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Review Article

Holographic Duality with a View Toward Many-Body Physics

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These are notes based on a series of lectures given at the KITP workshop *Quantum Criticality and the AdS/CFT Correspondence* in July, 2009. The goal of the lectures was to introduce condensed matter physicists to the AdS/CFT correspondence. Discussion of string theory and of supersymmetry is avoided to the extent possible.

1. Introductory Remarks

My task in these lectures is to engender some understanding of the following

Bold Assertion:

- (a) Some ordinary quantum field theories (QFTs) are secretly quantum theories of gravity.
- (b) Sometimes the gravity theory is classical, and therefore we can use it to compute interesting observables of the QFT.

Part (a) is vague enough that it really just raises the following questions: “which QFTs?” and “what the heck is a quantum theory of gravity?” Part (b) begs the question “when?!”

In trying to answer these questions, I have two conflicting goals: on the one hand, I want to convince you that some statement along these lines is *true*, and on the other hand I want to convince you that it is *interesting*. These goals conflict because our best evidence for the Assertion comes with the aid of supersymmetry and complicated technology from

string theory and applies to very peculiar theories which represent special cases of the correspondence, wildly overrepresented in the literature on the subject. Since most of this technology is completely irrelevant for the applications that we have in mind (which I will also not discuss explicitly except to say a few vague words at the very end), I will attempt to accomplish the first goal by way of showing that the correspondence gives sensible answers to some interesting questions. Along the way we will try to get a picture of its regime of validity.

Material from other review articles, including [1–7], has been liberally borrowed to construct these notes. In addition, some of the text source and most of the figures were pillaged from lecture notes from my class at MIT during Fall 2008 [8], some of which were created by students in the class.

2. Motivating the Correspondence

To understand what one might mean by a more precise version of the *Bold Assertion* above, we will follow for a little while the interesting logic of [1], which liberally uses hindsight, but does not use string theory.

Here are three facts which make the Assertion seem less unreasonable.

(1) First we must define what we mean by a quantum gravity (QG). As a working definition, let us say that a QG is a quantum theory with a dynamical metric. In enough dimensions, this usually means that there are local degrees of freedom. In particular, linearizing equations of motion (EoM) for a metric usually reveals a propagating mode of the metric, some spin-2 massless particle which we can call a “graviton”.

So at least the assertion must mean that there is some spin-two graviton particle that is somehow a composite object made of gauge theory degrees of freedom. This statement seems to run afoul of the Weinberg-Witten no-go theorem, which says the following.

Theorem 2.1 (Weinberg-Witten [9, 10]). *A QFT with a Poincaré covariant conserved stress tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum (i.e., with $P^\mu = \int d^D x T^{0\mu} \neq 0$).*

You may worry that the assumption of Poincaré invariance plays an important role in the proof, but the set of QFTs to which the Bold Assertion applies includes relativistic theories.

General relativity (GR) gets around this theorem because the total stress tensor (including the gravitational bit) vanishes by the metric EoM: $T^{\mu\nu} \propto \delta S / \delta g_{\mu\nu} = 0$. (Alternatively, the “matter stress tensor,” which does not vanish, is not general-coordinate invariant.)

Like any good no-go theorem, it is best considered a sign pointing away from wrong directions. The loophole in this case is blindingly obvious in retrospect: the graviton need not live in the same spacetime as the QFT.

(2) Hint number two comes from the Holographic Principle (a good reference is [11]). This is a far-reaching consequence of black hole thermodynamics. The basic fact is that a black hole must be assigned an entropy proportional to the *area* of its horizon (in Planck units). On the other hand, dense matter will collapse into a black hole. The combination of these two observations leads to the following crazy thing: the maximum entropy in a region of space is the area of its boundary, in Planck units. To see this, suppose that you have in a volume V (bounded by an area A) a configuration with entropy $S > S_{\text{BH}} = A/4G_N$ (where S_{BH} is the entropy of the biggest black hole fittable in V), but which has *less* energy. Then by throwing in more stuff (as arbitrarily nonadiabatically as necessary, i.e., you can increase the entropy),

since stuff that carries entropy also carries energy,¹ you can *make* a black hole. This would violate the second law of thermodynamics, and you can use it to save the planet from the humans. This probably means that you cannot do it, and instead we conclude that the black hole is the most entropic configuration of the theory in this volume. But its entropy goes like the *area*! This is much smaller than the entropy of a local quantum field theory on the same space, even with some UV cutoff, which would have a number of states $N_s \sim e^V$ (maximum entropy = $\ln N_s$). Indeed it is smaller (when the linear dimensions are large compared to the Planck length) than that of any system with local degrees of freedom, such as a bunch of spins on a spacetime lattice.

We conclude from this that a quantum theory of gravity must have a number of degrees of freedom which scales like that of a QFT in a smaller number of dimensions. This crazy thing is actually true, and the AdS/CFT correspondence [12, 13] is a precise implementation of it.

Actually, we already know some examples like this in low dimensions. An alternative, more general, definition of a quantum gravity is a quantum theory where we do not need to introduce the geometry of spacetime (i.e., the metric) as input. We know two ways to accomplish this.

- (a) Integrate over all metrics (fixing some asymptotic data). This is how GR works.
- (b) Do not ever introduce a metric. Such a thing is generally called a topological field theory. The best-understood example is Chern-Simons gauge theory in three dimensions, where the dynamical variable is a one-form field and the action is

$$S_{CS} \sim \int_M \text{tr} A \wedge dA + \dots, \quad (2.1)$$

(where the dots are extra stuff to make the non-Abelian case gauge invariant); note that there is no metric anywhere here. With option (b) there are no local degrees of freedom. But if you put the theory on a space with boundary, there are local degrees of freedom which live on the boundary. Chern-Simons theory on some three-manifold M induces a WZW model (a 2d CFT) on the boundary of M . So this can be considered an example of the correspondence, but the examples to be discussed below are quite a bit more dramatic, because there will be dynamics in the bulk.

(3) A beautiful hint as to the possible identity of the extra dimensions is this. Wilson taught us that a QFT is best thought of as being sliced up by length (or energy) scale, as a family of trajectories of the renormalization group (RG). A remarkable fact about this is that the RG equations for the behavior of the coupling constants as a function of RG scale u are *local* in scale:

$$u \partial_u g = \beta(g(u)). \quad (2.2)$$

The beta function is determined by the coupling constant evaluated at the energy scale u , and we do not need to know its behavior in the deep UV or IR to figure out how it's changing. This fact is basically a consequence of locality in ordinary spacetime. This opens the possibility that we can associate the extra dimensions suggested by the Holographic idea with energy scale. This notion of locality in the extra dimension actually turns out to be much *weaker* than what we will find in AdS/CFT (as discussed recently in [14]), but it is a good hint.

To summarize, we have three hints for interpreting the Bold Assertion:

- (1) The Weinberg-Witten theorem suggests that the graviton lives on a different space than the QFT in question.
- (2) The holographic principle says that the theory of gravity should have a number of degrees of freedom that grows more slowly than the volume. This suggests that the quantum gravity should live in more dimensions than the QFT.
- (3) The structure of the Renormalization Group suggests that we can identify one of these extra dimensions as the RG-scale.

Clearly the field theory in question needs to be strongly coupled. Otherwise, we can compute and we can see that there is no large extra dimension sticking out. This is an example of the extremely useful Principle of Conservation of Evil. Different weakly coupled descriptions should have nonoverlapping regimes of validity.²

Next we will make a simplifying assumption in an effort to find concrete examples. The simplest case of an RG flow is when $\beta = 0$ and the system is self-similar. In a Lorentz invariant theory (which we also assume for simplicity), this means that the following scale transformation $x^\mu \rightarrow \lambda x^\mu$ ($\mu = 0, 1, 2, \dots, d-1$) is a symmetry. If the extra dimension coordinate u is to be thought of as an energy scale, then dimensional analysis says that u will scale under the scale transformation as $u \rightarrow u/\lambda$. The most general $(d+1)$ -dimensional metric (one extra dimension) with this symmetry and Poincaré invariance is of the following form:

$$ds^2 = \left(\frac{\tilde{u}}{\tilde{L}}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\tilde{u}^2}{\tilde{u}^2} L^2. \quad (2.3)$$

We can bring it into a more familiar form by a change of coordinates, $\tilde{u} = (\tilde{L}/L)\tilde{u}$:

$$ds^2 = \left(\frac{u}{L}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} L^2. \quad (2.4)$$

This is AdS_{d+1} .³ It is a family of copies of Minkowski space, parametrized by u , whose size varies with u (see Figure 1). The parameter L is called the "AdS radius" and it has dimensions of length. Although this is a dimensionful parameter, a scale transformation $x^\mu \rightarrow \lambda x^\mu$ can be absorbed by rescaling the radial coordinate $u \rightarrow u/\lambda$ (by design); we will see below more explicitly how this is consistent with scale invariance of the dual theory. It is convenient to do one more change of coordinates, to $z \equiv L^2/u$, in which the metric takes the form

$$ds^2 = \left(\frac{L}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right). \quad (2.5)$$

These coordinates are better because fewer symbols are required to write the metric. z will map to the length scale in the dual theory.

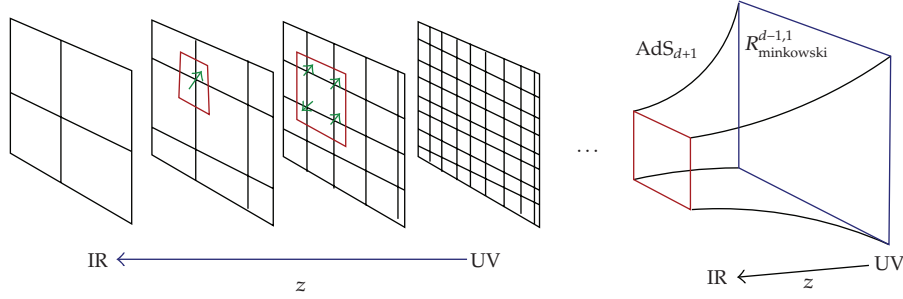


Figure 1: The extra (radial) dimension of the bulk is the resolution scale of the field theory. The left figure indicates a series of block spin transformations labelled by a parameter z . The right figure is a cartoon of AdS space, which organizes the field theory information in the same way. In this sense, the bulk picture is a hologram: excitations with different wavelengths get put in different places in the bulk image. The connection between these two pictures is pursued further in [15]. This paper contains a useful discussion of many features of the correspondence for those familiar with the real-space RG techniques developed recently from quantum information theory.

So it seems that a d -dimensional conformal field theory (CFT) should be related to a theory of gravity on AdS_{d+1} . This metric (2.5) solves the equations of motion of the following action (and many others)⁴:

$$S_{\text{bulk}}[g, \dots] = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} (-2\Lambda + \mathcal{R} + \dots). \quad (2.6)$$

Here, $\sqrt{g} \equiv \sqrt{|\det g|}$ makes the integral coordinate-invariant, and \mathcal{R} is the Ricci scalar curvature. The cosmological constant Λ is related by the equations of motion

$$0 = \frac{\delta S_{\text{bulk}}}{\delta g^{AB}} \implies R_{AB} + \frac{d}{L^2} g_{AB} = 0 \quad (2.7)$$

to the value of the AdS radius: $-2\Lambda = d(d-1)/L^2$. This form of the action (2.6) is what we would guess using Wilsonian naturalness (which in some circles is called the “Landau-Ginzburg-Wilson paradigm”): we include all the terms which respect the symmetries (in this case, this is general coordinate invariance), organized by decreasing relevantness, that is, by the number of derivatives. The Einstein-Hilbert term (the one with the Ricci scalar) is an irrelevant operator: $\mathcal{R} \sim \partial^2 g + (\partial g)^2$ has dimensions of length^{-2} , and so G_N here is a length^{d-1} , the Planck length: $G_N \equiv \ell_{\text{pl}}^{d-1} \equiv M_{\text{pl}}^{1-d}$ (in units where $\hbar = c = 1$). The gravity theory is classical if $L \gg \ell_{\text{pl}}$. In this spirit, the \dots on the RHS denotes more irrelevant terms involving more powers of the curvature. Also hidden in the \dots are other bulk fields which vanish in the dual of the CFT vacuum (i.e., in the AdS solution).

This form of the action (2.6) is indeed what comes from string theory at low energies and when the curvature (here, $\mathcal{R} \sim 1/L^2$) is small (compared to the *string tension*, $1/\alpha' \equiv 1/\ell_s^2$; this is the energy scale that determines the masses of excited vibrational modes of the string), at least in cases where we are able to tell. The main role of string theory in this business (at the moment) is to provide consistent ways of filling in the dots.

In a theory of gravity, the space-time metric is a dynamical variable, and we only get to specify the boundary behavior. The AdS metric above has a boundary at $z = 0$. This is a bit subtle. Keeping x^μ fixed and moving in the z direction from a finite value of z to $z = 0$ is actually infinite distance. However, massless particles in AdS (such as the graviton discussed above) travel along null geodesics; these reach the boundary in finite time. This means that in order to specify the future evolution of the system from some initial data, we have also to specify boundary conditions at $z = 0$. These boundary conditions will play a crucial role in the discussion below.

