbit and hyperfine interactions. The transition occurs when the strength of spin-orbit coupling exceeds the strength of the net hyperfine coupling and results in the complete suppression of nuclear spin pumping. Below the transition point, nuclear pumping is accompanied by a strong reduction in current due to the presence of nondecaying “dark states” in this regime. Due to its topological character, the transition is expected to be robust against dephasing of the electronic degrees of freedom.

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Since the first observation of spin blockade in vertical GaAs double quantum dots,1 spin-blockaded transport has been observed in a variety of systems such as lateral double quantum dots in GaAs, Si, and Si/SiGe heterostructures,2–4 InAs nanowires,5,6 and carbon nanotubes.7 Much of this work was driven by the need to better understand the coupled dynamics of electron and nuclear spins in double dot systems with potential applications in spintronics and quantum computation. A variety of interesting and surprising phenomena such as current bistabilities and hysteresis,2,5,7 very long time scale switching,8 and periodic oscillations9 have been observed, and their origins linked to the dynamical polarization of nuclear spins (DNP). Although the involvement of nuclear spins in these phenomena is clear, in many cases the underlying mechanisms remain a mystery.

Spin blockade of dc transport occurs in a two-electron double quantum dot when the electron spins form a triplet state that prohibits both electrons from occupying the same site. In this case, as shown in Fig. 1(a), residual current arises from mechanisms that do not conserve the electron spin such as the spin-orbit interaction, and the hyperfine coupling to nuclear spins in the host lattice. Because the hyperfine contact interaction conserves the total spin of all electrons plus nuclei, each hyperfine-mediated electron spin flip is accompanied by a nuclear spin flip in the opposite direction. In the presence of strong spin-orbit coupling, e.g., as in InAs systems,5,6 the physics of DNP can be very different from that studied previously in the absence of spin-orbit coupling.9,10 In particular, by making several transitions between singlet and triplet states using a combination of hyperfine and spin-orbit processes, the decay of a single electron spin can lead to a change in nuclear polarization by an amount which can have either sign, and a magnitude potentially even greater than one unit of angular momentum.11

In this paper we explore the rich quantum dynamics of coupled electron and nuclear spins that results from the coherent competition of hyperfine and spin-orbit decay channels in spin-blockaded quantum dots. We focus on the polarization transferred to the nuclear spin bath by the decay of a single electron in one of the blockaded triplet states. This quantity reveals a strikingly high sensitivity of the DNP production efficiency to the presence of the competing spin-orbit decay channel.

In order to study DNP in this regime, we develop a class of models which capture the essential physics of polarization transfer during electron spin decay. One of the main difficulties in describing nuclear pumping in the presence of spin-orbit coupling stems from the lack of a concrete conservation law which directly relates the changes of electron and nuclear spin polarizations. Any model of this process must account for the possibility of multiple electron spin transitions which can lead to a variety of final nuclear spin states. For a typical system containing $N = 10^6$ nuclear spins, the exponentially large Hilbert space makes exact analytical or numerical solutions difficult to obtain. However, as we show...
which are independent of the specific grouping of nuclear spins. The behavior obtained for $d>1$ is essentially analogous to that found in the $d=1$ case (the giant spin model). While the details of the behavior near the transition are sensitive to the particular decomposition into collective spins, the strong suppression of DNP in the spin-orbit dominated phase is found to be generic, suggesting that it will persist in more realistic models.

I. CONSTRUCTION OF THE MODEL

We begin by reviewing the relevant two-electron states of a spin-blockaded double quantum dot, see e.g., Ref. 1. For simplicity, suppose that the left and right dots each support a single orbital state $|L\rangle$ or $|R\rangle$. An applied potential bias approximately compensates the charging energy when two electrons occupy the right dot. Energetically, electrons which form a spin singlet can thus assume either the $(1,1)$ or $(0,2)$ charge configuration with one electron on each dot or both electrons on the right dot:

$$
| (1,1)_{S} \rangle = \frac{1}{2} (| LR \rangle + | RL \rangle) \otimes (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle),
$$

$$
| (0,2)_{S} \rangle = \frac{1}{\sqrt{2}} (| RR \rangle \otimes (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle)).
$$

When the electrons form a spin triplet, the Pauli exclusion principle forbids double occupation of the right dot.

Due to tunnel coupling, the singlet states $| (1,1)_{S} \rangle$ and $| (0,2)_{S} \rangle$, Eq. (1), hybridize to form “bonding” and “antibonding” states $| S \rangle$ and $| S' \rangle$. In the presence of an external magnetic field, the triplet splits into its three Zeeman sublevels. For concreteness, we consider decay of the state $| T_{S} \rangle$ when its energy is close to that of the singlet state $| S \rangle$, with all other states far away in energy [see Fig. 1(b)]. The relevant electronic states for our model are thus

$$
| T_{S} \rangle = \frac{1}{\sqrt{2}} (| LR \rangle - | RL \rangle) \otimes | \uparrow \uparrow \rangle,
$$

$$
| S \rangle = C_{11}| (1,1)_{S} \rangle + C_{02}| (0,2)_{S} \rangle.
$$

Because of spin-orbit coupling, interdot tunneling is accompanied by a spin-rotation which couples the triplet and singlet states $| T_{S} \rangle$ and $| S \rangle$ with an amplitude $u$. In the basis of Eq. (2), the purely electronic part of the Hamiltonian is written as

$$
\hat{H}_{0} = \begin{pmatrix}
\epsilon_{T_{S}} & u \\
u & \epsilon_{S}
\end{pmatrix}, \quad \epsilon_{S} = \epsilon_{S} - iy/2,
$$

where $\epsilon_{T_{S}}$ and $\epsilon_{S}$ are the energies of the states $| T_{S} \rangle$ and $| S \rangle$, respectively. The imaginary term $-iy/2$ in $\epsilon_{S}$ accounts for the decay of the singlet state due to coupling of its $| (0,2)_{S} \rangle$ component to the drain, see Fig. 1(a).

