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Systematic wireless network coding

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Abstract—We present a systematic network coding strategy for cooperative communication, in which some nodes may replicate-and-forward packets in addition to sending random linear combinations of the packets. We argue that if this strategy is used only at certain nodes in the network, the throughput will not be reduced relative to random linear network coding. Furthermore, if packets can traverse the entire network in their systematic (uncoded) form, per-packet delay can be reduced, decoding complexity can be reduced, and the potential to recover packets from incomplete coded blocks will be improved. We describe this approach and provide an analysis of the packet loss rate for fixed-rate coding on a multihop path.

I. INTRODUCTION

Network coding, first introduced in [1], is a useful technique for cooperative communication in multihop networks. In this approach, in order to deliver packets from a source to one or more destinations via multihopping, the source and intermediate nodes, rather than applying a replicate-and-forward strategy to the packets, will send some function (preferably a linear function) of the packets to downstream nodes toward the destination(s). By diffusing or mixing the packets in this way, unique information can be carried on multiple paths for delivery to the destination(s). Previous work in [2], [3] proposes a practical approach for carrying this out: nodes in the network can send random linear combinations of the packets and append the coefficients of the linear combination (which are chosen from a large finite field) to the header of the packet to allow for decoding at the destination(s). Although random linear network coding can provide improved throughput [2], [3] and allows for distributed implementation, it can have drawbacks. Consider random linear coding over a fixed-size group or *generation* of K packets. The destination(s) will need to receive at least K random linear combinations before *any* of the packets can be decoded, inducing a reception delay on a per-packet basis and a likelihood that *none* of the K packets can be recovered if fewer than K random linear combinations are received. Additionally, the source node may need to wait for K packets to arrive before it can send anything, which will also induce delay. A solution to the latter drawback is explored in [3] by beginning encoding and

transmission on a partial block and adding packets to the generation as they arrive.

In order to address these drawbacks, in this work we explore the use of a technique in which the source and intermediate nodes can replicate-and-forward packets in addition to sending random linear combinations. We refer to this technique as *systematic network coding* in that the packets transmitted by a node consist of the packets received by that node in addition to random linear combinations of the packets received by the node. This form of network coding differs from techniques considered in previous work [2], [3], where all packets transmitted by a node are linear combinations of the packets it has received. There are multiple potential benefits of systematic network coding. First, it can reduce the computations involved in the finite field arithmetic needed to construct random linear combinations and to decode packets. Furthermore, if packets can traverse the entire network in their systematic (uncoded) form, per-packet delay can be reduced, decoding complexity can be reduced, and the potential to recover packets from incomplete coded blocks will be improved.

Systematic network coding is best applied only at certain nodes in a network. Specifically, if systematic network coding is used at nodes on multiple diverse paths, then it can inhibit the ability of network coding to exploit path diversity. In the following section we argue that if a node has the property of being a *cut-node*, then systematic network coding at that will not reduce the rank of the encoding vectors of packets sent toward the destination(s). The source node of any flow is always a cut-node, and additionally, any node that connects disjoint sets of nodes in the network is a cut-node. Thus systematic coding is applicable to hierarchical networks in which one node serves as a bridge between different parts of the network. Furthermore, we argue that systematic coding can be used at nodes on one of multiple diverse paths without increasing the chance that nodes send redundant information. Systematic coding within a network has been explored in previous work. In [4], rateless systematic coding is applied to a tandem network and performance tradeoffs in terms of complexity, delay, and memory requirements are presented. In [5], the source node in a single-hop multicast flow makes use of systematic coding and this approach is shown to be beneficial in terms of delay performance. Systematic network coding is used in mobile devices in [6] and is shown to provide improved throughput over a traditional network coding

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approach. Our work differs from these previous works in that: (i) we present general conditions on node connectivity for applying systematic coding and these conditions are suited to our interest in hierarchical networks, and (ii) we analyze performance benefits for fixed (possibly adaptive) rate coding rather than rateless coding due to our interest in satellite networks with large propagation delays.

In Section II we formally describe the systematic network coding approach and provide conditions under which it can be applied. In Sections III and IV we explore the performance benefits offered by systematic coding in terms of packet loss rate and per-packet delay.