So we should amend our statement to say that a d -dimensional conformal field theory is related to a theory of gravity on spaces which are *asymptotically* AdS_{d+1} . Note that this case of negative cosmological constant (CC) turns out to be much easier to understand holographically than the naively-simpler (asymptotically-flat) case of zero CC. Let us not even talk about the case of positive CC (asymptotically de Sitter).

Different CFTs will correspond to such theories of gravity with different field content and different bulk actions, for example, different values of the coupling constants in S_{bulk} . The example which is understood best is the case of the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four dimensions. This is dual to maximal supergravity in AdS_5 (which arises by dimensional reduction of ten-dimensional IIB supergravity on $\text{AdS}_5 \times S^5$). In that case, we know the precise values of many of the coefficients in the bulk action. This will not be very relevant for our discussion below. An important conceptual point is that the values of the bulk parameters which are realizable will in general be discrete.⁵ This discreteness is hidden by the classical limit.

We will focus on the case of relativistic CFT for a while, but let me emphasize here that the name “AdS/CFT” is a very poor one: the correspondence is much more general. It can describe deformations of UV fixed points by relevant operators, and it has been extended to cases which are not even relativistic CFTs in the UV: examples include fixed points with dynamical critical exponent $z \neq 1$ [16], Galilean-invariant theories [17, 18], and theories which do more exotic things in the UV like the “duality cascade” of [19].

2.1. Counting of Degrees of Freedom

We can already make a check of the conjecture that a gravity theory in AdS_{d+1} might be dual to a QFT in d dimensions. The holographic principle tells us that the area of the boundary in Planck units is the number of degrees of freedom (dof), that is, the maximum entropy:

$$\frac{\text{Area of boundary}}{4G_N} \stackrel{?}{=} \text{number of dof of QFT} \equiv N_d. \quad (2.8)$$

Is this true [20]? Yes: both sides are equal to infinity. We need to regulate our counting.

Let’s regulate the field theory first. There are both UV and IR divergences. We put the thing on a lattice, introducing a short-distance cut-off ϵ (e.g., the lattice spacing) and we put it in a cubical box of linear size R . The total number of degrees of freedom is the number of cells $(R/\epsilon)^{d-1}$, times the number of degrees of freedom per lattice site, which we will call “ N^2 ”. The behavior suggested by the name we have given this number is found in well-understood examples. It is, however, clear (e.g., from the structure of known AdS vacua of string theory [21]) that other behaviors N^b are possible, and that’s why I made it a funny color and put it in quotes. So $N_d = (R^{d-1}/\epsilon^{d-1})N^2$.

The picture we have of AdS_{d+1} is a collection of copies of d -dimensional Minkowski space of varying size; the boundary is the locus $z \rightarrow 0$ where they get really big. The area of the boundary is

$$A = \int_{\mathbb{R}^{d-1}, z \rightarrow 0, \text{fixed } t} \sqrt{g} d^{d-1}x = \int_{\mathbb{R}^{d-1}, z \rightarrow 0} d^{d-1}x \frac{L^{d-1}}{z^{d-1}}. \quad (2.9)$$

As in the field theory counting, this is infinite for two reasons: from the integral over x and from the fact that z is going to zero. To regulate this integral, we integrate not to $z = 0$ but rather cut it off at $z = \epsilon$. We will see below a great deal more evidence for this idea that the boundary of AdS is associated with the UV behavior of the field theory, and that cutting off the geometry at $z = \epsilon$ is a UV cutoff (not identical to the lattice cutoff, but close enough for our present purposes). Given this,

$$A = \int_0^R d^{d-1}x \frac{L^{d-1}}{z^{d-1}} \Big|_{z=\epsilon} = \left(\frac{RL}{\epsilon} \right)^{d-1}. \quad (2.10)$$

The holographic principle then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^{d-1}}{4G_N} \left(\frac{R}{\epsilon} \right)^{d-1}. \quad (2.11)$$

We see that the scaling with the system size agrees—both hand side go like R^{d-1} . So AdS/CFT is indeed an implementation of the holographic principle. We can learn more from this calculation: In order for the prefactors of R^{d-1} to agree, we need to relate the AdS radius in Planck units $L^{d-1}/G_N \sim (LM_{\text{pl}})^{d-1}$ to the number of degrees of freedom per site of the field theory:

$$\boxed{\frac{L^{d-1}}{G_N} = N^2} \quad (2.12)$$

up to numerical prefactors.

2.2. Preview of the AdS/CFT Correspondence

Here is the ideology:

$$\begin{array}{ll} \text{fields in AdS} & \longleftrightarrow \text{local operators of CFT} \\ \text{spin} & \text{spin} \\ \text{mass} & \text{scaling dimension } \Delta \end{array} \quad (2.13)$$

In particular, for a scalar field in AdS, the formula relating the mass of the scalar field to the scaling dimension of the corresponding operator in the CFT is $m^2 L_{\text{AdS}}^2 = \Delta(\Delta - d)$, as we will show in Section 4.1.

One immediate lesson from this formula is that a simple bulk theory with a small number of light fields is dual to a CFT with a hierarchy in its spectrum of operator dimensions. In particular, there need to be a small number of operators with small (e.g., of order N^0) dimensions. If you are aware of explicit examples of such theories, please let me know.^{6,7} This is to be distinguished from the thus-far-intractable case where some whole tower of massive string modes in the bulk is needed.

Now let us consider some observables of a QFT (we'll assume Euclidean spacetime for now), namely vacuum correlation functions of local operators in the CFT:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle. \quad (2.14)$$

We can write down a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\begin{aligned} \mathcal{L}(x) &\longrightarrow \mathcal{L}(x) + \sum_A J_A(x) \mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x), \\ Z[J] &= \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{\text{CFT}}, \end{aligned} \quad (2.15)$$

where $J_A(x)$ are arbitrary functions (sources) and $\{\mathcal{O}_A(x)\}$ is some basis of local operators. The n -point function is then given by

$$\left\langle \prod_n \mathcal{O}_n(x_n) \right\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z \Big|_{J=0}. \quad (2.16)$$

Since \mathcal{L}_J is a *UV* perturbation (because it is a perturbation of the *bare* Lagrangian by *local* operators), in AdS it corresponds to a perturbation near the boundary, $z \rightarrow 0$. (Recall from the counting of degrees of freedom in Section 2.1 QFT with UV cutoff $E < 1/\epsilon \leftrightarrow$ AdS cutoff $z > \epsilon$.) The perturbation J of the CFT action will be encoded in the boundary condition on bulk fields.

The idea ([22, 23], often referred to as GKPW) for computing $Z[J]$ is then schematically

$$Z[J] \equiv \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{\text{CFT}} = \underbrace{Z_{\text{QG}}[\text{b.c. depends on } J]}_{=??} \Big|_{N \gg 1} \sim e^{-S_{\text{grav}}} \Big|_{\text{EOM, b.c. depend on } J}. \quad (2.17)$$

The middle object is the partition function of quantum gravity. We do not have a very useful idea of what this is, except in perturbation theory and via this very equality. In a limit where this gravity theory becomes classical, however, we know quite well what we are doing, and we can do the path integral by saddle point, as indicated on the RHS of (2.17).

An important point here is that even though we are claiming that the QFT path integral is dominated by a classical saddle point, this does not mean that the field theory degrees of freedom are free. How this works depends on what kind of large- N limit we take to make the gravity theory classical. This is our next subject.

3. When Is the Gravity Theory Classical?

So we have said that some QFT path integrals are dominated by saddle points⁸ where the degrees of freedom near the saddle are those of a gravitational theory in extra dimensions:

$$Z_{\text{some QFTs}}[\text{sources}] \approx e^{-S_{\text{bulk}}[\text{boundary conditions at } z \rightarrow 0]} \Big|_{\text{extremum of } S_{\text{bulk}}} . \quad (3.1)$$

The sharpness of the saddle (the size of the second derivatives of the action evaluated at the saddle) is equivalent to the classicalness of the bulk theory. In a theory of gravity, this is controlled by the Newton constant in front of the action. More precisely, in an asymptotically AdS space with AdS radius L , the theory is classical when

$$\frac{L^{d-1}}{G_N} \equiv "N^2" \gg 1. \quad (3.2)$$

This quantity, the AdS radius in Planck units $L^{d-1}/G_N \equiv (LM_{\text{pl}})^{d-1}$, is what we identified (using the holographic principle) as the number of degrees of freedom per site of the QFT.

In the context of our current goal, it is worth spending some time talking about different kinds of large-species limits of QFTs. In particular, in the condensed matter literature, the phrase "large- n " usually means that one promotes a two-component object to an n -component vector, with $O(n)$ -invariant interactions. This is probably *not* what we need to have a simple gravity dual, for the reasons described next.

3.1. Large n Vector Models

A simple paradigmatic example of this vector-like large- n limit (I use a different n to distinguish it from the matrix case to be discussed next) is a QFT of n scalar fields $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$ with the following action:

$$S[\varphi] = -\frac{1}{2} \int d^d x \left(\partial_\mu \vec{\varphi} \partial^\mu \vec{\varphi} + m^2 \vec{\varphi} \cdot \vec{\varphi} + \frac{\lambda_v}{n} (\vec{\varphi} \cdot \vec{\varphi})^2 \right). \quad (3.3)$$

The fields $\vec{\varphi}$ transform in the fundamental representation of the $O(n)$ symmetry group. Some foresight has been used to determine that the quartic coupling λ_v is to be held fixed in the large- n limit. An effective description (i.e., a well-defined saddle-point) can be found in terms of $\sigma \equiv \vec{\varphi} \cdot \vec{\varphi}$ by standard path-integral tricks, and the effective action for σ is

$$S_{\text{eff}}[\sigma] = -\frac{n}{2} \left[\int \frac{\sigma^2}{2\lambda} + \text{tr} \ln(-\partial^2 + m^2 + \sigma) \right]. \quad (3.4)$$

The important thing is the giant factor of n in front of the action which makes the theory of σ classical. Alternatively, the only interactions in this n vector model are “cactus” diagrams; this means that, modulo some self energy corrections, the theory is free.

So we have found a description of this saddle point *within* weakly coupled quantum field theory. The Principle of Conservation of Evil then suggests that this should not also be a simple, classical theory of gravity. Klebanov and Polyakov [24] have suggested what the (not simple) gravity dual might be.

3.2. 't Hooft Counting

“You can hide a lot in a large- N matrix.”

Steve Shenker

Given some system with a few degrees of freedom, there exist many interesting large- N generalizations, many of which may admit saddle-point descriptions. It is not guaranteed that the effective degrees of freedom near the saddle (sometimes ominously called “the masterfield”) are simple field theory degrees of freedom (at least not in the same number of dimensions). If they are not, this means that such a limit is not immediately useful, but it is not necessarily more distant from the physical situation than the limit of the previous subsection. In fact, we will see dramatically below that the 't Hooft limit described here preserves *more* features of the interacting small- N theory than the usual vector-like limit. The remaining problem is to find a description of the masterfield, and this is precisely what is accomplished by AdS/CFT.

Next we describe in detail a large- N limit (found by 't Hooft⁹) where the right degrees of freedom seem to be closed strings (and hence gravity). In this case, the number of degrees of freedom per point in the QFT will go like N^2 . Evidence from the space of string vacua suggests that there are many generalizations of this where the number of dofs per point goes like N^b for $b \neq 2$ [21]. However, a generalization of the 't Hooft limit is not yet well understood for other cases.¹⁰

Consider a (any) quantum field theory with *matrix* fields, $\Phi_{a=1, \dots, N}^{b=1, \dots, N}$. By matrix fields, we mean that their products appear in the Lagrangian only in the form of matrix multiplication, for example, $(\Phi^2)_a^c = \Phi_a^b \Phi_b^c$, which is a big restriction on the interactions. It means the interactions must be invariant under $\Phi \rightarrow U^{-1} \Phi U$; for concreteness we will take the matrix group to be $U \in U(N)$.¹¹ The fact that this theory has many more interaction terms than the vector model with the same number of fields (which would have a much larger $O(N^2)$ symmetry) changes the scaling of the coupling in the large N limit.

In particular, consider the *'t Hooft limit* in which $N \rightarrow \infty$ and $g \rightarrow 0$ with $\lambda = g^2 N$ held fixed in the limit. Is the theory free in this limit? The answer turns out to be no. The loophole is that even though the coupling goes to zero, the number of modes diverges. Compared to the vector model, the quartic coupling in the matrix model $g \sim 1/\sqrt{N}$ goes to zero slower than the coupling in the vector model $g_v \equiv \lambda_v/N \sim 1/N$.

We will be agnostic here about whether the $U(N)$ symmetry is gauged, but if it is not, there are many more states than we can handle using the gravity dual. The important role of the gauge symmetry for our purpose is to restrict the physical spectrum to gauge-invariant operators, like $\text{tr } \Phi^k$.