Without loss of generality, we take the spin-orbit coupling matrix element $u$ to be real. A microscopic derivation of the value of $u$ is beyond the scope of this work. An estimate of
its value can be obtained by noting that $u$ arises from a spin rotation which accompanies interdot tunneling. For single channel transport,\cite{14,15} whether of ballistic or tunneling type, the spin-rotation angle $\theta$ can be estimated as $\theta \approx d / \ell_{\text{SO}}$, where $d$ is the distance traveled (here the interdot separation), and $\ell_{\text{SO}}$ is the spin-orbit length. Assuming that tunneling is dominated by a single channel, we obtain the estimate $u \approx C_{02} \theta t_{\text{c}}$, where $t_{\text{c}}$ is the interdot tunnel coupling, and $C_{02}$ is the amplitude of $|0,2\rangle$ in the final state $|S\rangle$, see Eq. (2). The spin-orbit length $\ell_{\text{SO}}$ is material dependent, with values of a few tens of microns in GaAs,\cite{16} down to hundreds of nanometers in, e.g., InAs.\cite{9} In addition, the value of $u$ is sensitive to the orientation of the dots relative to the crystallographic axes and to the direction of the applied magnetic field.

The hyperfine interaction between electron and nuclear spins also couples the triplet and singlet states. In a two-electron system, the hyperfine Hamiltonian

$$\hat{H}_{\text{HF}} = A \sum_n \mathbf{S}_n \cdot [\mathbf{S}_1 \delta(\mathbf{x}_1 - \mathbf{x}_n) + \mathbf{S}_2 \delta(\mathbf{x}_2 - \mathbf{x}_n)]$$  \hspace{1cm} (4)

couples the spin $\mathbf{S}_{1,2}$ of each electron to each nuclear spin $\mathbf{I}_n$, with weight proportional to the probability to find the electron at the location $\mathbf{x}_n$ of nucleus $n$. Here, for simplicity, we consider a single species of nuclear spin but a generalization to multiple species is straightforward. The spin-flip terms $\hat{S}_z \hat{I}_z$ couple the electron states with spin projections along the $z$ axis differing by one unit of angular momentum.

In the basis of Eqs. (2) and (3), Hamiltonian (4) takes the form

$$\hat{H}_{\text{HF}} = \begin{pmatrix} \frac{1}{2} \sum_n A_n \hat{S}_n^+ \hat{I}_n^- + \frac{C_{11}}{2\sqrt{2}} \sum_n \eta_n A_n \hat{S}_n^+ \hat{I}_n^- \\ \frac{C_{11}}{2\sqrt{2}} \sum_n \eta_n A_n \hat{S}_n^+ \hat{I}_n^- \end{pmatrix}$$  \hspace{1cm} (5)

where

$$A_n = A \rho(\mathbf{x}_n)$$  \hspace{1cm} (6)

is the hyperfine coupling weighted by the local electron density $\rho(\mathbf{x}_n) = \langle \psi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{x}_n) \psi \rangle$ [see Fig. 1(c)], and, due to antisymmetry of the wave function, $\eta_n = +1$\text{(-1)} if nucleus $n$ is located in the left (right) dot. Because the hyperfine interaction is local, the off-diagonal matrix elements of Hamiltonian (4) between $|T\rangle$ and $|S\rangle$ are proportional to the amplitude of $|1,1\rangle$ in $|S\rangle$, i.e., to the parameter $C_{11}$ in Eq. (2).

The sign factors $\eta_n$ indicate that the difference between transverse nuclear polarizations in the left and right dots couples the electron triplet and singlet levels. The mathematical annoyance of alternating signs can be removed by applying a $\pi$ rotation about the $z$ axis to all spins in the right dot via the operator $\hat{U} = e^{-i\mathbf{S}_2 \cdot \hat{\mathbf{S}}_2} \mathbf{I}_2$, where the sum is taken over all spins in the right dot. In the rotated frame, the Hamiltonian $\hat{H}_{\text{HF}} = \hat{U} \hat{H}_{\text{HF}} \hat{U}^\dagger$ takes the simpler form

$$\hat{H}_{\text{HF}}' = \begin{pmatrix} \frac{1}{2} \sum_n A_n \hat{S}_n^+ \hat{I}_n^- + \frac{C_{11}}{2\sqrt{2}} \sum_n \eta_n A_n \hat{S}_n^+ \hat{I}_n^- \\ \frac{C_{11}}{2\sqrt{2}} \sum_n \eta_n A_n \hat{S}_n^+ \hat{I}_n^- \end{pmatrix}$$  \hspace{1cm} (7)

The factor $1 / \sqrt{2}$ arises from the normalization of the states in Eqs. (1) and (2). Up to these numerical prefactors, the transformed Hamiltonian is equivalent to that describing the hyperfine interaction for a single electron in a quantum dot with an electron density profile consistent with the distribution of couplings $\{A_n\}$.