II. SYSTEMATIC NETWORK CODING

A. Description

We first describe the approach using notation similar to that in [3]. The network is modeled by a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where \mathcal{V} is the set of nodes and \mathcal{E} the set of directed links between nodes. The links in the network may correspond to lossless links or they may correspond to erasure links, where packets are dropped with some probability. Consider a group of K packets originating at a source node $s \in \mathcal{V}$ and destined for a set of nodes $\mathcal{T} \subseteq \mathcal{V}$. Unicast transmission corresponds to $|\mathcal{T}| = 1$ and multicast transmission corresponds to $|\mathcal{T}| > 1$. We denote the source packets as $\mathbf{s} = [s_1, \dots, s_K]$ and each packet s_i is a vector of symbols over a finite field \mathbb{F}_q , where q is a power of two. Nodes in the network will cooperate to deliver the K source packets to all nodes in \mathcal{T} . We assume that at least one path in \mathcal{G} exists between s and each node $t \in \mathcal{T}$. Packets sent for the $s - \mathcal{T}$ flow can traverse multiple paths; those paths may coincide at some links and they may be disjoint at other links. In this work we consider the situation in which there is only one session or flow that is active in the network and intra-session coding can be carried out by coding among the K packets within the $s - \mathcal{T}$ flow. Our results can also be applied to networks serving multiple flows and performing intra-session coding among those flows.

Consider the coding and transmission of packets at a node $v \in \mathcal{V}$. With random linear network coding, packets traversing the network will be linear combinations of the source packets \mathbf{s} ; the vector of K coefficients used in forming a random linear combination is referred to as the *encoding vector* for that packet and is assumed to be appended to the packet as in [2], [3]. Let $\mathbf{x}_v = [x_1, \dots, x_k]$ denote a set of k packets that are received at node v . We assume that nodes can monitor their incoming packets and will discard packets that are not innovative (i.e., a packet will be discarded if its encoding vector is not linearly independent of the encoding vectors of packets that v has already received); as such, $k \leq K$ and the encoding vectors of packets in \mathbf{x}_v are all linearly independent. Node v can perform coding on the packets \mathbf{x}_v and will transmit a set of n packets, which we denote $\mathbf{y}_v = [y_1, \dots, y_n]$. In general we do not assume that the n packets y_1, \dots, y_n are linearly independent; node v may wish to send $n > k$ packets with redundant information in order to overcome erasures on

the links to downstream nodes. The coding performed at node v is given by the *local code generator matrix* L_v where

$$\mathbf{y}_v = \mathbf{x}_v L_v. \quad (1)$$

Any set of packets traversing the network can also be written in terms of a *global code generator matrix* denoted by G , whose columns are the encoding vectors of the packets. For instance, we can write

$$\mathbf{y}_v = \mathbf{s} G_v = \mathbf{x}_v L_v \quad (2)$$

where G_v is formed by the product of the local code generator matrices for the nodes on the path(s) between s and v . The random linear network coding technique introduced in [2], [3] corresponds to a local code generator matrix L_v in which *all* elements of the matrix are chosen randomly and uniformly from \mathbb{F}_q . In contrast, systematic network coding corresponds to a local code generator matrix given by the following structure.

$$L_v^{(s)} = \begin{cases} \begin{bmatrix} I_{k \times k} & P_{k \times (n-k)} \end{bmatrix}, & k < n \\ \begin{bmatrix} I_{n \times n} \\ 0_{(k-n) \times n} \end{bmatrix}, & k \geq n, \end{cases} \quad (3)$$

where $I_{i \times i}$ is the $i \times i$ identity matrix, $0_{i \times j}$ is a $i \times j$ matrix of zeros, and $P_{i \times j}$ is a $i \times j$ matrix of elements chosen uniformly from \mathbb{F}_q . The identity matrix contained in $L_v^{(s)}$ means that $\min(n, k)$ packets are simply forwarded rather than coded at node v . We note that $\text{rank}(L_v^{(s)}) = \min(n, k)$.

B. Sufficiency

The benefit of network coding for cooperative communication is that packets are *mixed* as they traverse the network and that packets traversing disparate paths will, with high probability, consist of independent sets of information. As such, the systematic coding technique described in (3) could potentially inhibit the cooperative capabilities of network coding in that it simply forwards some packets - this can result in a loss in throughput from the random linear network coding strategy in [2], [3]. We consider use of the local code generator matrix in (3) at nodes that do not contribute to path diversity. Nodes at which we are interested in applying systematic network coding are defined as follows.

Definition 1. *A node v is a cut-node for the $s - \mathcal{T}$ flow if the removal of v causes all of v 's downstream destinations \mathcal{T}_v , where $\mathcal{T}_v \subseteq \mathcal{T}$, to become disconnected from the source s . Conversely, if there is a forward path from v to t_i , for any $t_i \in \mathcal{T}_v$, where the removal of v does not disconnect t_i from s , then v is not a cut-node.*

For a unicast flow, $|\mathcal{T}| = 1$, a cut-node as defined above is equivalent to an articulation point of a graph [7]. The set of cut-nodes includes the source node s as well as nodes that serve as gateways between disjoint parts of a network. The following proposition states that the use of systematic coding at a cut-node is sufficient to propagate information through the network without loss of rank in the global code generator matrix.