The fields can have all kinds of spin labels and global symmetry labels, but we will just call them Φ . In fact, the location in space can also for the purposes of the discussion of this

section be considered as merely a label on the field (which we are suppressing). So consider a schematic Lagrangian of the following form:

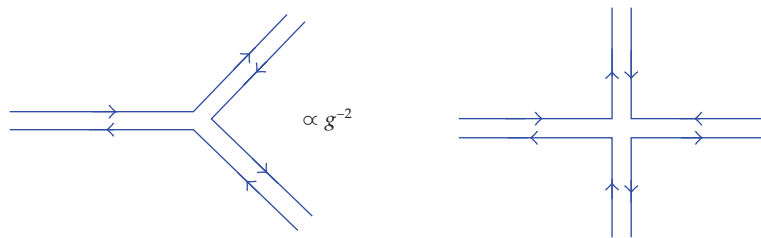
$$\mathcal{L} \sim \frac{1}{g^2} \text{Tr} \left((\partial\Phi)^2 + \Phi^2 + \Phi^3 + \Phi^4 + \dots \right). \tag{3.5}$$

I suppose that we want Φ to be Hermitian so that this Lagrangian is real, but this will not be important for our considerations.

We will now draw some diagrams which let us keep track of the N -dependence of various quantities. It is convenient to adopt the *double line* notation, in which oriented index lines follow conserved color flow. We denote the propagator:¹²

$$\langle \Phi_b^a \Phi_c^d \rangle \propto g^2 \delta_c^a \delta_b^d \equiv g^2 \tag{3.6}$$


and the vertices by



$$\propto g^{-2} \tag{3.7}$$

Created by Brian Swingle.

To see the consequences of this more concretely, let us consider some vacuum-to-vacuum diagrams (see Figures 3 and 4 for illustration). We will keep track of the color structure and not worry even about how many dimensions we are in (the theory could even be zero-dimensional, such as the matrix integral which constructs the Wigner-Dyson distribution).

A general diagram consists of propagators, interaction vertices, and index loops, and gives a contribution:

$$\text{diagram} \sim \left(\frac{\lambda}{N} \right)^{\text{no. of prop.}} \left(\frac{N}{\lambda} \right)^{\text{no. of int. vert.}} N^{\text{no. of index loops}}. \tag{3.8}$$

For example, the diagram in Figure 2 has 4 three-point vertices, 6 propagators, and 4 index loops, giving the final result $N^2 \lambda^2$. In Figure 3 we have a set of *planar graphs*, meaning that we can draw them on a piece of paper without any lines crossing; their contributions take the general form $\lambda^n N^2$. However, there also exist *nonplanar graphs*, such as the one in Figure 4, whose contributions are down by (an even number of) powers of N . One thing that is great about this expansion is that the diagrams which are harder to draw are less important.

We can be more precise about how the diagrams are organized. Every double-line graph specifies a triangulation of a 2-dimensional surface Σ . There are two ways to construct the explicit mapping.

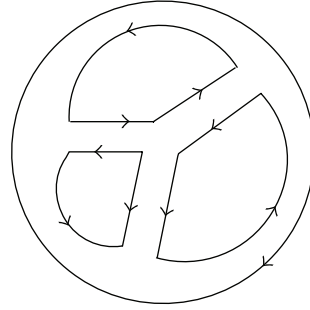


Figure 2: This diagram consists of 4 three-point vertices, 6 propagators, and 4 index loops.

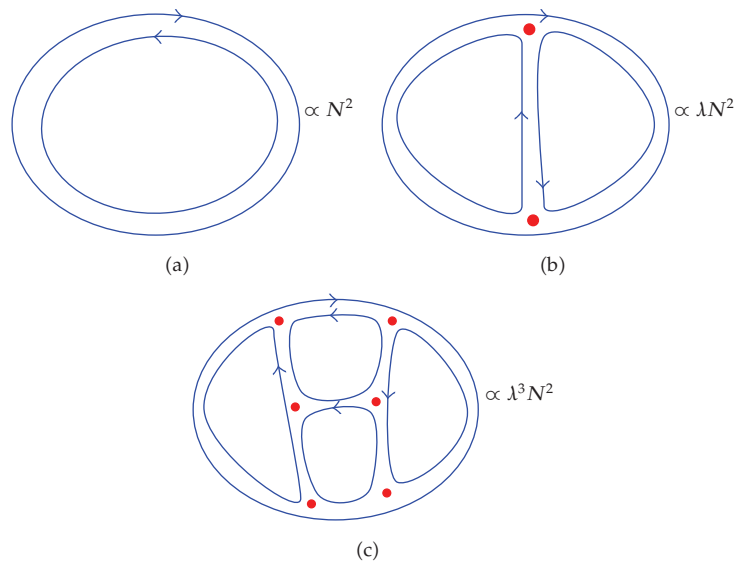


Figure 3: Planar graphs that contribute to the vacuum \rightarrow vacuum amplitude.

Method 1 (direct surface). Fill in index loops with little plaquettes.

Method 2 (dual surface). (1) draw a vertex¹³ in every index loop and (2) draw an edge across every propagator.

These constructions are illustrated in Figures 5 and 6.

If E = number of propagators, V = number of vertices, and F = number of index loops, then the diagram gives a contribution $N^{F-E+V} \lambda^{E-V}$. The letters refer to the “direct” triangulation of the surface in which interaction vertices are triangulation vertices. Then we interpret E as the number of edges, F as the number of faces, and V as the number of vertices in the triangulation. In the dual triangulation there are dual faces \tilde{F} , dual edges \tilde{E} , and dual vertices \tilde{V} . The relationship between the original and dual variables is $E = \tilde{E}$, $V = \tilde{F}$, and $F = \tilde{V}$. The exponent $\chi = F - E + V = \tilde{F} - \tilde{E} + \tilde{V}$ is the Euler character and it is a topological invariant of two-dimensional surfaces. In general it is given by $\chi(\Sigma) = 2 - 2h - b$ where h is the number of handles (the genus) and b is the number of boundaries. Note that the exponent of

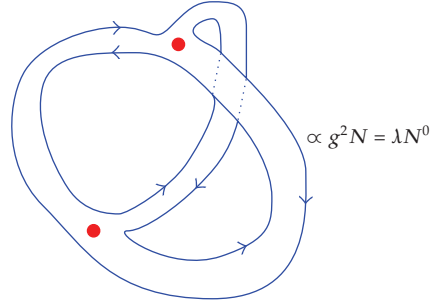


Figure 4: Non-planar (but still oriented!) graph that contributes to the vacuum \rightarrow vacuum amplitude. Created by Wing-Ko Ho.

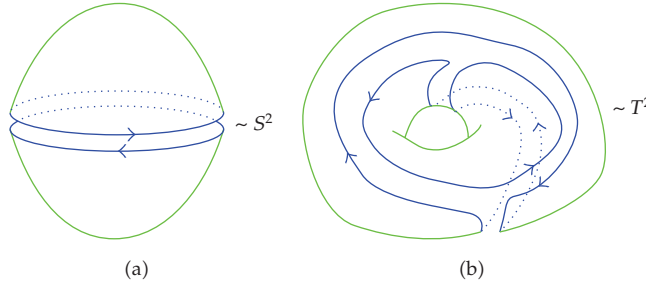


Figure 5: Direct surfaces constructed from the vacuum diagram in (a) Figure 3(a) and (b) Figure 4.

λ , $E - V$ or $\tilde{E} - \tilde{F}$ is not a topological invariant and depends on the triangulation (Feynman diagram).

Because the N -counting is topological (depending only on $\chi(\Sigma)$), we can sensibly organize the perturbation series for the effective action $\ln Z$ in terms of a sum over surface topology. Because we are computing only vacuum diagrams for the moment, the surfaces we are considering have no boundaries $b = 0$ and are classified by their number of handles h ($h = 0$ is the two-dimensional sphere, $h = 1$ is the torus, and so on). We may write the effective action (the sum over connected vacuum-to-vacuum diagrams) as

$$\ln Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{\ell=0}^{\infty} c_{\ell,h} \lambda^{\ell} = \sum_{h=0}^{\infty} N^{2-2h} \mathfrak{F}_h(\lambda), \tag{3.9}$$

where the sum over topologies is explicit.

Now we can see some similarities between this expansion and perturbative string expansions.¹⁴ $1/N$ plays the role of the string coupling g_s , the amplitude joining and splitting of the closed strings. In the large N limit, this process is suppressed and the theory is classical. Closed string theory generically predicts gravity, with Newton's constant $G_N \propto g_s^2$; so this reproduces our result $G_N \sim N^{-2}$ from the holographic counting of degrees of freedom (this time, without the quotes around it).

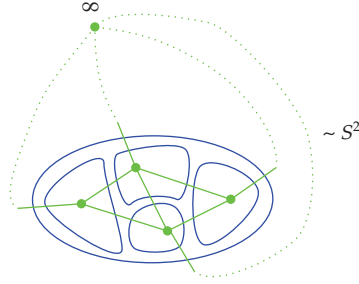


Figure 6: Dual surface constructed from the vacuum diagram in Figure 3(c). Note that points at infinity are identified.

It is reasonable to ask what plays the role of the worldsheet coupling: there is a 2d QFT living on the worldsheet of the string, which describes its embeddings into the target space; this theory has a weak-coupling limit when the target-space curvature L^{-2} is small, and it can be studied in perturbation theory in powers of ℓ_s/L , where ℓ_s^{-2} is the string tension. We can think of λ as a sort of chemical potential for edges in our triangulation. Looking back at our diagram counting we can see that if λ becomes large then diagrams with lots of edges are important. Thus large λ encourages a smoother triangulation of the worldsheet which we might interpret as fewer quantum fluctuations on the worldsheet. We expect a relation of the form $\lambda^{-1} \sim \alpha'$ which encodes our intuition about large λ suppressing fluctuations. This is what is found in well-understood examples.

This story is very general in the sense that all matrix models define something like a theory of two-dimensional fluctuating surfaces via these random triangulations. The connection is even more interesting when we remember all the extra labels we have been suppressing on our field Φ . For example, the position labeling where the field Φ sits plays the role of embedding coordinates on the worldsheet. Other indices (spin, etc.) indicate further worldsheet degrees of freedom. However, the microscopic details of the worldsheet theory are not so easily discovered. It took about fifteen years between the time when 't Hooft described the large- N perturbation series in this way and the first examples where the worldsheet dynamics were identified (these old examples are reviewed in, e.g., [25]).

As a final check on the nontriviality of the theory in the 't Hooft limit, let us see if the 't Hooft coupling runs with scale. For argument let us think about the case when the matrices are gauge fields and $L = -(1/g_{YM}^2) \text{tr} F_{\mu\nu} F^{\mu\nu}$. In d dimensions, the behavior through one loop is

$$\mu \partial_\mu g_{YM} \equiv \beta_g \sim \frac{4-d}{2} g_{YM} + b_0 g_{YM}^3 N. \quad (3.10)$$

(b_0 is a coefficient which depends on the matter content and vanishes for $\mathcal{N} = 4$ SYM.) So we find that $\beta_\lambda \sim ((4-d)/2)\lambda + b_0 \lambda^2$. Thus λ can still run in the large N limit and the theory is nontrivial.

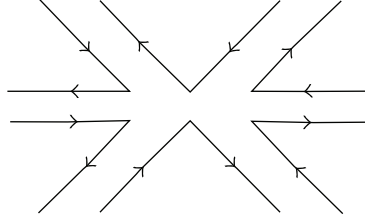


Figure 7: New vertex for an operator insertion of $\text{Tr}(\Phi^k)$ with $k = 6$.

3.3. *N*-Counting of Correlation Functions

Let us now consider the N -counting for correlation functions of local gauge-invariant operators. Motivated by gauge invariance and simplicity, we will consider “single trace” operators, operators $\mathcal{O}(x)$ that look like

$$\mathcal{O}(x) = c(k, N) \text{Tr}(\Phi_1(x) \cdots \Phi_k(x)) \tag{3.11}$$

and which we will abbreviate as $\text{Tr}(\Phi^k)$. We will keep k finite as $N \rightarrow \infty$.¹⁵ There are two little complications here. We must be careful about how we normalize the fields Φ and we must be careful about how we normalize the operator \mathcal{O} . The normalization of the fields will continue to be such that the Lagrangian takes the form $L = (1/g_{YM}^2)\mathcal{L} = (N/\lambda)\mathcal{L}$ with $\mathcal{L}(\Phi)$ containing no explicit factors of N . To fix the normalization of \mathcal{O} (to determine the constant $c(k, N)$) we will demand that when acting on the vacuum, the operator \mathcal{O} creates states of finite norm in the large- N limit, that is, $\langle \mathcal{O}\mathcal{O} \rangle_c \sim N^0$ where the subscript c stands for connected.

To determine $c(k, N)$ we need to know how to insert single trace operators into the t’Hooft counting. Each single-trace operator in the correlator is a new vertex which is required to be present in every contributing diagram. This vertex has k legs where k propagators can be attached and looks like a big squid. An example of such a new vertex appears in Figure 7 which corresponds to the insertion of the operator $\text{Tr}(\Phi^6)$. For the moment we do not associate any explicit factors of N with the new vertex. Let us consider the example $\langle \text{Tr}(\Phi^4) \text{Tr}(\Phi^4) \rangle$. We need to draw two four point vertices for the two single trace operators in the correlation function. How are we to connect these vertices with propagators? The dominant contribution comes from disconnected diagrams like the one shown in Figure 8. The leading disconnected diagram has four propagators and six index loops and so gives a factor $\lambda^4 N^2 \sim N^2$. On the other hand, the leading connected diagram shown in Figure 9 has four propagators and four index loops and so only gives a contribution $\lambda^4 \sim N^0$. (A way to draw the connected diagram in Figure 9 which makes the N -counting easier is shown in Figure 10 where we have deformed the two four point operator insertion vertices so that they are “ready for contraction”.)