II. GIANT SPIN MODEL ($d=1$)

Many interesting aspects of the problem can be understood by focusing on the special case where the couplings $A_n$ are the same for all nuclei inside the dots, $A_n = \bar{A}$. This distribution arises for a boxlike electron-density profile, which is uniform within the dots and zero outside. Since in this case the square of the total nuclear spin operator, $\hat{S}^2 = (\sum_n \mathbf{I}_n)^2$, commutes with the Hamiltonian, all nuclei in the system act together coherently as one “giant” spin of fixed total angular momentum $I$. More precisely, because the evolution operator is block diagonal in the basis of $\hat{S}^2$ eigenstates, we can investigate the behavior for a fixed total angular momentum $I$, and then average the results over an appropriate distribution of $I$ values.

What determines the size $I$ of the giant spin? Recall the rules of angular momentum addition from basic quantum mechanics, focusing on the case of spin-1/2 for simplicity. If we bring together an even (odd) number $N$ of spin-1/2 moments $\{\mathbf{I}_n\}$, the resultant total spin can take any non-negative integer (odd-half-integer) value up to $N/2$. Each total spin value $I$ can, in general, be obtained in many ways depending on how the various spins are combined. Analysis shows that each total spin value $I$ occurs with a multiplicity\cite{17} $\Omega(I) = \frac{\bar{A}^N}{N!\sqrt{2\pi}} \frac{1}{\sqrt{N/2}}$ determined by the combinatorics of angular momentum addition.

When $N$ is large, the distribution of multiplicities approaches a simple form

$$\Omega(I) \approx (2I + 1)e^{-I(I+1)/2}, \quad I_0 = \sqrt{N/2}. \hspace{1cm} (8)$$

This expression can be derived either by approximating the factorials in the exact expression using Stirling’s formula or by a “quasiclassical” argument. The latter treats the sum of a large number of spins, $\mathbf{I} = \sum_n \mathbf{I}_n$, as a sum of statistically independent random vector variables. In the limit of large $N$, according to the central limit theorem, the quantity $\mathbf{I}$ has a three-dimensional Gaussian distribution. The width of the Gaussian is determined by the condition $\langle \mathbf{F} \rangle = \sum_n \langle \mathbf{I}_n^2 \rangle = (3/4)N$. The probability density for the total spin vector $\mathbf{I}$ goes as $dP(\mathbf{I}) \approx e^{-I^2/2N} d_I d_Id_Id_Id_Id_Id_Id_I$. For the distribution of the absolute values, this gives the Maxwell-Boltzmann distribution $dP(I) \approx e^{-I^2/2N} dI$. Taking into account the $(2I+1)$-fold degeneracy of each total-spin-$I$ manifold, we obtain the expression given in Eq. (8).

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For nuclear spin species with spins greater than 1/2, e.g., the spin-3/2 nuclei of Ga and As or the spin-9/2 nucleus of In, the same basic rules of angular momentum addition apply. Explicit formulas for the combinatorial multiplicity factors can still be found,\textsuperscript{17} however the expressions are more cumbersome than in the spin-1/2 case. Nonetheless, the qualitative features and the shape of the distribution in the large-$N$ limit are the same as for the spin-1/2 case. In particular, the multiplicities for the spin-$s$ case are given by $\Omega_h(I) \propto (2I+1)e^{-IP/(2s(I+1))N}$.

For fixed $I$, the configuration space of the system is defined by the electron states $|T_+\rangle$ and $|S\rangle$, and by the $z$ projection $m$ of total nuclear spin, $\hat{I}m = m|m\rangle$, with $-I \leq m \leq I$. Combining Eqs. (3) and (7), the Hamiltonian for this system can be written as

$$H_{1D} = \begin{pmatrix} \Delta \varepsilon + \frac{1}{2} \hat{A}^+ \hat{A} - i \gamma/2 & 0 \\ 0 & \hat{A}^+ \hat{A} - i \gamma/2 \end{pmatrix},$$

(9)

where $\Delta \varepsilon$ is the triplet-singlet detuning and $\hat{A}^\pm$ is the raising (lowering) operator for the giant spin.

Below we neglect the polarization-dependent Overhauser shift $\hat{A}$ by absorbing its mean value into the definition of $\Delta \varepsilon$. For polarizations which are not too large, this approximation is justified by the fact that the typical off-diagonal matrix elements of $H_{1D}$ are of the order $\hat{A}/\sqrt{N}$ while the Overhauser shift only changes by an amount of order $\hat{A}$ when the nuclear polarization changes by one unit of angular momentum. For a typical dot containing $N = 10^6$ nuclear spins, the variation of the Overhauser shift thus imposes only a small perturbation on the dynamics of the system. In a similar spirit, we also ignore the nuclear Zeeman energy, which is assumed to be small compared with the inverse lifetime of the blockaded state.