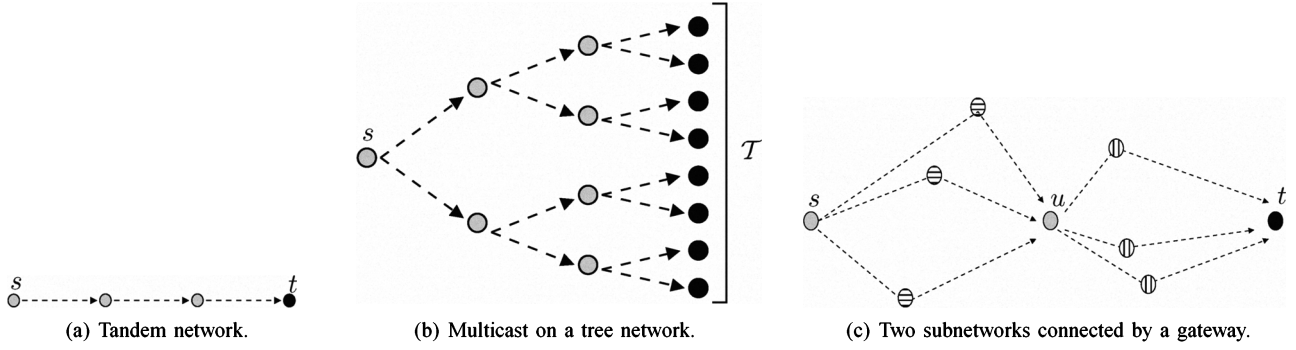


Fig. 1. Example networks. Destination nodes are shaded black. Cut nodes are shaded gray. Note that in 1(c), one of the horizontal-striped nodes and one of the vertical-striped nodes can perform systematic coding.

Proposition 1. For a cut-node $v \in \mathcal{V}$, use of the local code generator matrix $L_v^{(s)}$ does not reduce the dimension of the space spanned by the encoding vectors of the packets propagated to the downstream neighbors of v .

Proof: Let G_v^U denote the $K \times k$ global code generator matrix corresponding to the k packets received at node v from its upstream neighbors. The dimension of the space spanned by the encoding vectors of the packets received at v is $\text{rank}(G_v^U) = k$. By the definition of a cut-node, the information propagated to the downstream neighbors of v is given entirely by $y_v = sG_v^U L_v$. In general, for an $m \times k$ matrix A and a $k \times l$ matrix B , it holds that [8],

$$\text{rank}(A) + \text{rank}(B) - k \leq \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)). \quad (4)$$

Let $A = G_v^U$ and $B = L_v$. For $L_v = L_v^{(s)}$ as in (3), the left and right hand sides of (4) are both equal to $\min(k, n)$ and the upper bound holds with equality. ■

Proposition 1 implies that in comparison to the schemes in [2], [3] where the local code generator matrix consists of all elements chosen uniformly from \mathbb{F}_q , the use of systematic network coding at cut-nodes will not reduce the number of packets decoded at the destination(s). In addition to cut-nodes, systematic network coding can be applied at other nodes in the network. Let \mathcal{C} denote the set of all possible encoding vectors for generations of size K over \mathbb{F}_q , i.e., the set of all K -length vectors with elements from \mathbb{F}_q . As a general rule, encoding vectors are chosen independently and uniformly from the set \mathcal{C} ; however, nodes on one path can deviate from this rule and employ systematic network coding.

Proposition 2. Let \mathcal{V}_S denote a set of nodes on the $s - \mathcal{T}$ path that share one or more successor nodes. The use of systematic network coding at **one** of the nodes in \mathcal{V}_S does not increase the probability that the nodes send redundant information to their successors.

Proof: The probability that nodes send redundant information is the probability that they choose linearly dependent encoding vectors. Let A denote the encoding vector chosen at a node $v \in \mathcal{V}_S$ and B denote the encoding vector chosen at any node in $\mathcal{V}_S \setminus v$. Assume that B is chosen independently

and uniformly over \mathcal{C} . The probability that the two nodes send redundant information is $\Pr(B = cA)$, where c is a non-zero constant from \mathbb{F}_q . Then $\Pr(B = cA) = \Pr(B = cA | A = a)$ regardless of whether A is chosen uniformly over \mathcal{C} or A is chosen deterministically as in systematic network coding. ■

From Propositions 1 and 2, we state the following sufficiency condition. Systematic network coding at nodes that meet this condition is sufficient to achieve the throughput of random linear network coding as in [2], [3].

Sufficiency condition for systematic network coding: Any successor node along the forward $s - \mathcal{T}$ path can have at most one predecessor using systematic network coding without decreasing the rank of the encoding vectors of packets received at the destination(s). Thus any node v can use systematic network coding if it holds true that it is the only predecessor using systematic network coding for all of its successors.