The fact that disconnected diagrams win in the large N limit is general and goes by the name “large- N factorization”. It says that single trace operators are basically classical objects in the large- N limit $\langle \mathcal{O}\mathcal{O} \rangle \sim \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + \mathcal{O}(1/N^2)$.

The leading connected contribution to the correlation function is independent of N and so $\langle \mathcal{O}\mathcal{O} \rangle_c \sim c^2 N^0$. Requiring that $\langle \mathcal{O}\mathcal{O} \rangle_c \sim N^0$ means that we can just set $c \sim N^0$. Having fixed the normalization of \mathcal{O} we can now determine the N -dependence of higher-

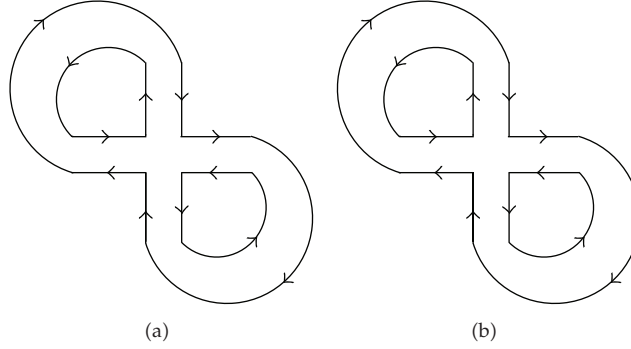


Figure 8: Disconnected diagram contributing to the correlation function $\langle \text{Tr}(\Phi^4) \text{Tr}(\Phi^4) \rangle$. Created by Brian Swingle.

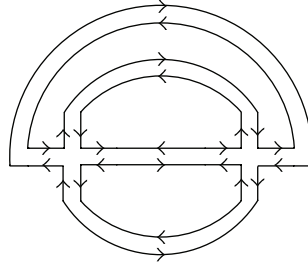


Figure 9: Connected diagram contributing to the correlation function $\langle \text{Tr}(\Phi^4) \text{Tr}(\Phi^4) \rangle$.

order correlation functions. For example, the leading connected diagram for $\langle \mathcal{O}^3 \rangle$ where $\mathcal{O} = \text{Tr}(\Phi^2)$ is just a triangle and contributes a factor $\lambda^3 N^{-1} \sim N^{-1}$. In fact, quite generally we have $\langle \mathcal{O}^n \rangle_c \sim N^{2-n}$ for the leading contribution.

So the operators \mathcal{O} (called glueballs in the context of QCD) create excitations of the theory that are free at large N —they interact with coupling $1/N$. In QCD with $N = 3$, quarks and gluons interact strongly, and so do their hadron composites. The role of large- N here is to make the color-neutral objects weakly interacting, in spite of the strong interactions of the constituents. So this is the sense in which the theory is classical: although the dimensions of these operators can be highly nontrivial (examples are known where they are irrational [26]), the dimensions of their products are additive at leading order in N .

Finally, we should make a comment about the N -scaling of the generating functional $Z = e^{-W} = \langle e^{-N \sum_A \lambda_A \mathcal{O}^A} \rangle$. We have normalized the sources so that each λ_A is an 't Hooft-like coupling, in that it is finite as $N \rightarrow \infty$. The effective action W is the sum of connected vacuum diagrams, which at large- N is dominated by the planar diagrams. As we have shown, their contributions go like N^2 . This agrees with our normalization of the gravity action,

$$S_{\text{bulk}} \sim \frac{L^{d-1}}{G_N} I_{\text{dimensionless}} \sim N^2 I_{\text{dimensionless}}. \quad (3.12)$$

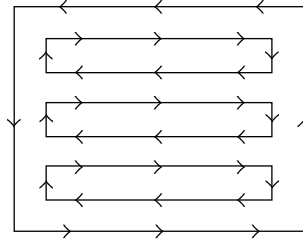


Figure 10: A redrawing of the connected diagram shown in Figure 9.

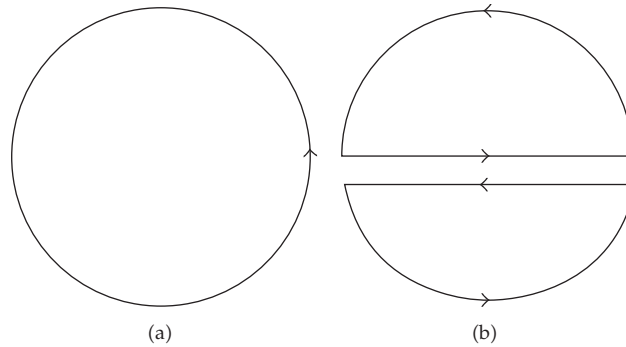


Figure 11: A quark vacuum bubble and a quark vacuum bubble with “gluon” exchange. Created by Brian Swingle.

3.4. Simple Generalizations

We can generalize the analysis performed so far without too much effort. One possibility is the addition of fields, “quarks”, in the fundamental of $U(N)$. We can add fermions $\Delta L \sim \bar{q}\gamma^\mu D_\mu q$ or bosons $\Delta L \sim |D_\mu q|^2$. Because quarks are in the fundamental of $U(N)$, their propagator consists of only a single line. When using Feynman diagrams to triangulate surfaces we now have the possibility of surfaces with boundary. Two quark diagrams are shown in Figure 11 both of which triangulate a disk. Notice in particular the presence of only a single outer line representing the quark propagator. We can conclude that adding quarks into our theory corresponds to admitting open strings into the string theory. We can also consider “meson” operators like $\bar{q}q$ or $\bar{q}\Phi^k q$ in addition to single trace operators. The extension of the holographic correspondence to include this case [27] has had many applications [28, 29], which are not discussed here for lack of time.

Another direction for generalization is to consider different matrix groups such as $SO(N)$ or $Sp(N)$. The adjoint of $U(N)$ is just the fundamental times the antifundamental. However, the adjoint representations of $SO(N)$ and $Sp(N)$ are more complicated. For $SO(N)$ the adjoint is given by the antisymmetric product of two fundamentals (vectors), and for $Sp(N)$ the adjoint is the symmetric product of two fundamentals. In both of these cases, the lines in the double-line formalism no longer have arrows. As a consequence, the lines in the propagator for the matrix field can join directly or cross and then join as shown in Figure 12. In the string language the worldsheet can now be unoriented, an example being given by a matrix field vacuum bubble where the lines cross giving rise to the worldsheet $\mathbb{R}P^2$.

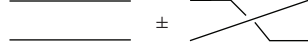


Figure 12: Propagator for $SO(N)$ (+) or $Sp(N)$ (-) matrix models.

4. Vacuum CFT Correlators from Fields in AdS

Our next goal is to evaluate $\langle e^{-\int \phi_0 \mathcal{O}} \rangle_{\text{CFT}} \equiv e^{-W_{\text{CFT}}[\phi_0]}$, where ϕ_0 is some small perturbation around some reference value associated with a CFT. You may not be interested in such a quantity in itself, but we will calculate it in a way which extends directly to more physically relevant quantities (such as real-time thermal response functions). The general form of the AdS/CFT conjecture for the generating functional is the GKPW equation [22, 23]

$$\langle e^{-\int \phi_0 \mathcal{O}} \rangle_{\text{CFT}} = Z_{\text{strings in AdS}}[\phi_0]. \quad (4.1)$$

This thing on the RHS is not yet a computationally effective object; the currently-practical version of the GKPW formula is the classical limit:

$$W_{\text{CFT}}[\phi_0] = -\ln \langle e^{\int \phi_0 \mathcal{O}} \rangle_{\text{CFT}} \simeq \text{extremum}_{\phi|_{z=\epsilon} \sim \phi_0} (N^2 I_{\text{grav}}[\phi]) + O\left(\frac{1}{N^2}\right) + O\left(\frac{1}{\sqrt{\lambda}}\right). \quad (4.2)$$

There are many things to say about this formula.

- (i) In the case of matrix theories like those described in the previous section, the classical gravity description is valid for large N and large λ . In some examples there is only one parameter which controls the validity of the gravity description. In (4.2) we have made the N -dependence explicit: in units of the AdS radius, the Newton constant is $L^{d-1}/G_N = N^2$. I_{grav} is some dimensionless action.
- (ii) We said that we are going to think of ϕ_0 as a small perturbation. Let us then make a perturbative expansion in powers of ϕ_0 :

$$W_{\text{CFT}}[\phi_0] = W_{\text{CFT}}[0] + \int d^D x \phi_0(x) G_1(x) + \frac{1}{2} \iint d^D x_1 d^D x_2 \phi_0(x_1) \phi_0(x_2) G_2(x_1, x_2) + \dots, \quad (4.3)$$

where

$$G_1(x) = \langle \mathcal{O}(x) \rangle = \left. \frac{\delta W}{\delta \phi_0(x)} \right|_{\phi_0=0}, \quad (4.4)$$

$$G_2(x) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_c = \left. \frac{\delta^2 W}{\delta \phi_0(x_1) \delta \phi_0(x_2)} \right|_{\phi_0=0}.$$

Now if there is no instability, then ϕ_0 is small implies ϕ is small. For one thing, this means that we can ignore interactions of the bulk fields in computing two-point functions. For n -point functions, we will need to know terms in the bulk action of degree up to n in the fields.

- (i) Anticipating divergences at $z \rightarrow 0$, we have introduced a cutoff in (4.2) (which will be a UV cutoff in the CFT) and set boundary conditions at $z = \epsilon$. They are in quotes because they require a bit of refinement (this will happen in Section 4.1).
- (ii) Equation (4.2) is written as if there is just one field in the bulk. Really there is a ϕ for every operator \mathcal{O} in the dual field theory. For such a pair, we will say “ ϕ couples to \mathcal{O} ” at the boundary. How to match up fields in the bulk and operators in the QFT? In general this is hard and information from string theory is useful. Without specifying a definite field theory, we can say a few general things.
- (1) We can organize both hand sides into representations of the conformal group. In fact only conformal primary operators correspond to “elementary fields” in the gravity action, and their descendants correspond to derivatives of those fields. More about this loaded word “elementary” in a moment.
 - (2) Only “single-trace” operators (like the $\text{tr}\Phi^k$ s of the previous section) correspond to “elementary fields” ϕ in the bulk. The excitations created by multitrace operators (like $(\text{tr}\Phi^k)^2$) correspond to multiparticle states of ϕ (in this example, a 2-particle state). Here I should stop and emphasize that this word “elementary” is well defined because we have assumed that we have a weakly coupled theory in the bulk, and hence the Hilbert space is approximately a Fock space, organized according to the number of particles in the bulk. A well-defined notion of single-particle state depends on large- N —if N is not large, it is not true that the overlap between $\text{tr}\Phi^2 \text{tr}\Phi^2|0\rangle$ and $\text{tr}\Phi^4|0\rangle$ is small.¹⁶
 - (3) There are some simple examples of the correspondence between bulk fields and boundary operators that are determined by symmetry. The stress-energy tensor $T_{\mu\nu}$ is the response of a local QFT to local change in the metric, $S_{\text{bdy}} \ni \int \gamma_{\mu\nu} T^{\mu\nu}$.

Here we are writing $\gamma_{\mu\nu}$ for the metric on the boundary. In this case

$$g_{\mu\nu} \longleftrightarrow T_{\mu\nu}. \quad (4.5)$$

Gauge fields in the bulk correspond to currents in the boundary theory:

$$A_\mu^a \longleftrightarrow J_a^\mu, \quad (4.6)$$

that is, $S_{\text{bdy}} \ni \int A_\mu^a J_a^\mu$. We say this mostly because we can contract all the indices to make a singlet action. In the special case where the gauge field is massless, the current is conserved.

- (iii) Finally, something that needs to be emphasized is that changing the *Lagrangian* of the CFT (by changing ϕ_0) is accomplished by changing the *boundary condition* in the bulk. The bulk equations of motion remain the same (e.g., the masses of the bulk fields do not change). This means that actually changing the bulk action corresponds to something more drastic in the boundary theory. One context in which it is useful to think about varying the bulk coupling constant is in thinking about the renormalization group. We motivated the form $\int (2\Lambda + \mathcal{R} + \dots)$ of the bulk

action by Wilsonian naturalness, which is usually enforced by the RG; so this is a delicate point. For example, soon we will compute the ratio of the shear viscosity to the entropy density, η/s , for the plasma made from *any* CFT that has an Einstein gravity dual; the answer is always $1/4\pi$. Each such CFT is what we usually think of as a universality class, since it will have some basin of attraction in the space of nearby QFT couplings. Here we are saying that a whole *class* of universality classes exhibits the same behavior.

What is special about these theories from the QFT point of view? Our understanding of this “bulk universality” is obscured by our ignorance about quantum mechanics in the bulk. Physicists with what could be called a monovacuist inclination may say that what is special about them is that they exist.¹⁷ The issue, however, arises for interactions in the bulk which are quite a bit less contentious than gravity; so this seems unlikely to me to be the answer.