Decay of the blocked triplet state occurs through electron spin-flip transitions to the state $|S\rangle$, which is broadened due to its coupling to the drain lead. These transitions can be mediated by either the hyperfine interaction or the spin-orbital interaction. The hyperfine process is accompanied by a change of the $z$ projection of nuclear spin, $\Delta m = \pm 1$, whereas for the spin-orbital process $\Delta m = 0$. As illustrated in Fig. 2(a), the resulting coherent dynamics in the combined Hilbert space of electron and nuclear degrees of freedom can thus be viewed as a hopping problem on a one-dimensional bipartite lattice.

In the basis $\{|m\rangle \otimes |T_+\rangle, |S\rangle\}$, the state of the system $|\psi\rangle$ is described by the amplitudes $\psi_m^L = \langle m | T_+ | \psi \rangle$ and $\psi_m^S = \langle m S | \psi \rangle$, and evolves according to the equations of motion (with $\hbar = 1$)

$$i \frac{d}{dt} \psi_m^L = \Delta \varepsilon \psi_m^L + u \psi_m^S + v_m \psi_{m+1}^L,$$

$$i \frac{d}{dt} \psi_m^S = -i \gamma/2 \psi_m^L + u \psi_m^L + v_{m-1} \psi_{m-1}^L,$$

(10)

with

![FIG. 3. (Color online) Results for the giant spin model, Eq. (10).](image)

(a) Dependence of hyperfine matrix element $v_m$ on the $z$ projection of total nuclear spin for a giant spin of length $I$. For moderate values of the spin-orbit coupling, $u = u_0 < v_{\text{max}}$, the system possesses critical points $\pm m_0$, see Eq. (14), where the spin-orbit and hyperfine couplings are roughly equal. (b) Average polarization change and inverse lifetime versus initial polarization. For all traces we take $\Delta \varepsilon = 0$ and $\gamma = v_{\text{max}}$. When spin-orbit coupling is strong, $u = u_1 = 1.2v_{\text{max}}$, nuclear spin pumping is suppressed. For moderate values of the spin-orbit coupling, $u = u_0 = 2/3v_{\text{max}}$, $\langle \Delta m \rangle = 1$ for initial states with $v_{m_0} > u_0$, and $\langle \Delta m \rangle = 0$ for $v_{m_0} < u_0$. Near the critical points $m_0 = m_0/I$, the system possesses “dark states” with extremely long lifetimes. We include a decay term $-i \gamma/2$ with $\gamma = 10^{-3}$ $\gamma$ in the triplet energy to cut off the divergence of the lifetime near the critical points.

$$v_m = \hat{A}/(I + 1) - m(m + 1).$$

(11)

The hopping amplitudes $v_m$, which originate from the transverse hyperfine field, attain a maximum value $v_{\text{max}} = \hat{A}/(I + 1)$ for unpolarized states $m = 0$, and become small near maximum polarization $|m| \leq I$, see Fig. 3(a).

Suppose the system is initially in the blockaded electron spin state with nuclear polarization $m_0$. What is the average change in nuclear polarization $\Delta m = m - m_0$ caused by the decay of the electron spin? To formulate the problem more precisely, we consider the situation where an electron is injected into the triplet state $|T_+\rangle$ at time $t=0$, with an initial
nuclear spin state characterized by total angular momentum $I$ and $z$ projection $m_0$. The system then executes a “quantum walk” under the equations of motion, Eq. (10), with initial state

$$\psi_m^T = \delta_{m,m_0}, \quad \psi_m^s = 0.$$  \quad (12)

The wave packet describing the quantum walker will spread throughout the lattice and leak out through its components on the $S$ sites, decaying completely as $t \to \infty$. The value of $m$ at the site from which the system decays determines the final value of nuclear polarization left behind when the electron escapes. Given the probability $P_m$ for the system to decay from each singlet site $m$, we would like to evaluate

$$\langle \Delta m \rangle = \sum_m (m - m_0)P_m, \quad P_m = \int_0^\infty \gamma |\psi_m^b(t)|^2 dt.$$ \quad (13)

This expression for $P_m$ results from the fact that the non-Hermitian equations of motion in Eq. (10) lead to decay which is a sum over local terms describing decay from each site of the lattice, $\frac{d}{dt} \langle \psi \rangle = -\sum_m \gamma |\psi_m^b|^2$. Because the system decays completely as $t \to \infty$, $\sum_m P_m = 1$.

To explore the behavior of this model, we have solved Eq. (10) numerically with the initial condition given in Eq. (12) for giant spins with $I = 25$, $I = 50$, and $I = 100$. The polarization transfer $\langle \Delta m \rangle$, Eq. (13), is plotted for each initial polarization $m_0$ in the upper panel of Fig. 3(b). In addition, we also show the inverse of the average dwell time $\bar{t} = -\int_0^\infty \frac{dt}{\gamma |\psi|^2}$ for each case in the bottom panel of Fig. 3(b).