It is clear that cut-nodes satisfy this sufficiency condition for systematic network coding. In Fig. 1 we show example networks and the cut nodes for those networks are identified. Figures 1(a) and 1(b) show examples in which all transmitting nodes are cut nodes. Figure 1(c) shows a network in which only two nodes are cut nodes; this example demonstrates the possibility of employing systematic coding at nodes other than cut-nodes. For instance, it may be beneficial to employ a technique in which *every* node on *one* path between s and t performs systematic coding (this path may be chosen as e.g., the path with the lowest aggregate loss rate or the path with the smallest delay) and cooperating nodes on other paths employ non-systematic coding. Note that not all nodes on the chosen path may meet the sufficiency condition. In this case, two possibilities are that (i) systematic network coding is employed only at nodes on the chosen path that meet the sufficiency condition, or (ii) systematic network coding is employed at all nodes on the chosen path, though this may result in an excess of non-innovative packets. Also note that this does not necessarily require *central* control, as non-cut-nodes can be chosen or elected to use systematic codes by only coordinating with their one-hop neighbors.

III. PERFORMANCE ON A SINGLE HOP

In this section we quantify the performance benefits of systematic network coding for communication over a single lossy link. The performance is quantified in terms of the packet loss rate and the per-packet delay. We consider a situation in which a single source node (a cut-node) has K packets that it would like to transmit to a single destination node. Transmission is carried out over an erasure link in which p denotes the probability that a transmitted packet is received without error; with probability $1 - p$ the packet is dropped on the channel and cannot be received. Coding is performed over the finite field \mathbb{F}_q and can be performed in either a non-systematic or a systematic manner. We assume that in the case of non-systematic coding, decoding is only performed once the decoder has gathered a full-rank matrix of encoding vectors, i.e., there is no aggressive or “earliest decoding” as in [3]. This implies that if a transmitting node generates a systematic (or uncoded) packet while performing non-systematic coding, which happens with probability Kq^{-K} , then the destination will not be able to recover that packet unless it collects a full-rank matrix of encoding vectors. Because of these assumptions, our analysis is most appropriate for large finite fields (e.g., $q = 2^8$). We assume throughout that random linear combinations of packets are formed by choosing coefficients uniformly from \mathbb{F}_q .

Regarding notation, in the following we will denote a binomial random variable X as $X \sim \text{bin}(N, p)$, indicating that $\Pr(X = x) = \binom{N}{x} p^x (1 - p)^{N-x}$, $x = 0, 1, \dots, N$. We also denote a geometrically distributed random variable X as $X \sim \text{geom}(p)$, where $\Pr(X = x) = p(1 - p)^{x-1}$, $x \geq 1$.

A. Packet loss rate

We first characterize the packet loss rate for fixed-rate coding. (Note: for rateless coding the packet loss rate is zero.) This is relevant to a scenario in which the source node has only partial knowledge of the channel - for instance, the source may know p but not the realization of the channel - and can arise for a system with limited feedback and/or large propagation delay. In this case the source will attempt to overcome erasures on the channel by sending E redundant packets in addition to the K information packets, where E is a deterministic value. There is a non-zero probability that some or all of the packets are never received at the destination. We define the *packet loss rate* λ as the ratio between the number of packets received at the destination and K ; λ takes values between zero and one.

In the case of **non-systematic** coding, in every transmission a random linear combination over \mathbb{F}_q of the K packets is sent. Let N denote the number of transmissions during which a transmitted packet is not erased; clearly $N \sim \text{bin}(K + E, p)$. Also let $Z^{(ns)}$ denote the number of original packets decoded upon completion of $K + E$ transmissions; $Z^{(ns)}$ is either zero (if the receiver cannot recover a rank- K matrix of encoding

vectors) or K . The distribution of $Z^{(ns)}$ is found as

$$\begin{aligned} \Pr(Z^{(ns)}=K) &= \Pr(\text{a random } K \times N \text{ matrix has rank } K) \\ &= \sum_{n=0}^{K+E} \Pr(K \times n \text{ matrix has rank } K) \Pr(N=n) \\ &= \sum_{n=K}^{K+E} \prod_{i=0}^{K-1} (1 - q^{i-n}) \Pr(N=n) \end{aligned} \quad (5)$$

where $\Pr(N = n)$ is given by the binomial distribution. The loss rate, which we denote $\lambda^{(ns)}$, has expected value

$$E[\lambda^{(ns)}] = 1 - \Pr(Z^{(ns)} = K). \quad (6)$$

For **systematic** coding, the source will send K uncoded packets followed by E random linear combinations. Let U denote the number of uncoded (systematic) packets received at the destination, $U \sim \text{bin}(K, p)$, and C denote the number of random linear combinations received at the destination, $C \sim \text{bin}(E, p)$. Since the channel is memoryless, U and C are independent. Also let $Z^{(s)}$ denote the total number of packets decoded at the destination. In this case $Z^{(s)}$ can take any integer value between 0 and K ; however $Z^{(s)}$ can take values $1, 2, \dots, K - 1$ *only* if uncoded packets are received and the random linear combinations received are not sufficient to decode the entire block. In particular, for $x = 1, \dots, K - 1$,

$$\begin{aligned} \Pr(Z^{(s)}=x) &= \Pr(U=x) \\ &\times (1 - \Pr(\text{a random } (K-x) \times C \text{ matrix has rank } K-x)) \end{aligned}$$