4.1. Wave Equation near the Boundary and Dimensions of Operators

The metric of AdS (in Poincaré coordinates, so that the constant- z slices are just copies of Minkowski space) is

$$ds^2 = L^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2} \equiv g_{AB} dx^A dx^B, \quad A = 0, \dots, d, \quad x^A = (z, x^\mu). \quad (4.7)$$

As the simplest case to consider, let’s think about a scalar field in the bulk. An action for such a scalar field suggested by Naturalness is

$$S = -\frac{\mathfrak{K}}{2} \int d^{d+1}x \sqrt{g} \left[g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 + b\phi^3 + \dots \right]. \quad (4.8)$$

Here \mathfrak{K} is just a normalization constant; we are assuming that the theory of ϕ is weakly coupled and one may think of \mathfrak{K} as proportional to N^2 . For this metric $\sqrt{g} = \sqrt{|\det g|} = (L/z)^{d+1}$. Our immediate goal is to compute a two-point function of the operator \mathcal{O} to which ϕ couples, so we will ignore the interaction terms in (4.8) for a while. Since ϕ is a scalar field we can rewrite the kinetic term as

$$g^{AB} \partial_A \phi \partial_B \phi = (\partial\phi)^2 = g^{AB} D_A \phi D_B \phi, \quad (4.9)$$

where D_A is the covariant derivative, which has the nice property that $D_A(g_{BC}) = 0$, so we can move the D s around the g s with impunity. By integrating by parts we can rewrite the action in a useful way:

$$S = -\frac{\mathfrak{K}}{2} \int d^{d+1}x \left[\partial_A \left(\sqrt{g} g^{AB} \phi \partial_B \phi \right) - \phi \partial_A \left(\sqrt{g} g^{AB} \partial_B \phi \right) + \sqrt{g} \left(m^2 \phi^2 + \dots \right) \right] \quad (4.10)$$

and finally by using Stokes' theorem we can rewrite the action as

$$S = -\frac{\mathfrak{R}}{2} \int_{\partial\text{AdS}} d^d x \sqrt{g} g^{zB} \phi \partial_B \phi - \frac{\mathfrak{R}}{2} \int \sqrt{g} \phi (-\square + m^2) \phi + O(\phi^3), \quad (4.11)$$

where we define the scalar Laplacian $\square \phi = (1/\sqrt{g}) \partial_A (\sqrt{g} g^{AB} \partial_B) \phi = D^A D_A \phi$. Note that we wrote all these covariant expressions without ever introducing Christoffel symbols.

We can rewrite the boundary term more covariantly as

$$\int_{\mathcal{M}} \sqrt{g} D_A J^A = \int_{\partial\mathcal{M}} \sqrt{\gamma} n_A J^A. \quad (4.12)$$

The metric tensor γ is defined as

$$ds^2 \Big|_{z=\epsilon} \equiv \gamma_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{\epsilon^2} \eta_{\mu\nu} dx^\mu dx^\nu; \quad (4.13)$$

that is, it is the induced metric on the boundary surface $z = \epsilon$. The vector n_A is a unit vector normal to boundary ($z = \epsilon$). We can find an explicit expression for it:

$$n_A \propto \frac{\partial}{\partial z} g_{AB} n^A n^B \Big|_{z=\epsilon} = 1 \implies n = \frac{1}{\sqrt{g_{zz}}} \frac{\partial}{\partial z} = \frac{z}{L} \frac{\partial}{\partial z}. \quad (4.14)$$

From this discussion we have learned the following.

- (i) The equation of motion for small fluctuations of ϕ is $(-\square + m^2)\underline{\phi} = 0$.¹⁸
- (ii) If $\underline{\phi}$ solves the equation of motion, the on-shell action is just given by the boundary term.

Next we will derive the promised formula relating bulk masses and operator dimensions

$$\Delta(\Delta - d) = m^2 L^2 \quad (4.15)$$

by studying the AdS wave equation near the boundary.

Let us take advantage of translational invariance in d dimensions, $x^\mu \rightarrow x^\mu + a^\mu$, to Fourier decompose the scalar field:

$$\phi(z, x^\mu) = e^{ik_\mu x^\mu} f_k(z), \quad k_\mu x^\mu \equiv -\omega t + \vec{k} \cdot \vec{x}. \quad (4.16)$$

In the Fourier basis, substituting (4.16) into the wave equation $(-\square + m^2)\phi = 0$ and using the fact that the metric only depends on z , the wave equation is

$$0 = \left(g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{g}} \partial_z (\sqrt{g} g^{zz} \partial_z) + m^2 \right) f_k(z) \quad (4.17)$$

$$= \frac{1}{L^2} \left[z^2 k^2 - z^{d+1} \partial_z (z^{-d+1} \partial_z) + m^2 L^2 \right] f_k(z), \quad (4.18)$$

where we have used $g^{\mu\nu} = (z/L)^2 \delta^{\mu\nu}$. The solutions of (4.18) are Bessel functions; we can learn a lot without using that information. For example, look at the solutions near the boundary (i.e., $z \rightarrow 0$). In this limit we have power law solutions, which are spoiled by the $z^2 k^2$ term. To see this, try using $f_k = z^\Delta$ in (4.18):

$$\begin{aligned} 0 &= k^2 z^{2+\Delta} - z^{d+1} \partial_z (\Delta z^{-d+\Delta}) + m^2 L^2 z^\Delta \\ &= (k^2 z^2 - \Delta(\Delta - d) + m^2 L^2) z^\Delta, \end{aligned} \quad (4.19)$$

and for $z \rightarrow 0$ we get

$$\Delta(\Delta - d) = m^2 L^2. \quad (4.20)$$

The two roots of (4.20) are

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}. \quad (4.21)$$

Comments

- (i) The solution proportional to z^{Δ_-} is bigger near $z \rightarrow 0$.
- (ii) $\Delta_+ > 0$ for all m , therefore z^{Δ_+} decays near the boundary for any value of the mass.
- (iii) $\Delta_+ + \Delta_- = d$.

We want to impose boundary conditions that allow solutions. Since the leading behavior near the boundary of a generic solution is $\phi \sim z^{\Delta_-}$, we impose

$$\phi(x, z) \Big|_{z=\epsilon} = \phi_0(x, \epsilon) = \epsilon^{\Delta_-} \phi_0^{\text{Ren}}(x), \quad (4.22)$$

where ϕ_0^{Ren} is a renormalized source field. With this boundary condition ϕ_0^{Ren} is a finite quantity in the limit $\epsilon \rightarrow 0$.

Wavefunction Renormalization of \mathcal{O} (Heuristic but useful)

Suppose that

$$\begin{aligned} S_{\text{bdy}} &\ni \int_{z=\epsilon} d^d x \sqrt{\bar{\gamma}} \phi_0(x, \epsilon) \mathcal{O}(x, \epsilon) \\ &= \int d^d x \left(\frac{L}{\epsilon}\right)^d \left(\epsilon^{\Delta_-} \phi_0^{\text{Ren}}(x)\right) \mathcal{O}(x, \epsilon), \end{aligned} \quad (4.23)$$

where we have used $\sqrt{\bar{\gamma}} = (L/\epsilon)^d$. Demanding this to be finite as $\epsilon \rightarrow 0$ we get

$$\begin{aligned} \mathcal{O}(x, \epsilon) &\sim \epsilon^{d-\Delta_-} \mathcal{O}^{\text{Ren}}(x) \\ &= \epsilon^{\Delta_+} \mathcal{O}^{\text{Ren}}(x), \end{aligned} \quad (4.24)$$

where in the last line we have used $\Delta_+ + \Delta_- = d$. Therefore, the scaling dimension of \mathcal{O}^{Ren} is $\Delta_+ \equiv \Delta$. We will soon see that $\langle \mathcal{O}(x) \mathcal{O}(0) \rangle \sim 1/|x|^{2\Delta}$, confirming that Δ is indeed the scaling dimension.

We are solving a second-order ODE; therefore we need two conditions in order to determine a solution (for each k). So far we have imposed one condition at the boundary of AdS.

- (i) For $z \rightarrow \epsilon$, $\phi \sim z^{\Delta_-} \phi_0 +$ (terms subleading in z).

In the Euclidean case (we discuss real time in the next subsection), we will also impose

- (ii) ϕ regular in the interior of AdS (i.e., at $z \rightarrow \infty$).

Comments on Δ

- (1) The ϵ^{Δ_-} factor is independent of k and x , which is a consequence of a local QFT (this fails in exotic examples).
- (2) *Relevantness*: If $m^2 > 0$: This implies $\Delta \equiv \Delta_+ > d$, so \mathcal{O}_Δ is an irrelevant operator. This means that if you perturb the CFT by adding \mathcal{O}_Δ to the Lagrangian, then its coefficient is some mass scale to a negative power:

$$\Delta S = \int d^d x (\text{mass})^{d-\Delta} \mathcal{O}_\Delta, \quad (4.25)$$

where the exponent is negative; so the effects of such an operator go away in the IR, at energies $E < \text{mass}$. $\phi \sim z^{\Delta_-} \phi_0$ is this coupling. It grows in the UV (small z). If ϕ_0 is a finite perturbation, it will back-react on the metric and destroy the asymptotic AdS-ness of the geometry: extra data about the UV will be required.

$m^2 = 0 \leftrightarrow \Delta = d$ means that \mathcal{O} is marginal.

If $m^2 < 0$, then $\Delta < d$; so \mathcal{O} is a relevant operator. Note that in AdS, $m^2 < 0$ is ok if m^2 is not too negative. Such fields with $m^2 > -|m_{\text{BF}}|^2 \equiv -(d/2L)^2$ are called ‘‘Breitenlohner-Freedman- (BF-) allowed tachyons’’. The reason you might think that $m^2 < 0$ is bad is

that usually it means an instability of the vacuum at $\phi = 0$. An instability occurs when a normalizable mode grows with time without a source. But for $m^2 < 0$, $\phi \sim z^{\Delta_-}$ decays near the boundary (i.e., in the UV). This requires a gradient energy of order $\sim 1/L$, which can stop the field from condensing.

To see what is too negative, consider the formula for the dimension, $\Delta_{\pm} = d/2 \pm \sqrt{(d/2)^2 + m^2 L^2}$. For $m^2 < m_{\text{BF}}^2$, the dimension becomes complex.

- (3) The formula relating the mass of a bulk field and the dimension of the associated operator depends on their spin. For example, for a massive gauge field in AdS with action

$$S = - \int_{\text{AdS}} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_{\mu} A^{\mu} \right), \quad (4.26)$$

the boundary behavior of the wave equation implies that $A_{\mu} \stackrel{z \rightarrow 0}{\sim} z^{\alpha}$ with

$$\alpha(\alpha - d + 2) = m^2 L^2. \quad (4.27)$$

For the particular case of A_{μ} massless this can be seen to lead to $\Delta(j^{\mu}) = d - 1$, which is the dimension of a conserved current in a CFT. Also, the fact that $g_{\mu\nu}$ is massless implies

$$\Delta(T^{\mu\nu}) = d, \quad (4.28)$$

which is required by conformal Ward identities.

4.2. Solutions of the AdS Wave Equation and Real-Time Issues

An approach which uses the symmetries of AdS [23] is appealing. However, it is not always available (e.g., if there is a black hole in the spacetime). Also, it can be misleading: as in quantum field theory, scale-invariance is inevitably broken by the UV cutoff.

Return to the scalar wave equation in momentum space:

$$0 = \left[z^{d+1} \partial_z \left(z^{-d+1} \partial_z \right) - m^2 L^2 - z^2 k^2 \right] f_k(z). \quad (4.29)$$

We will now stop being agnostic about the signature and confront some issues that arise for real time correlators. If $k^2 > 0$, that is, k^{μ} is spacelike (or Euclidean), the solution is

$$f_k(z) = A_K z^{d/2} K_{\nu}(kz) + A_I z^{d/2} I_{\nu}(kz), \quad (4.30)$$

where $\nu = \Delta - d/2 = \sqrt{(d/2)^2 + m^2 L^2}$. In the interior of AdS ($z \rightarrow \infty$), the Bessel functions behave as

$$K_{\nu}(kz) \sim e^{-kz}, \quad I_{\nu}(kz) \sim e^{kz}. \quad (4.31)$$

So we see that the regularity in the interior uniquely fixes $f_{\underline{k}}$ and hence the bulk-to-boundary propagator. Actually there is a theorem (the Graham-Lee theorem) addressing this issue for gravity fields (and not just linearly in the fluctuations); it states that if you specify a Euclidean metric on the boundary of a Euclidean AdS (which we can think of as topologically a S^d by adding the point at infinity in \mathbb{R}^d) modulo conformal rescaling, then the metric on the space inside of the S^d is uniquely determined. A similar result holds for gauge fields.

In contrast to this, in Lorentzian signature with timelike k^2 , that is, for on-shell states with $\omega^2 > \vec{k}^2$, there exist two linearly independent solutions with the same leading behavior at the UV boundary. In terms of $q \equiv \sqrt{\omega^2 - \vec{k}^2}$, the solutions are

$$z^{d/2} K_{\pm\nu}(iqz) \sim e^{\pm iqz} \quad (z \rightarrow \infty); \quad (4.32)$$

so these modes oscillate in the far IR region of the geometry.¹⁹ This ambiguity reflects the many possibilities for real-time Green's functions in the QFT. One useful choice is the retarded Green's function, which describes causal response of the system to a perturbation. This choice corresponds to a choice of boundary conditions at $z \rightarrow \infty$ describing stuff falling into the horizon [30], that is, moving towards larger z as time passes. There are three kinds of reasons for this prescription.²⁰

- (i) Both the retarded Green's functions and stuff falling through the horizon describe things that *happen*, rather than *unhappen*.
- (ii) You can check that this prescription gives the correct analytic structure of $G_R(\omega)$ ([30] and all the hundreds of papers that have used this prescription).
- (iii) It has been derived from a holographic version of the Schwinger-Keldysh prescription [31–33].