Two very different situations arise depending on the relationship between the spin-orbit coupling matrix element $u$ and the maximum hyperfine coupling matrix element $v_{\text{max}}$. In case I, indicated by the dotted line $u = u_I > v_{\text{max}}$ in Fig. 3(a), spin-orbit coupling dominates the dynamics for any initial polarization $m_0$, and nuclear spin pumping is strongly suppressed [see Fig. 3(b)]. For $u = u_I < v_{\text{max}}$, however, the dynamics can be dominated either by hyperfine coupling or by spin-orbit coupling, depending on the value of the initial polarization. The system possesses critical points $m = \pm m_*$, with

$$m_* \approx \pm I \left(1 - \left(\frac{u}{v_{\text{max}}}\right)^2\right)^{1/2},$$ \quad (14)

where, locally, the strengths of hyperfine and spin-orbit coupling are nearly equal. For initial polarizations satisfying $|m_0| < m_*$, the dynamics are hyperfine-dominated and approximately one unit of angular momentum is transferred to the nuclear spin subsystem per electron. Outside the critical points, i.e., for $|m_0| > m_*$, spin-orbit coupling dominates and polarization transfer is strongly suppressed [see Fig. 3(b)].

As the length of the giant spin increases, the distinction between the behaviors in these two regimes becomes more sharply defined. In particular, $\langle \Delta m \rangle$ becomes sharply quantized to 1 for $|m_0| < m_*$. Near the critical points, the dwell time of an electron in the system can become very long [see lower panel of Fig. 3(b)].

The behavior in all of these regimes can be understood in terms of a simplified model. If the quantum walk in Eq. (10) only explores a window of sites which is small compared with the scale of the giant spin, $2I + 1$, then we may approximate the $m$-dependent hopping amplitudes $v_m$ by a single amplitude $v = v_{m_0}$ which characterizes the strength of the transverse hyperfine field when the giant spin has $z$ projection $m_0$. In this same spirit, we also extend the lattice to infinity, $-\infty < m < \infty$. These approximations make the quantum walk translationally invariant and allow us to find an exact analytical solution to the dynamics.

The translationally invariant model described above is identical to the one dimensional non-Hermitian quantum walk which was studied in Ref. 13. In that work, we found that the expected displacement $\langle \Delta m \rangle$ achieved before decay is quantized as an integer,

$$\langle \Delta m \rangle = \begin{cases} 1 & v > u, \\ 0 & u > v. \end{cases}$$ \quad (15)

The quantized value of $\langle \Delta m \rangle$ is determined by the winding of the phase between two components of the Bloch eigenstates of the bipartite one-dimensional system as the momentum $k$ is taken through the Brillouin zone. Equivalently, the value of $\langle \Delta m \rangle$ can be determined directly from the winding number of the complex amplitude $A_k = u + ve^{ik}$. In the regime where this winding number is zero, corresponding to the situation where spin-orbit coupling $u$ is stronger than the hyperfine coupling $v$, the result $\langle \Delta m \rangle = 0$ indicates that no angular momentum is pumped into the nuclear spin subsystem. As we will see below, this behavior is quite general and persists for more refined models which go beyond the giant spin approximation.

### III. TOPOLOGICAL TRANSITION

The behavior of the giant spin model, displayed in Fig. 3(b), closely resembles that of the translationally invariant model of Ref. 13. Due to the underlying topological structure of the problem, the quantization of the average displacement, Eq. (15), is a robust phenomenon. While for the giant spin model some distortions of the behavior are observed in the highly polarized regions, $|m| \approx I$, where $v_m$ varies strongly with $m$, the primary features have direct analogs in the translationally invariant model. In particular, the striking suppression of decay near the upper critical point can be traced to the divergence of the lifetime which accompanies the topological transition between winding and nonwinding phases (see Ref. 13).

The topological transition is manifested in the nontranslationally invariant system through the presence of a topologically protected “dark” edge state which is localized at the phase boundary between winding and nonwinding phases. This state has zero overlap with the electron singlet state and thus does not decay. Physically, the extended lifetime results from the fact that, near the critical point, the effective hyperfine and spin-orbit fields responsible for causing electron spin transitions can cancel.

One important implication of the topological character of the transition between winding and nonwinding phases is
that the associated behavior is protected against certain types of decoherence and noise. Indeed, as discussed for the translationally invariant model in Ref. 13, any local perturbation which distinguishes only the site type, decaying or nondecaying, with no dependence on the site index $m$, leaves both the winding number and the displacement in Eq. (13) invariant. For more general nontranslationally invariant perturbations, however, which carry “which-site” information to the environment, the displacement in Eq. (13) ceases to be quantized.

Interestingly, noise in the electron part of the Hamiltonian, Eq. (3), which is blind to the nuclear polarization state and thus translationally invariant in the language of Ref. 13, usually represents the main type of noise in quantum dots. Time variation in $\Delta e$ and $\gamma$ can arise, for example, due to charge fluctuations around the dot which lead to switching fluctuations of the singlet and triplet electron energies, $e_{S}$ and $e_{T}$, and fluctuations in the tunnel coupling to the leads. Crucially, such fluctuations only very weakly affect the hyperfine Hamiltonian, Eq. (5), since the hyperfine coupling strength depends on the electron wave function profile in each dot but not on the electron energy. Note that noise in the hyperfine Hamiltonian, if present, would be of the which-site type for the Knight-shift terms, and of the nonlocal (off-diagonal) type for the transverse coupling terms, and would thus lead to dephasing. Yet, since noise in a double quantum dot is dominated by fluctuations in electron energy levels, which do not carry which-site-type information, we expect the topologically protected phenomena such as quantization of nuclear spin pumping to be readily observable.