Then

$$\begin{aligned} \Pr(Z^{(s)}=x) &= \\ \Pr(U=x) &\left(1 - \sum_{c=K-x}^E \Pr(C=c) \prod_{i=0}^{K-x-1} (1 - q^{i-c}) \right). \end{aligned} \quad (7)$$

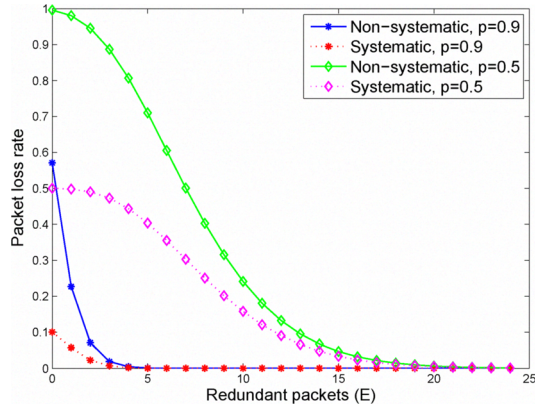
By similar arguments, we can write

$$\begin{aligned} \Pr(Z^{(s)} = K) &= \\ \sum_{x=0}^K \Pr(U=x) &\sum_{c=K-x}^E \Pr(C=c) \prod_{i=0}^{K-x-1} (1 - q^{i-c}). \end{aligned} \quad (8)$$

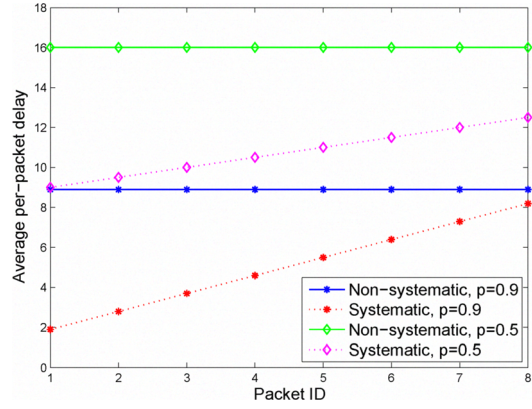
In this case we denote the loss rate by $\lambda^{(s)}$ and its expected value is given by

$$E[\lambda^{(s)}] = 1 - \frac{1}{K} \sum_{z=1}^K z \Pr(Z^{(s)} = z). \quad (9)$$

Figure 2(a) displays numerical examples of the loss rate as a function of E , the number of redundant packets. As expected, systematic coding provides a smaller expected loss rate than non-systematic coding.



(a) Average loss rate versus redundant packets for fixed-rate coding.



(b) Average per-packet delay versus packet ID for rateless coding.

Fig. 2. Comparison of systematic and non-systematic coding in terms of average packet loss rate and per-packet delay for $K = 8$, $q = 2^8$.

B. Per-packet delay

We now characterize the per-packet delay of non-systematic and systematic coding. In this case we consider the use of rateless codes, i.e., it is assumed that there is feedback on the channel and transmissions are made continually until the source receives an acknowledgment indicating that all K packets have been decoded at the receiver. (Note that the per-packet delay for fixed-rate coding can be infinite.) We denote the delay of packet i within a generation by D_i , $i = 1, \dots, K$. We assume that all packets in a generation are available at the source node when transmission begins; D_i measures the time from the start of transmission until packet i is decoded at the destination. Also we assume that when random linear combinations of packets are sent, decoding is performed only when the receiver has collected a full-rank matrix of encoding vectors.