The fact that stuff goes past the horizon and does not come out is what breaks time-reversal invariance in the holographic computation of G^R . Here, the ingoing choice is $\phi(t, z) \sim e^{-i\omega t + iqz}$, since as t grows, the wavefront moves to larger z . This specifies the solution which computes the causal response to be $z^{d/2} K_{+\nu}(iqz)$.

The same prescription, adapted to the black hole horizon, will work in the finite temperature case.

One last thing we must deal with before proceeding is to define what we mean by a “normalizable” mode, or solution, when we say that we have many normalizable solutions for $k^2 < 0$ with a given scaling behavior. In Euclidean space, ϕ is normalizable when $S[\phi] < \infty$. This is because when we are thinking about the partition function $Z[\phi] = \sum_{\phi} e^{-S[\phi]}$, modes with boundary conditions which force $S[\phi] = \infty$ would not contribute. In real-time, we say that ϕ is normalizable if $E[\phi] < \infty$ where

$$E[\phi] = \int_{\Sigma} d^{d-1}x dz \sqrt{h} T_{\mu\nu}[\phi] n^{\mu} \xi^{\nu} = \int_{x^0=\text{constant}} d^{d-1}x dz \sqrt{h} T_0^0[\phi], \quad (4.33)$$

where Σ is a given spatial slice, h is the induced metric on that slice, n^μ is a normal unit vector to Σ , and ξ^μ is a timelike killing vector. T_{AB} is defined as

$$T_{AB}[\phi] = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{AB}} S_{\text{Bulk}}[\phi]. \quad (4.34)$$

4.3. Bulk-to-Boundary Propagator in Momentum Space

We return to considering spacelike k in this section. Let us normalize our solution of the wave equation by the condition $f_k(z = \epsilon) = 1$. This means that its Fourier transform is a δ -function in position space $\delta^d(x)$ when evaluated at $z = \epsilon$, not at the actual boundary $z = 0$. The solution, which we can call the ‘‘regulated bulk-to-boundary propagator’’, is then

$$\underline{f}_{\underline{k}}(z) = \frac{z^{d/2} K_\nu(kz)}{e^{d/2} K_\nu(k\epsilon)}. \quad (4.35)$$

The general position space solution can be obtained by Fourier decomposition:

$$\underline{\phi}^{[\phi_0]}(x) = \int d^d k e^{ikx} \underline{f}_{\underline{k}}(z) \phi_0(k, \epsilon). \quad (4.36)$$

The ‘‘on-shell action’’ (i.e., the action evaluated on the saddle-point solution) is (using (4.11))

$$\begin{aligned} S[\underline{\phi}] &= -\frac{\mathfrak{R}}{2} \int d^d x \sqrt{\gamma} \underline{\phi} n \cdot \partial \underline{\phi} \\ &= -\frac{\mathfrak{R}}{2} \int d^d x \int d^d k_1 \int d^d k_2 e^{i(k_1+k_2)x} \phi_0(k_1, \epsilon) \phi_0(k_2, \epsilon) L^{d-1} z^{-d} \underline{f}_{\underline{k}_1}(z) z \partial_z \underline{f}_{\underline{k}_2}(z) \Big|_{z=\epsilon} \\ &= -\frac{\mathfrak{R} L^{d-1}}{2} \int d^d k \phi_0(k, \epsilon) \phi_0(-k, \epsilon) \mathcal{F}_\epsilon(k), \end{aligned} \quad (4.37)$$

and therefore²¹

$$\langle \mathcal{O}(k_1) \mathcal{O}(k_2) \rangle_c^\epsilon = -\frac{\delta}{\delta \phi_0(k_1)} \frac{\delta}{\delta \phi_0(k_2)} S = (2\pi)^d \delta^d(k_1 + k_2) \mathcal{F}_\epsilon(k_1). \quad (4.38)$$

Here $\mathcal{F}_\epsilon(k)$ (sometimes called the ‘‘flux factor’’) is

$$\mathcal{F}_\epsilon(k) = z^{-d} \underline{f}_{\underline{k}}(z) z \partial_z \underline{f}_{\underline{k}}(z) \Big|_{z=\epsilon} + (k \longleftrightarrow -k) = 2\epsilon^{-d+1} \partial_z \left(\frac{z^{d/2} K_\nu(kz)}{e^{d/2} K_\nu(\epsilon z)} \right) \Big|_{z=\epsilon}. \quad (4.39)$$

The small- u (near boundary) behavior of $K_\nu(u)$ is

$$\begin{aligned} K_\nu(u) &= u^{-\nu} \left(a_0 + a_1 u^2 + a_2 u^4 + \dots \right) \quad (\text{leading term}) \\ &+ u^\nu \ln u \left(b_0 + b_1 u^2 + b_2 u^4 + \dots \right) \quad (\text{subleading term}), \end{aligned} \quad (4.40)$$

where the coefficients of the series a_i, b_i depend on ν . For noninteger ν , there would be no $\ln u$ in the second line, and so we make it pink. Of course, we saw in the previous subsection (with very little work) that *any* solution of the bulk wave equation has this kind of form (the boundary is a regular singular point of the ODE). We could determine the a s and b s recursively by the same procedure. This is just like a scattering problem in 1d quantum mechanics. The point of the Bessel function here is to choose which values of the coefficients a, b give a function which has the correct behavior at the other end, that is, at $z \rightarrow \infty$. Plugging the asymptotic expansion of the Bessel function into (4.39),

$$\begin{aligned} \mathcal{F}_\epsilon(k) &= 2\epsilon^{-d+1} \partial_z \left(\frac{(kz)^{-\nu+d/2} (a_0 + \dots) + (kz)^{\nu+d/2} \ln kz (b_0 + \dots)}{(k\epsilon)^{-\nu+d/2} (a_0 + \dots) + (k\epsilon)^{\nu+d/2} \ln k\epsilon (b_0 + \dots)} \right) \Big|_{z=\epsilon} \\ &= 2\epsilon^{-d} \left[\left\{ \frac{d}{2} - \nu \left(1 + c_2 (\epsilon^2 k^2) + c_4 (\epsilon^4 k^4) + \dots \right) \right\} \right. \\ &\quad \left. + \left\{ \nu \frac{2b_0}{a_0} (\epsilon k)^{2\nu} \ln(\epsilon k) \left(1 + d_2 (\epsilon k)^2 + \dots \right) \right\} \right] \\ &\equiv (\text{I}) + (\text{II}), \end{aligned} \quad (4.41)$$

where (I) and (II) denote the first and second groups of terms of the previous line.

(I) is a Laurent series in ϵ with coefficients which are positive powers of k (i.e., analytic in k at $k = 0$). These are contact terms, that is, short distance goo that we do not care about and can subtract off. We can see this by noting that

$$\int d^d k e^{-ikx} (\epsilon k)^{2m} \epsilon^{-d} = \epsilon^{2m-d} \square_x^m \delta^d(x) \quad (4.42)$$

for $m > 0$. The ϵ^{2m-d} factor reinforces the notion that ϵ , which is an IR cutoff in AdS, is a UV cutoff for the QFT.

The interesting bit of $\mathcal{F}(k)$, which gives the $x_1 \neq x_2$ behavior of the correlator (4.38), is nonanalytic in k :

$$(\text{II}) = -2\nu \cdot \frac{b_0}{a_0} k^{2\nu} \ln(k\epsilon) \cdot \epsilon^{2\nu-d} \left(1 + \mathcal{O}(\epsilon^2) \right), \quad \left(\frac{b_0}{a_0} = \frac{(-1)^{\nu-1}}{2^{2\nu} \nu \Gamma(\nu)^2} \text{ for } \nu \in \mathbb{Z} \right). \quad (4.43)$$

To get the factor of 2ν , one must expand both the numerator and the denominator in (4.41); this important subtlety was pointed out in [34].²² The Fourier transformation of the leading term of (II) is given by

$$\int d^d k e^{-ikx} (\text{II}) = \frac{2\nu\Gamma(\Delta_+)}{\pi^{d/2}\Gamma(\Delta_+ - d/2)} \frac{1}{x^{2\Delta_+}} e^{2\nu-d}. \quad (4.44)$$

Note that the AdS radius appears only through the overall normalization of the correlator (4.38), in the combination $\mathfrak{R}L^{d-1}$.

Now let us deal with the pesky cutoff dependence. Since $e^{2\nu-d} = e^{-2\Delta_-}$ if we let $\phi_0(k, \epsilon) = \phi_0^{\text{Ren}}(k)\epsilon^{\Delta_-}$ as before, the operation

$$\frac{\delta}{\delta\phi_0(k, \epsilon)} = e^{-\Delta_-} \frac{\delta}{\delta\phi_0^{\text{Ren}}(k)} \quad (4.45)$$

removes the potentially divergent factor of $e^{-2\Delta_-}$. We also see that for $\epsilon \rightarrow 0$, the $\mathcal{O}(\epsilon^2)$ terms vanish.

If you are bothered by the infinite contact terms (I), there is a prescription to cancel them, appropriately called Holographic Renormalization [35]. Add to S_{bulk} the local, intrinsic boundary term:

$$\begin{aligned} \Delta S = S_{\text{c.t.}} &= \frac{\mathfrak{R}}{2} \int_{\text{bdy}} d^d x \left(-\Delta_- L^{d-1} \epsilon^{2\Delta_- - d} \left(\phi_0^{\text{Ren}}(x) \right)^2 \right) \\ &= -\Delta_- \frac{\mathfrak{R}}{2L} \int_{\partial\text{AdS}, z=\epsilon} \sqrt{\gamma} \phi^2(z, x), \end{aligned} \quad (4.46)$$

and redo the preceding calculation. Note that this does not affect the equations of motion, nor does it affect $G_2(x_1 \neq x_2)$.

4.4. The Response of the System to an Arbitrary Source

Next we will derive a very important formula for the response of the system to an arbitrary source. The preceding business with the on-shell action is cumbersome, and is inconvenient for computing real-time Green's functions. The following result [36–40] circumvents this confusion.

The solution of the equations of motion, satisfying the conditions we want in the interior of the geometry, behaves near the boundary as

$$\phi(z, x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right); \quad (4.47)$$

this formula defines the coefficient ϕ_1 of the subleading behavior of the solution. First we recall some facts from classical mechanics. Consider the dynamics of a particle in one

dimension, with action $S[x] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t))$. The variation of the action with respect to the initial value of the coordinate is the momentum:

$$\frac{\delta S}{\delta x(t_i)} = \Pi(t_i), \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}}. \quad (4.48)$$

Thinking of the radial direction of AdS as time, the following is a mild generalization of (4.48):

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \rightarrow 0} \left(\frac{z}{L} \right)^{\Delta_-} \Pi(z, x) \Big|_{\text{finite}}, \quad (4.49)$$

where $\Pi \equiv \partial \mathcal{L} / \partial (\partial_z \phi)$ is the bulk field-momentum, with z thought of as time. There are two minor subtleties.

- (1) The factor of z^{Δ_-} arises because ϕ_0 differs from the boundary value of ϕ by a factor: $\phi \sim z^{\Delta_-} \phi_0$, so $\partial / \partial \phi_0 = z^{-\Delta_-} (\partial / \partial \phi (z = \epsilon))$.
- (2) Π itself in general (for $m \neq 0$) has a term proportional to the source ϕ_0 , which diverges near the boundary; this is related to the contact terms in the previous description. Do not include these terms in (4.49). Finally, then, we can evaluate the field momentum in the solution (4.47) and find²³

$$\langle \mathcal{O}(x) \rangle = \mathfrak{R} \frac{2\Delta - d}{L} \phi_1(x). \quad (4.50)$$

This is an extremely important formula. We already knew that the leading behavior of the solution encoded the source in the dual theory, that is, the perturbation of the *action* of the QFT. Now we see that the coefficient of the subleading falloff encodes the response [36]. It tells us how the *state* of the QFT changes as a result of the perturbation.²⁴

This formula applies in the real-time case [41]. For example, to describe linear response, $\delta \langle \mathcal{O} \rangle = \delta \phi_0 G + O(\delta \phi_0^2)$, then (4.50) says that

$$G(\omega, k) = \mathfrak{R} \frac{2\Delta - d}{L} \frac{\phi_1(\omega, k)}{\phi_0(\omega, k)}. \quad (4.51)$$

Which kind of Green's function we get depends on the boundary conditions we impose in the interior.