It is straightforward to generalize the above analysis of decoherence to the giant spin model, in which the parameters of the effective tight binding problem, Eq. (10), are slowly varying as a function of position. Based on the above argument, we expect the overall dependence of the displacement on $m$, pictured in Fig. 3, to remain unchanged in the presence of noise. In particular, the transition between winding and nonwinding phases will occur at the same critical values of $m$ given by Eq. (14) as in noiseless case. This can be seen microscopically by considering the dark states. Because these states have zero amplitude on all decaying sites, they remain intact in the presence of noise in $\Delta e$ and $\gamma$, which is true both for the translationally invariant problem, and for the problem with position dependent couplings, Eq. (11). Robustness of the dark states signals that the transition, pictured in Fig. 4, is topologically protected with respect to dephasing of electron dynamics.

While the gross features of dynamics in the giant spin problem are similar to those of the translationally invariant problem, there are several interesting differences. In particular, although suppression of decay is seen near both critical points, the effect is much more dramatic near the upper critical point where the dark state is stable; near the lower, unstable critical point, an analogous exponentially growing (delocalized) dark state can be found.

The average displacement $\langle \Delta m \rangle$ displays small “overshoots” near the critical points, where $\langle \Delta m \rangle \neq 0,1$ [see Fig. 3(b)]. These features can be understood with intuition gained from the translationally invariant model. Semiclassically, the site-dependent hopping amplitudes $v_{m}$ give the “walker” a position-dependent effective mass (i.e., band gap) which goes through a minimum at the critical point. As a result, the walker experiences a force which attracts it to the critical point. For $m_{0} \ll m_{\text{c}}$, where each electron already has a tendency to transfer one unit of angular momentum to the nuclear spin subsystem, this additional force results in a transfer of more than one unit of angular momentum per electron, $\langle \Delta m \rangle > 1$.19 Similarly, above the critical point, the attraction leads to a negative polarization, $\langle \Delta m \rangle < 0$. The shape of $\langle \Delta m \rangle$ around the lower critical point can be understood analogously.

In experiments, the length $I$ of the giant spin and its $z$ projection $m_{0}$ in the initial state are, in general, random variables picked from the thermal distribution. As described above in the discussion surrounding Eq. (8), for a completely random collection of spins, the length $I$ of the net spin is distributed according to the “Maxwell-Boltzmann”-type distribution $p(I) \propto (2I+1)^{5}c^{-1/I_{0}}$ while $m_{0}$ is uniformly distrib-
uted for each choice of $I$, $p(m_0)=$const. In Fig. 4(a) we show the numerically obtained expected polarization transfer $\langle \Delta m \rangle$ and inverse lifetime $1/\tau$; averaged over all initial conditions $m_0$, for a giant spin with fixed total spin $I=50$. The existence of dark states near the critical points is manifested in the suppressed current for $u<v_{\text{max}}$. Compared with the translationally invariant case, where $\langle \Delta m \rangle$ displays a sharp step as a function of $u/(u+v)$, here the step is rounded into the phase where $u\approx v_{\text{max}}$. When spin-orbit coupling dominates, however, i.e., for $u>v_{\text{max}}$, the suppression of nuclear spin pumping is nearly complete.

The shape of the rounded step can also be understood simply within the context of the translationally invariant quantum walk model. For each initial condition $m_0$, we take $\langle \Delta m \rangle=1$ if $|m_0|<m_*$ (hyperfine dominates), or $\langle \Delta m \rangle=0$ if $|m_0|>m_*$ (spin-orbit dominates). After averaging over all initial polarizations $m_0$, we find

\[ \langle \Delta m \rangle = 0 \times \left(1 - \frac{m_*}{I}\right) + 1 \times \frac{m_*}{I} = \frac{m_*}{I}, \]

where $m_*$ is given by Eq. (14). Expression (16) is plotted as a dotted line in Fig. 4(a). After averaging over the distribution of $I$ values, the transition is broadened, see Fig. 4(b). However, the nuclear spin pumping rate is still exponentially suppressed in the strong spin-orbit region, $u/(u+Al_0) \approx 0.75$. The good agreement with exact numerics for the giant spin model further indicates that the intuition gained from the translationally invariant model provides a useful tool for understanding the behavior in the more realistic situation.

IV. BEYOND GIANT SPIN ($d>1$)

Faced with the surprising prediction of complete suppression of DNP in the spin-orbit dominated phase, it is natural to wonder to what extent this result relies on the assumption of uniform hyperfine coupling (i.e., on the validity of the giant spin model). The giant spin approximation tightly constrains the dynamics by truncating the dimension of the Hilbert space from $2^d$ down to $O(N)$, where $N$ is the number of nuclear spins in the system. This appears to be a rather severe approximation. To address this concern, we now explore a more general class of models, derived in a similar spirit, which allow us to investigate the effects of nonuniform coupling.