For **non-systematic** coding, all K packets will be decoded at the same time and the per-packet delay is constant with respect to i . In this case we let $D_i^{(ns)}$ denote the per-packet delay. The value of the delay is given by the sum of K terms Y_k , $k = 0, \dots, K-1$, where Y_k denotes the number of transmissions needed for a new linearly independent packet to be received given that the receiver has already collected k linearly independent packets. This argument is more thoroughly described in [9]. Then for all $i = 1, \dots, K$,

$$E[D_i^{(ns)}] = \sum_{k=0}^{K-1} E[Y_k], \quad Y_k \sim \text{geom}(p(1-q^{k-K})) \quad (10)$$

$$\approx \frac{K}{p}. \quad (11)$$

For **systematic** coding, the delay of packet i is given by i if the packet is received when transmitted in its uncoded form, or it is given by some value at least as large as $K+1$ if the receiver must solve a system of equations to recover that packet. Clearly, the delay of packet i takes value i with probability p , which is the probability of successful

transmission of that packet in uncoded form. If packet i is not received in uncoded form, which happens with probability $1-p$, then the delay is given by K plus the time needed to receive enough coded packets to decode all K packets. In the second case, we condition on the value of the number of successful transmissions of uncoded packets U_{K-1} , where $U_{K-1} \sim \text{bin}(K-1, p)$. Conditioned on $U_{K-1} = j$, the time needed to receive enough coded packets to recover all K is given by the sum of $K-j$ terms Y_k where [9]

$$Y_k \sim \text{geom}(p(1-q^{k-(K-j)})), \quad k = 0, \dots, K-j-1. \quad (12)$$

Then the expected per-packet delay is given as

$$E[D_i^{(s)}] = pi + (1-p) \left(K + \sum_{j=0}^{K-1} \Pr(U_{K-1} = j) \sum_{k=0}^{K-j-1} E[Y_k] \right). \quad (13)$$

Figure 2(b) plots numerical examples of the per-packet delay for non-systematic and systematic coding as a function of i , which is referred to as the packet ID. Again, as expected, we observe that systematic coding provides improved performance, particularly for a lossy channel.

IV. PACKET LOSS RATE ON A MULTIHOP PATH

In this section we compute the packet loss rate performance for systematic coding over a single multihop path. The path consists of $L+1$ nodes connected by L directed links. A set of K packets is to be transmitted from source node 0 via nodes $1, \dots, L-1$ to destination node L . The L transmitting nodes on the path are all cut-nodes. We let p_i , $i = 1, \dots, L$, denote the probability that a transmitted packet is received on the link between nodes $i-1$ and i and E_i , $i = 1, \dots, L$ denote the number of redundant packets sent on that link. The number of redundant packets E_i can be any non-negative integer, the E_i packets are sent immediately after the original K packets, and the loss rate performance for $E_i \rightarrow \infty$ corresponds to the

performance for rateless coding on the link between nodes $i - 1$ and i . Additionally, Z_i denotes the number of packets that can be decoded at node i ; it can take values in $[0, K]$. The results on packet loss rate described here can apply to, for instance, the tandem network shown in Fig. 1(a) or to one of the paths in the multicast tree shown in Fig. 1(b).

A. Non-systematic coding

In this scheme, each transmitting node will send random linear combinations of the original K packets. Node $i - 1$ will transmit $K + E_i$ random linear combinations of the packets it has received. Specifically, node $i - 1$ will *re-encode* the linear combinations it has received. In order for destination node L to be able to decode the original K packets, each node on the path must also have gathered enough packets in order to be able to decode the original K . We have the following result.

Lemma 1. *The packet loss rate for non-systematic coding on a path of L hops is given by*

$$E[\lambda^{(ns)}] = 1 - \prod_{i=1}^L \sum_{n=K}^{K+E_i} \prod_{j=0}^{K-1} (1 - q^{j-n}) \Pr(N_i = n) \quad (14)$$

where $N_i \sim \text{bin}(K + E_i, p_i)$.

Proof: Let $Z_L^{(ns)}$ denote the number of packets decoded at node L under the systematic coding scheme. We have

$$\begin{aligned} E[\lambda^{(ns)}] &= 1 - \Pr(Z_L^{(ns)} = K) \\ &= 1 - \Pr(Z_L^{(ns)} = K | Z_{L-1}^{(ns)} = K) \Pr(Z_{L-1}^{(ns)} = K) \\ &= 1 - \prod_{i=1}^L \Pr(Z_i^{(ns)} = K | Z_{i-1}^{(ns)} = K), \quad Z_0^{(ns)} = K \\ &= 1 - \prod_{i=1}^L \sum_{n=K}^{K+E_i} \Pr(N_i = n) \Pr(K \times n \text{ matrix has rank } K) \end{aligned}$$

where in the final equality, N_i denotes the number of coded packets (random linear combinations) received at node i . Clearly $N_i \sim \text{bin}(K + E_i, p_i)$. The result follows. ■

B. Systematic coding

In this case each transmitting node forwards the first K packets it receives (without re-encoding) and then sends E_i random linear combinations of the packets it has received. The scheme is carried out as described in (3). The first K packets received by a node may consist of both uncoded (systematic) packets as well as random linear combinations of packets; the random linear combinations received among the first K will *not* be re-encoded. Once again, in order for destination node L to be able to decode all K original packets, each of its upstream nodes must also have collected enough packets to be able to decode the original K packets. However, with this scheme, the destination may be able to recover a partial block, or fewer than K packets, by collecting uncoded (systematic) packets. The packet loss rate is quantified below.