4.5. A Useful Visualization

We are doing classical field theory in the bulk, that is, solving a boundary value problem. We can describe the expansion about an extremum of classical action in powers of ϕ_0 in terms of tree level Feynman graphs. External legs of the graphs correspond to the wavefunctions of asymptotic states. In the usual example of QFT in flat space, these are plane waves. In

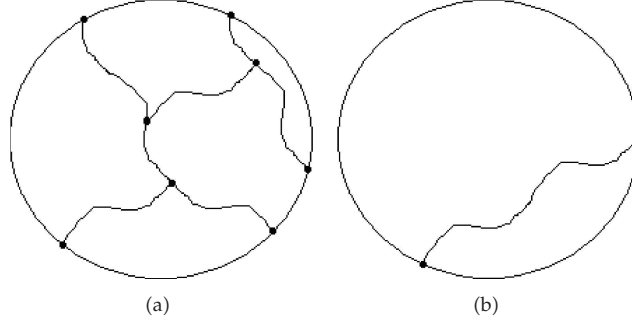


Figure 13: Feynman graphs in AdS. We did the one with two external legs in Section 4.3. Created by Francesco D'Éramo.

the expansion we are setting up in AdS, the external legs of graphs are associated with the boundary behavior of ϕ (bulk-to-boundary propagators). These diagrams are sometimes called “Witten diagrams,” after [23].

4.6. n -Point Functions

Next we will briefly talk about connected correlation functions of three or more operators. Unlike two-point functions, such observables are sensitive to the details of the bulk interactions, and we need to make choices. For definiteness, we will consider the three-point functions of scalar operators dual to scalar fields with the following bulk action:

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left[\sum_{i=1}^3 \left((\partial\phi_i)^2 + m_i^2 \phi_i^2 \right) + b\phi_1\phi_2\phi_3 \right]. \quad (4.52)$$

The discussion can easily be extended to n -point functions with $n > 3$.

The equations of motion are

$$\left(\square - m_1^2 \right) \phi_1(z, x) = b\phi_2\phi_3 \quad (4.53)$$

and its permutations. We solve this perturbatively in the sources, ϕ_0^i :

$$\begin{aligned} \underline{\phi}^1(z, x) &= \int d^d x_1 K^{\Delta_1}(z, x; x_1) \phi_0^1(x_1) \\ &+ b \int d^d x' dz' \sqrt{g} G^{\Delta_1}(z, x; z', x') \int d^d x_1 \int d^d x_2 K^{\Delta_2}(z', x'; x_1) \phi_0^2(x_1) K^{\Delta_3}(z', x'; x_2) \phi_0^3(x_2) \\ &+ \mathcal{O}(b^2 \phi_0^3), \end{aligned} \quad (4.54)$$

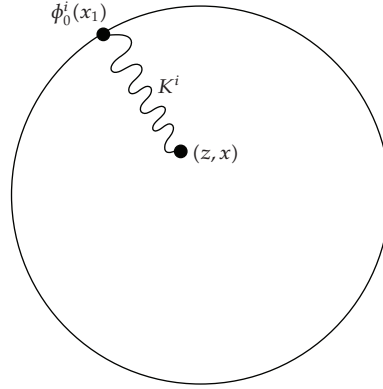


Figure 14: Witten diagram 1, created by Daniel Park.

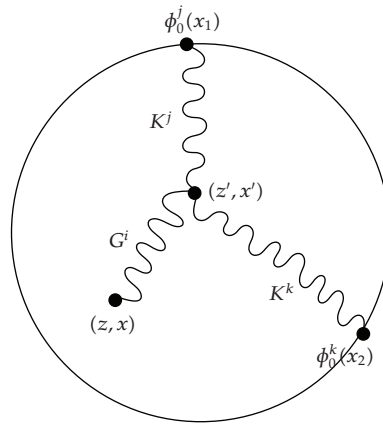


Figure 15: Witten diagram 2, created by Daniel Park.

with similar expressions for $\underline{\phi}^{2,3}$. We need to define the quantities K, G appearing in (4.54). K^Δ is the bulk-to-boundary propagator for a bulk field dual to an operator of dimension Δ . We determined this in the previous subsection: it is defined to be the solution to the homogeneous scalar wave equation $(\square - m^2)K^\Delta(z, x; x') = 0$ which approaches a delta function at the boundary,²⁵ $K^\Delta(z, x; x') \xrightarrow{z \rightarrow 0} z^\Delta \delta(x, x')$. So K is given by (4.36) with $\phi_0(k) = e^{-ikx'}$. $G^{\Delta_i}(z, x; z', x')$ is the bulk-to-bulk propagator, which is the *normalizable* solution to the wave equation with a source

$$(\square - m_i^2)G^{\Delta_i}(z, x; z', x') = \frac{1}{\sqrt{g}}\delta(z - z')\delta^d(x - x') \quad (4.55)$$

(so that $(\square - m_i^2) \int \sqrt{g} G J = J$ for a source J).

The first and second terms in (4.54) are summarized by the Witten diagrams in Figures 14 and 15. A typical higher-order diagram would look something like Figure 16. This result can be inserted into our master formula (4.50) to find G_3 .

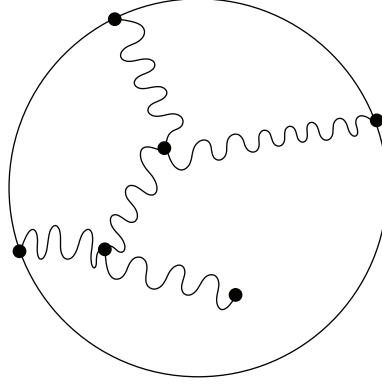


Figure 16: Witten diagram 3, created by Daniel Park.

4.7. Which Scaling Dimensions Are Attainable?

Let us think about which Δ are attainable by varying m .²⁶ $\Delta_+ \geq d/2$ is the smallest dimension we have obtained so far, but the bound from unitarity is lower: $\Delta_+ \geq (d-2)/2$. There is a range of values for which we have a choice about which of $z^{\Delta_{\pm}}$ is the source and which is the response. This comes about as follows.

ϕ is normalizable if $S[\phi] < \infty$. With the action we have been using

$$S_{\text{Bulk}} = \int_{\epsilon} d^{d+1}x \sqrt{g} (g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2), \quad (4.56)$$

with $\sqrt{g} = z^{-d-1}$, our boundary conditions demand that $\phi \sim z^{\Delta} (1 + \mathcal{O}(z^2))$ with $\Delta = \Delta_+$ or Δ_- ,

$$g^{zz} (\partial_z \phi)^2 = (z \partial_z \phi)^2 \sim \Delta^2 z^{2\Delta} \quad (4.57)$$

and hence,

$$g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 \simeq \Delta^2 z^{2\Delta} + k^2 z^{2\Delta+2} + m^2 z^2 = (\Delta^2 + m^2) z^{2\Delta} (1 + \mathcal{O}(z^2)) \quad (4.58)$$

in the limit $z \rightarrow 0$. Since for $\Delta = \Delta_{\pm}$, $\Delta^2 + m^2 = -d\Delta \neq 0$,

$$S_{\text{Bulk}}[z^{\Delta}] \sim \int_{\epsilon} dz z^{-d-1} (-d\Delta) z^{2\Delta} (1 + \mathcal{O}(z^2)) \propto \frac{1}{2\Delta - d} e^{2\Delta - d}. \quad (4.59)$$

We emphasize that we have only specified the boundary behavior of ϕ , and it is not assumed that ϕ satisfies the equation of motion. We see that

$$S_{\text{Bulk}}[z^{\Delta}] < \infty \iff \Delta > \frac{d}{2}. \quad (4.60)$$

This does not saturate the lower bound from unitarity on the dimension of a scalar operator, which is $\Delta > (d-2)/2$; the bound coincides with the engineering dimension of a free field.

We can change which fall-off is the source by adding a boundary term to the action (4.56) [37]:

$$S_{\text{Bulk}}^{KW} = \int_{\epsilon} d^{d+1}x \sqrt{g} \phi (-\square + m^2) \phi = S_{\text{Bulk}} - \int_{\partial \text{AdS}} \sqrt{\gamma} \phi n \cdot \partial_z \phi. \quad (4.61)$$

For this action we see that

$$\begin{aligned} S_{\text{Bulk}}^{KW} [\phi \sim z^{\Delta} (1 + \mathcal{O}(z^2))] \\ &= \int_{\epsilon} dzz^{-d-1} z^{\Delta} (1 + \mathcal{O}(z^2)) \left[(-\Delta(\Delta-d) + m^2) z^{\Delta} (1 + \mathcal{O}(z^2)) + k^2 z^{2\Delta+2} \right] \\ &\sim \int_{\epsilon} dzz^{-d-1+2\Delta+2} \sim e^{2\Delta-d+2} < \infty \end{aligned} \quad (4.62)$$

is equivalent to

$$\Delta \geq \frac{d-2}{2} \quad (4.63)$$

which is exactly the unitarity bound. We see that in this case both Δ_{\pm} give normalizable modes for $\nu \leq 1$. Note that it is actually Δ_- that gives the value which saturates the unitarity bound, that is, when

$$\Delta_- = \left(\frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \right)_{m^2=1-d^2/4} = \frac{d-2}{2}. \quad (4.64)$$

The coefficient of z^{Δ_+} would be the source in this case.

We have found a description of a different boundary CFT from the same bulk action, which we have obtained by adding a boundary term to the action. The effect of the new boundary term is to lead us to impose Neumann boundary conditions on ϕ , rather than Dirichlet conditions. The procedure of interchanging the two is similar to a Legendre transformation.

4.8. Geometric Optics Limit

When the dimension of our operator \mathcal{O} is very large, the mass of the bulk field is large in units of the AdS radius:

$$m^2 L^2 = \Delta(\Delta-d) \sim \Delta^2 \gg 1 \implies mL \gg 1. \quad (4.65)$$

This means that the path integral which computes the solution of the bulk wave equation has a sharply peaked saddle point, associated with geodesics in the bulk. That is, we can think

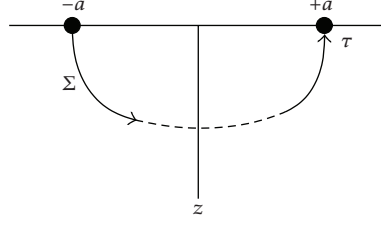


Figure 17: The curve $\Sigma \subset \text{AdS}$ connecting the two-points in C . The arrows on the curve indicate the orientation, created by Vijay Kumar.

of the solution of the bulk wave equation from a first-quantized point of view, in terms of particles of mass m in the bulk. For convenience, consider the case of a complex operator \mathcal{O} , so that the worldlines of these particles are oriented. Then

$$\langle \mathcal{O}(+a)\bar{\mathcal{O}}(-a) \rangle = Z[\pm a] \stackrel{m \gg L^{-1}}{\sim} \exp(-S[\underline{z}]). \quad (4.66)$$

($\bar{\mathcal{O}}$ is the complex conjugate operator.) The middle expression is the Feynman path integral $Z[\pm a] = \int [dz(\tau)] \exp(-S[z])$; the action for a point particle of mass m whose world-line is Σ is given by $S[z] = m \int_{\Sigma} ds$. In the limit of large m , we have

$$Z[\pm a] \sim \exp(-S[\underline{z}]), \quad (4.67)$$

where we have used the saddle point approximation; \underline{z} is the geodesic connecting the points $\pm a$ on the boundary.

We now compute $Z[\pm a]$ in the saddle point approximation. The metric restricted to Σ is given by $ds^2|_{\Sigma} = (L^2/z^2)(1+z'^2)d\tau^2$, where $z' := dz/d\tau$. This implies that the action is

$$S[z] = \int d\tau \frac{L}{z} \sqrt{1+z'^2}. \quad (4.68)$$

The geodesic can be computed by noting that the action $S[z]$ does not depend on τ explicitly. This implies, we have a conserved quantity:

$$\begin{aligned} h &= z' \frac{\partial \mathcal{L}}{\partial z'} - \mathcal{L} = \frac{L}{z} \frac{1}{\sqrt{1+z'^2}} \\ \implies z'^2 &= \left(\frac{L^2}{h^2} - z^2 \right) \frac{1}{z^2}. \end{aligned} \quad (4.69)$$

The above is a first-order differential equation with solution $\tau = \sqrt{z_{\max}^2 - z^2}$, with $z(\tau = 0) = z_{\max} = L/h = a$. This is the equation of a semicircle. Substituting the solution back into the action gives

$$\begin{aligned} S[\underline{z}] &= \int d\tau \frac{L}{\underline{z}} \sqrt{1 + \underline{z}'^2} = 2L \int_{\epsilon}^a \frac{dz}{\sqrt{a^2 - z^2}} \frac{a}{z} \\ &= 2L \log \frac{2a}{\epsilon} + (\text{terms} \rightarrow 0 \text{ as } \epsilon \rightarrow 0). \end{aligned} \quad (4.70)$$

You might think that since we are computing this in a conformal field theory, the only scale in the problem is a and therefore the path integral should be independent of a . This argument fails in the case at hand because there are two scales: a and ϵ , the UV cutoff. The scale transformation is anomalous and this is manifested in the ϵ dependence of $S[\underline{z}]$:

$$\langle \mathcal{O}(+a) \bar{\mathcal{O}}(-a) \rangle = Z[\pm a] \stackrel{m \gg L^{-1}}{\sim} \exp(-S[\underline{z}]) \sim \frac{1}{a^{2mL}}. \quad (4.71)$$

This is exactly what we expect for the two-point function, since $m^2 L^2 = \Delta(\Delta - d) \approx \Delta^2$ in the large Δ limit. Without this anomaly, the two-point function of the operator \mathcal{O} would be independent of a forcing $\Delta = 0$, which is impossible in a unitary CFT. This is similar to the anomaly that gives a nonzero scaling dimension of $k^2/4$ to the operator $\exp(ikX)$ for the 2d free boson CFT which has classical scaling dimension zero. Graham and Witten [42] showed that such a scale anomaly exists for surface observables for any even k . This relation between correlators of large Δ observables and geodesics in the bulk theory offers a probe of the bulk spacetime [43, 44].