As illustrated in Fig. 1(c), hyperfine coupling is generally strong near the center of the dots, where electron density is maximal, and weak near the edges, where electron density is small. To improve upon the uniform coupling model, consider dividing the nuclei into two groups, one of “strongly coupled” spins, and the other of “weakly coupled” spins. In this approximation, depicted in Fig. 2(b), the nuclear spins within each group form two separately conserved collective spins of lengths $I_1$ and $I_2$. Here the coupled dynamics of electron and nuclear spins can be viewed as a hopping model on a two-dimensional lattice, where the two dimensions index the $z$ projections of the two collective spins, $m_1$ and $m_2$. This model captures both the transfer of polarization from the electron spins to the nuclear spins, and the electron-mediated transfer of polarization between the two groups of nuclear spins.

Continuing this reasoning, we can further refine the model by approximating the smooth electron density profile by $d$ shells of constant density $\rho_a$, with $a=1,\ldots,d$ (see red dotted line in Fig. 1(c) and, e.g., Ref. 25). In this case, the nuclear spins couple to form $d$ collective spins $I_1,\ldots,I_d$. The corresponding polarization transfer dynamics can be viewed as a hopping problem on a $d$-dimensional lattice indexed by the polarizations $m_1,\ldots,m_d$ of the $d$ collective spins [see Fig. 2(c)]. This “quantum walk” is described by equations of motion analogous to Eq. (10) with the position variable $m$ replaced by a vector $m=(m_1,\ldots,m_d)$

\[ i\dot{\psi}_m^T = \Delta e \psi_m^T + u \psi_m^T + \sum_{a} v_{m}^{(a)} \psi_{m+e_a}, \]

\[ i\dot{\psi}_m^{-} = -i\gamma/2 \psi_m^T + u \psi_m^T + \sum_{a} v_{-m}^{(a)} \psi_{-m-e_a}, \]

where $v_{m}^{(a)}=A_{m}/I_d^2(I_d+1)-m_{a}(m_{a}+1)$ is the hyperfine coupling to collective spin $I_a$ with $A_a=A_{m}$, and $e_a$ is the unit vector along the axis describing the polarization of $I_a$.

Our goal is now to calculate the polarization $\langle \Delta m \rangle$ transferred into the nuclear spin bath

\[ \langle \Delta m \rangle = \sum_{m} (m-m_{0})P_{m}. \quad P_{m} = \int_{0}^{\infty} \gamma|\psi_m(t)|^2 dt \]

under the dynamics of Eq. (17) with initial condition

\[ \psi_m^T = \delta_{m,m_{0}}, \quad \psi_m^{-} = 0. \]

In particular, we will be interested in determining which features of the results are independent of the particular grouping into collective spins and which survive as the level of refinement, $d$, is increased.

Based on the success of the translationally invariant approximation to the quantum walk in the giant spin model (i.e., the case $d=1$), we begin by replacing the $m$-dependent hopping amplitudes $v_{m}^{(a)}$ by constants $v_{m}^{(a)}=v_{m_{0}}^{(a)}$. Note that the approximations associated with making hopping translationally invariant and with extending the lattice of states for each collective spin to infinity become more severe as the size of each spin decreases. Thus, although we will proceed for arbitrary dimension $d$, this number should be considered small compared to the total number of nuclear spins in the double dot.

The next step is to pass to the momentum representation, $\psi_{m}^{\pm} = \frac{1}{(2\pi)^{d/2}} \int dp de^{ik_{e} \cdot k_{m}} \psi_{m}^{\pm}(p)$, where the integral is taken over the Brillouin zone $-\pi \leq k_{a} \leq \pi$, with $a \in \{1,2,\ldots,d\}$. These Fourier states correspond to coherent nuclear spin states with the transverse component of each collective spin $a$ pointing along the azimuthal angle $k_{a}$. Due to the translational invariance of the system, the equations of motion in the Fourier representation break up into $2 \times 2$ blocks, one for each momentum $k$.
The outer integral is taken over the $d-1$ momenta $k_{\beta\neq\alpha}$. The expression inside the braces is identical to Eq. (18) for the displacement in the one-dimensional model. As shown there, the value of this integral is quantized as either 0 or 1 depending on the winding of $\theta_k=\arg(A_k)$ as $k_\alpha$ is taken around the Brillouin zone.

For the one-dimensional case, quantization means that the expected change in polarization per electron through the system is either 1 if $\theta_k$ wraps the origin ($v>u$, hyperfine coupling exceeds spin-orbit coupling) or 0 if it does not ($u>v$, spin-orbit coupling exceeds hyperfine coupling). Roughly speaking, the winding of $\theta_k$ therefore distinguishes whether or not the hyperfine coupling is strong enough for the electron to flip the nuclear spin.

To understand the meaning of Eq. (22) in the multidimensional case, it is helpful to view the integral in the following way: for each fixed configuration of $d-1$ collective spins described by the $d-1$ angles $\{k_{\beta\neq\alpha}\}$, the expression inside the braces is either 0 or 1 depending on, for the given field of the other spins and strength of spin-orbit coupling, whether the electron’s hyperfine coupling $v^{(0)}$ to the remaining spin is strong enough to induce a spin flip [i.e., whether or not $\theta_k$ winds the origin as $k_\alpha$ is varied from $-\pi$ to $\pi$, see Fig. 5(a)]. The integral over the remaining $d-1$ variables simply counts the “phase space” over which this condition is satisfied. This result is represented graphically for the case $d=2$ in Fig. 5(b).