Lemma 2. *The packet loss rate for systematic coding on a path of L hops is given by*

$$E[\lambda_L^{(s)}] = 1 - \frac{1}{K} \sum_{z=1}^K z \Pr(Z_L^{(s)} = z) \quad (15)$$

where the marginal distribution of $Z_L^{(s)}$ can be computed from the joint distribution of the pair $(Z_L^{(s)}, U_L)$ given by the following recursive equations. For $u_i = 0, \dots, K - 1$ and $u_{i-1} = u_i, \dots, K$

$$\Pr(Z_i^{(s)} = u_i, U_i = u_i | Z_{i-1}^{(s)} < K, U_{i-1} = u_{i-1}) = f(u_i | u_{i-1}), \quad (16)$$

and

$$\begin{aligned} \Pr(Z_i^{(s)} = u_i, U_i = u_i | Z_{i-1}^{(s)} = K, U_{i-1} = u_{i-1}) &= f(u_i | u_{i-1}) \\ &\times \left(1 - \sum_{c_i=K-u_i}^{K+E_i-u_{i-1}} f(c_i | u_{i-1}) \prod_{j=0}^{K-u_i-1} (1 - q^{j-c_i}) \right), \quad (17) \end{aligned}$$

and for $u_i = 0, \dots, K$ and $u_{i-1} = u_i, \dots, K$,

$$\begin{aligned} \Pr(Z_i^{(s)} = K, U_i = u_i | Z_{i-1}^{(s)} = K, U_{i-1} = u_{i-1}) &= f(u_i | u_{i-1}) \\ &\times \left(\sum_{c_i=K-u_i}^{K+E_i-u_{i-1}} f(c_i | u_{i-1}) \prod_{j=0}^{K-u_i-1} (1 - q^{j-c_i}) \right), \quad (18) \end{aligned}$$

where $f(u_i | u_{i-1})$ denotes the probability mass function (pmf) of a $\text{bin}(u_{i-1}, p_i)$ random variable evaluated at u_i , $f(c_i | u_{i-1})$ denotes the pmf of a $\text{bin}(K + E_i - u_{i-1}, p_i)$ random variable evaluated at c_i , and $U_0 = Z_0^{(s)} = K$ with probability one.

Proof: Let $Z_i^{(s)}$ denote the number of packets that node i is able to decode, U_i denote the number of uncoded (systematic) packets received at node i , and C_i denote the number of coded packets (random linear combinations) received at node i . Note that the C_i packets received at node i are not necessarily linearly independent. Conditioned on $U_{i-1} = u_{i-1}$, then $U_i \leq u_{i-1}$ and $U_i \sim \text{bin}(u_{i-1}, p_i)$. Also, given $U_{i-1} = u_{i-1}$, then $C_i \sim \text{bin}(K + E_i - u_{i-1}, p_i)$. In order for $Z_i^{(s)} = K$, it must be true that $Z_{i-1}^{(s)} = K$. We have,

$$\begin{aligned} \Pr(Z_i^{(s)} = K, U_i = u_i | Z_{i-1}^{(s)} = K, U_{i-1} = u_{i-1}) &= f(u_i | u_{i-1}) \\ &\times \sum_{c_i=K-u_i}^{K+E_i-u_{i-1}} f(c_i | u_{i-1}) \Pr((K-u_i) \times c_i \text{ matrix is full rank}). \end{aligned}$$

From the equation above, the result in (18) follows. Next, $Z_i^{(s)} < K$ indicates that node i is only able to recover the U_i uncoded (systematic) packets it has received. If $Z_{i-1}^{(s)} = K$, then $Z_i^{(s)} < K$ only if node i does not receive $K - U_i$ linearly independent random linear combinations. By the same reasoning as used above,

$$\begin{aligned} \Pr(Z_i^{(s)} = u_i, U_i = u_i | Z_{i-1}^{(s)} = K, U_{i-1} = u_{i-1}) &= f(u_i | u_{i-1}) \\ &\times \left(1 - \sum_{c_i=K-u_i}^{K+E_i-u_{i-1}} f(c_i | u_{i-1}) \Pr((K-u_i) \times c_i \text{ matrix is full rank}) \right) \end{aligned}$$