A. Special case: $d=2$

Recently, the relative dynamics of nuclear spins in the two dots of spin-blockaded double quantum dots has attracted considerable experimental\cite{26,27} and theoretical\cite{25,28,29,30,31} attention. From a practical point of view, understanding the behavior of the difference of polarization between the two dots

$$\langle \Delta m_1 \rangle = \sum_{\alpha} \left( \begin{array}{cc} 1 & 0 \\ \frac{\theta_1}{\pi} & 0 \end{array} \right) \langle \Delta m_2 \rangle = \begin{cases} 0 & v' > u + v'' \\ \frac{\theta_2}{\pi} & |u - v'| \leq v'' \\ 0 & v' < u - v'' \end{cases}$$

Expression (23) is plotted in Fig. 5(c) as a function of spin-orbit coupling strength $u$ for the fixed ratio $v' / v'' = 2$. For strong spin-orbit coupling $u > v' + v''$, neither collective
spin is pumped at all: we find complete suppression of DNP in the spin-orbit dominated phase just as in the single giant spin model. Interestingly, for weak spin-orbit coupling we find a quantization \( \langle \Delta m_n \rangle = 1, \langle \Delta m_n \rangle = 0 \), which indicates that on average the collective spin with stronger hyperfine coupling to the electron absorbs the full angular momentum, while the more weakly coupled spin is not pumped at all. For an asymmetric system composed of one large dot and one small dot, electron density and thus average hyperfine coupling is larger on the smaller dot. Thus in the limit of weak spin-orbit coupling, \( u \ll u' \), nuclear pumping may be highly asymmetric, with DNP initially produced primarily on the smaller of the two dots.

B. General results

In contrast to the one-dimensional translationally invariant model, Fig. 5(c) shows that in higher dimensions \( \langle \Delta m_n \rangle \) is not strictly quantized as an integer. The breakdown of quantization results from the appearance of “mixed” phases where the winding number along one dimension of the Brillouin zone is 1 for some values of the remaining momenta, and 0 for the others. However, for the quantum walk in Eq. (17) in any dimension \( d \) there is always a “nonwinding” phase with \( u > \Sigma \omega^{(\alpha)} \), where all winding numbers are 0 for all values of \( \mathbf{k} \) in the Brillouin zone. In this phase, \( \langle \Delta m_n \rangle = 0 \) for all \( \alpha \). In fact, either through graphical methods or with a few lines of algebra, one can see that, as \( u \) is increased for fixed \( \{ \omega^{(\alpha)} \} \), all of the \( \langle \Delta m_n \rangle \) vanish simultaneously at the point \( u = \Sigma \omega^{(\alpha)} \). Thus very generally, strong spin-orbit coupling can lead to a dramatic suppression of nuclear spin pumping.

Furthermore, the expected polarizations \( \langle \Delta m_n \rangle \) are determined purely by geometrical constraints imposed by the set of hyperfine matrix elements \( \omega^{(\alpha)} \). Importantly, just as in the case of uniform hyperfine coupling discussed in Secs. II and III, the results are insensitive to the values of \( \Delta e \) and \( \gamma \). As discussed above, the quantized displacement and dark states are thus robust against electrical noise which makes \( \Delta e \) and \( \gamma \) time-dependent without carrying which-site-type information about the value of nuclear polarization to the environment.

In other words, the suppression of pumping results from coherence in the nuclear spin state and is robust against dephasing of the electron singlet and triplet states.

V. CONCLUSIONS

In this work we have analyzed the coupled dynamics of electron and nuclear spins in spin blocked quantum dots by mapping it onto a non-Hermitian quantum walk. By making the approximation of uniform hyperfine coupling to all nuclear spins, the problem can be treated as a one-dimensional hopping problem. Through a combination of both numerical and analytical approaches, we find that nuclear spin pumping is strongly suppressed whenever the strength of spin-orbit coupling exceeds the magnitude of the hyperfine coupling. Analytical results for a higher dimensional hopping problem which approximates the system with spatially nonuniform hyperfine coupling indicate that the suppression of pumping is a generic phenomenon.

The transition in the nuclear spin pumping efficiency is accompanied by an abrupt change in average current through the double dot. The behavior near the transition is dominated by “dark states” which result from complete destructive interference between hyperfine and spin-orbit processes. Due to the presence of the dark states, the expected lifetime of the spin-blockaded triplet state is significantly longer in the regime where hyperfine and spin-orbit couplings compete than in the regimes where only one of the two mechanisms is present.

Many interesting features in spin pumping dynamics can be seen as manifestations of the topological character of the transition. In particular, this framework helps us to understand the effect of different types of noise and dephasing on the system. We find that the most important noise sources for quantum dots do not destroy the transition, and thus expect the topologically protected dark states and quantized pumping to be readily accessible in experiments.

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18 Here we use the fact that $I(I+1) = 1$ for $I \geq 1$.
19 This result does not violate conservation of angular momentum, as the spin-orbit interaction allows the spins to exchange angular momentum with orbital degrees of freedom and eventually to the lattice.