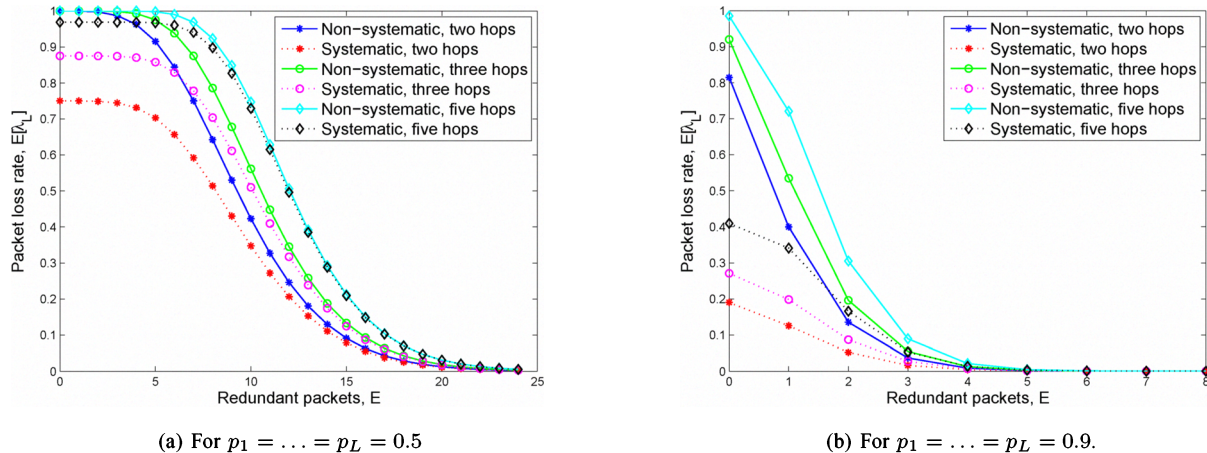


Fig. 3. Expected packet loss rate versus redundant packets on a multihop path. In these results $K = 8$, $q = 2^8$, $p_1 = \dots = p_L$, and $E_1 = \dots = E_L = E$.

and (17) follows. Finally, in the case that $Z_{i-1}^{(s)} < K$ and $Z_i^{(s)} < K$, node i will only be able to recover its uncoded (systematic) packets and this outcome is independent of how many coded packets i receives; in this case (16) applies. ■

C. Numerical results

Two numerical examples of the packet loss rate on a multihop path are shown in Fig. 3. We have computed these results assuming that $p_1 = \dots = p_L$ and $E_1 = \dots = E_L = E$. The figures display the values of the loss rate for $L = 2, 3$, and 5. As expected, the packet loss rate increases with L and systematic coding provides a consistently smaller loss rate than non-systematic coding.

V. CONCLUSIONS

This work presents a strategy that is a blend of the traditional replicate-and-forward technique used in packet networks and the technique of random linear coding of packets for cooperative communication. We propose that replicate-and-forward be used at cut-nodes in the network, and that forwarding is followed by the transmission of “parity packets” or random linear combinations. Furthermore, systematic coding can be used at other nodes (in addition to cut-nodes) and we have provided specific conditions for which systematic network coding at a node will not increase the number of redundant packets sent through the network relative to non-systematic network coding. We have provided quantitative results on the performance benefits of systematic network coding in terms of the packet loss rate and per-packet delay. There can also be drawbacks to the use of systematic network coding: it may increase the incidence of out-of-order packet delivery and may require additional coordination to determine the coding strategy used at each node. These costs, in addition to the benefits, of systematic network coding will depend on many factors, including the size and topology of the network and the specific manner in which systematic network coding is

implemented. Our future work will more thoroughly explore these issues.

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REFERENCES

- [1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R.W. Yeung, “Network information flow,” *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1204-1216, July 2000.
- [2] T. Ho, R. Koetter, M. Medard, D. R. Karger, and M. Effros, “The benefits of coding over routing in a randomized setting,” *Proc. IEEE International Symposium on Information Theory*, 2003.
- [3] P. A. Chou, Y. Wu, and K. Jain, “Practical network coding,” *Proc. Allerton Conference*, 2003.
- [4] P. Pakzad, C. Fragouli, and A. Shokrollahi, “Coding schemes for line networks,” *Proc. IEEE International Symposium on Information Theory*, 2005.
- [5] J. Barros, R. A. Costa, D. Munaretto, and J. Widmer, “Effective delay control in online network coding,” *Proc. IEEE Infocom*, 2009.
- [6] M. Pedersen, J. Heide, F. H. P. Fitzek, and T. Larsen, “Network coding for mobile devices - systematic binary random rateless codes,” *Proc. IEEE International Conference on Communication (ICC)*, 2009.
- [7] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, Second edition, MIT Press, Boston, 2001.
- [8] R. A. Horn and C. A. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.
- [9] R. Cogill, B. Shrader, and A. Ephremides, “Stability analysis of random linear coding across multicast sessions,” *Proc. IEEE International Symposium on Information Theory*, 2008.