4.9. Comment on the Physics of the Warp Factor

Recall that the bulk radial coordinate behaves like a spectrograph separating the theory into energy scales. $z = 0$ is associated with the UV while $z \rightarrow \infty$ describes the IR.

One way to think about the implementation of this relationship between the radial coordinate of the bulk geometry and the field theory energy scale is as follows. The geometry dual to a more general Lorentz-invariant state is

$$ds^2 = w^2(z) \left(-dt^2 + d\vec{x}^2 + dz^2 \right), \quad (4.72)$$

where $w(z)$ is called the ‘‘warp factor’’. The warp factor for AdS is simply L/z . The coordinates t, \vec{x} in (4.72) are the field theory time and space coordinates. This means that the size of a field theory object is related to the size of its holographic image by the relation:

$$(\text{size})_{\text{FT}} = \frac{1}{w(z)} (\text{proper size}) \quad (4.73)$$

and similarly the energy of a field theory object is related to the proper energy of the corresponding bulk excitation by

$$E_{\text{FT}} \sim i\partial_t = w(z)E_{\text{proper}}. \quad (4.74)$$

The fact that AdS goes forever in the z direction is related to the fact that the dual theory is a conformal field theory with degrees of freedom at all energies. We can interpret the existence of modes at arbitrarily low energy as the statement that the warp factor has a zero at $z = \infty$. More precisely, there are $\mathcal{O}(N^2)$ degrees of freedom at every energy scale in the CFT. One concrete manifestation of this is the fact that we found a continuum of solutions of $(-\square + m^2)\phi = 0$ —in particular, for any k^μ we could match the required boundary condition in the IR region, $z \rightarrow \infty$.

In the gravity dual of a QFT with an energy gap, the warp factor has a *minimum*. This will impose a boundary condition on the bulk wave equation at the IR end of the geometry. The problem becomes like QM of a 1d particle trapped at both ends and therefore has a discrete spectrum.

There exist gravity solutions [19, 45, 46] which are dual to field theories with logarithmically running couplings which become strong in the IR, leading to a gap to almost all excitations (except some order N^0 number of Goldstone bosons), as in QCD.

5. Finite Temperature and Density

AdS is scale invariant. It is a solution dual to the *vacuum* of a CFT. The correspondence we have developed can also describe systems which are not scale invariant. This includes QFTs with a scale in the Lagrangian, or where a scale is generated quantum mechanically. A simpler thing to describe is the dual of a CFT in an ensemble which introduces a scale. A saddle point of the bulk path integral describing such a state would correspond to a geometry which approaches AdS near the boundary, and solves the same equations of motion:

$$0 = R_{AB} + \frac{d}{L^2} g_{AB} \propto \frac{\delta S_{\text{bulk}}}{\delta g^{AB}}, \quad (5.1)$$

but does something else in the interior.

5.1. Interjection on Expectations for CFT at Finite Temperature

In particular we mean a d dimensional relativistic CFT. The partition function is

$$Z(\tau) = \text{tr} e^{-H/T} = e^{-F/T} \quad (5.2)$$

with free energy F , on a space with geometry $S_{th}^1 \times \Sigma_{d-1}$ where the S_{th}^1 has radius $1/T$, $\tau \sim \tau + 1/T$ and Σ_{d-1} is some $(d-1)$ -manifold. We can give Σ_{d-1} finite volume as an IR regulator. The temperature is a deformation of the IR physics (modes with $\omega \gg T = E_{KK}$ do not notice).

For large $V = \text{Vol}(\Sigma_{d-1})$, then

$$F = -cVT^d \quad (5.3)$$

which follows from extensivity of F and dimensional analysis. This is the familiar Stefan-Boltzmann behavior for blackbody radiation. The pressure is $P = -\partial_V F$. Note that in a relativistic theory, just putting it at finite temperature is enough to cause stuff to be present, because of the existence of antiparticles. It is the physics of this collection of CFT stuff that we would like to understand.

The prefactor c in (5.3) is a measure the number of degrees-of-freedom-per-site, that is, the number of species of fields in the CFT, which we called “ N^2 ” above.

5.2. Back to the Gravity Dual

The desired object goes by many names, such as planar black hole, Poincaré black hole, and black brane. . . . Let us just call it a black hole in AdS_{d+1} . The metric is

$$ds^2 = \frac{L^2}{z^2} \left(-f dt^2 + d\vec{x}^2 + \frac{dz^2}{f} \right), \quad (5.4)$$

$$f = 1 - \frac{z^d}{z_H^d}.$$

We again put the \vec{x} coordinates on a finite volume space, for example, in box of volume V , $x \sim x + V^{1/d}$. Notice that if we set the emblackening factor $f = 1$, we recover the Poincaré AdS metric, and in fact f approaches 1 as $z \rightarrow 0$, demonstrating that this is an IR deformation.

This metric solves Einstein’s equations with a cosmological constant $\Lambda = -d(d-1)/2L^2$ and asymptotes to Poincaré AdS. It has a horizon at $z = z_H$, where the emblackening factor f vanishes linearly. This means that events at $z > z_H$ cannot influence the boundary near $z = 0$.

The fact that the horizon is actually translation invariant (i.e., it is a copy of \mathbb{R}^{d-1} , rather than a sphere) leads some people to call this a “black brane”.

In general, horizons describe thermally mixed states. Here we can see the connection more directly: this solution (5.4) is the extremum of the Euclidean gravity action dual to the QFT path integral with thermal boundary conditions. Recall that the boundary behavior of the bulk metric acts as a source for the boundary stress tensor—changing the boundary behavior of the bulk metric is the same as changing the metric of the space on which we put the boundary theory. This is to say that if

$$ds_{\text{bulk}}^2 \stackrel{z \rightarrow 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g_{\mu\nu}^{(0)} dx^\mu dx^\nu, \quad (5.5)$$

then, up to a factor, the boundary metric is $g_{\mu\nu}^{(0)}$. This includes any periodic identifications we might make on the geometry, such as making the Euclidean time periodic.

A study of the geometry near the horizon will give a relationship between the temperature and the parameter z_H appearing in the metric (5.4). The near-horizon ($z \sim z_H$) metric is

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2, \quad (5.6)$$

where $\rho^2 = (2/\kappa)(L/z_H^2)(z_H - z) + \mathcal{O}(z_H - z)^2$, and $\kappa \equiv |f'(z_H)|/2 = d/2z_H$ is called the “surface gravity”. If we analytically continue this geometry to Euclidean time, $t \rightarrow i\tau$, it becomes

$$ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad (5.7)$$

which looks like \mathbb{R}^{d-1} times a Euclidean plane in polar coordinates $\rho, \kappa\tau$. There is a deficit angle in this plane unless τ is periodic according to

$$\kappa\tau \simeq \kappa\tau + 2\pi. \quad (5.8)$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral. Therefore we identify the temperature as $T = \kappa/(2\pi) = d/(4\pi z_H)$.

Meanwhile the area of the horizon (the set of points with $z = z_H, t$ fixed) is

$$A = \int_{z=z_H, \text{fixed } t} \sqrt{g} d^{d-1}x = \left(\frac{L}{z_H}\right)^{d-1} V. \quad (5.9)$$

Therefore the entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N} \frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi} (\pi T)^{d-1} V = \frac{\pi^2}{2} N^2 V T^{d-1}. \quad (5.10)$$

Here I have used the relation $L^{d-1}/4G_N = N^2/2\pi$. (The factor of 2π is the correct factor for the case of the duality involving the $\mathcal{N} = 4$ SYM, and so this factor of N^2 is not red.) The entropy *density* is

$$s_{\text{BH}} = \frac{S_{\text{BH}}}{V} = \frac{a_{\text{BH}}}{4G_N}, \quad (5.11)$$

where $a_{\text{BH}} \equiv A/V$ is the “area density” (area per unit volume!) of the black hole.

We would like to identify this entropy with that of the CFT at finite temperature. The power of T is certainly consistent with the constraints of scale invariance. It is not a coincidence that the number of degrees-of-freedom-per-site L^{d-1}/G_N appears in the prefactor. In the concrete example of $\mathcal{N} = 4$ SYM in $d = 3 + 1$, we can compute this prefactor in the weak-coupling limit $\lambda \rightarrow 0$ and the answer is [47, 48]

$$F(\lambda = 0) = \frac{3}{4} F(\lambda = \infty); \quad (5.12)$$

the effect of strong coupling is to reduce the effective number of degrees-of-freedom-per-site by an order-one factor. Similar behavior is also seen in lattice simulations of QCD [49].

To support the claim that this metric describes the saddle point of the partition sum of a CFT in thermal equilibrium, consider again the partition function:

$$Z_{\text{CFT}} \equiv e^{-\beta F} = e^{-S_g[\underline{g}]}, \quad (5.13)$$

where \underline{g} is the Euclidean saddle-point metric.²⁷ S_g is the on-shell gravity action for the black hole solution and is equal to

$$S_g = S_{\text{EH}} + S_{\text{GH}} + S_{ct} \quad (5.14)$$

$$S_{\text{EH}} = -\frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{L^2} \right)$$

is just the usual bulk gravity action. In addition to this there are two boundary terms. The ‘‘Gibbons-Hawking’’ term is an extrinsic boundary term which affects which boundary conditions we impose on the metric, in the same way that the $\int_{\partial\text{AdS}} \phi n \cdot \partial\phi$ term in the scalar case changed the scalar boundary conditions from Neumann to Dirichlet. Its role is to guarantee that when we impose Dirichlet boundary conditions on the metric by specifying $g_{\mu\nu}^{(0)}$, the action evaluated on a solution is stationary under an arbitrary variation of the metric satisfying that boundary condition.²⁸ Its specific form is:

$$S_{\text{GH}} = -2 \int_{\partial M} d^d x \sqrt{\gamma} \Theta, \quad (5.15)$$

where Θ is the extrinsic curvature of the boundary

$$\Theta \equiv \gamma^{\mu\nu} \nabla_\mu n_\nu = \frac{n^z}{2} \gamma^{\mu\nu} \partial_z \gamma_{\mu\nu}, \quad (5.16)$$

where n^A is an outward-pointing unit normal to the boundary $z = \epsilon$, and we have defined γ by

$$ds^2 \stackrel{z \rightarrow 0}{\approx} L^2 \frac{dz^2}{z^2} + \gamma_{\mu\nu} dx^\mu dx^\nu. \quad (5.17)$$

Finally,

$$S_{ct} = \int_{\partial M} d^d x \sqrt{\gamma} \frac{2(d-1)}{L} + \dots \quad (5.18)$$

is a local, intrinsic boundary counter-term, as we need to subtract some divergences as $z \rightarrow 0$, just like in the calculation of vacuum correlation functions of local operators. The \dots are terms proportional to the intrinsic curvature of the boundary metric $g_{\mu\nu}^{(0)}$, which we have taken to be flat. See [50–52] for more details.

Thus by plugging in the AdS planar black hole solution (the saddle point) we obtain the free energy. Specializing to $d = 3 + 1$ and using the $\mathcal{N} = 4$ SYM normalization for the rest of this section, we find

$$-\frac{F}{V} = \frac{L^2}{16\pi G_N} \frac{1}{z_H^4} = \frac{\pi^2}{8} N^2 T^4. \quad (5.19)$$

This is consistent with the Bekenstein-Hawking entropy calculation above:

$$S_{\text{BH}} = -\partial_T F. \quad (5.20)$$

We can also check the first law of thermodynamics:

$$dE + PdV = T_{\text{BH}} dS_{\text{BH}} + \Omega dJ + \Phi dQ, \quad (5.21)$$

where the pressure is $P = -\partial_V F$. This is actually a strong check (at least on the correctness of our numerical factors), because it relates horizon quantities such as T, S to global quantities such as the free energy F . The red terms on the RHS of (5.21) would be present if we were studying a black hole with angular momentum J , or with charge Q as in the next subsection.

We can also compute the expectation value of the field theory stress tensor $T^{\mu\nu}$. We just use the usual GKPW prescription, using the fact that $T^{\mu\nu}$ couples to the induced metric on the boundary $\gamma_{\mu\nu}$. The energy is $E = -V\sqrt{\gamma}T_t^t$ and the pressure is $P = V\sqrt{\gamma}T_x^x$. The answers agree with the form we found from thermodynamics:

$$\left(\frac{E}{V}\right) = \frac{N^2\pi^2}{8} \frac{3}{2z_H^4}, \quad P = \frac{N^2\pi^2}{8} \frac{1}{2z_H^4}, \quad (5.22)$$

which satisfies $E = 3P$, so $T_\mu^\mu = 0$ as required in a $3 + 1$ -dimensional relativistic CFT. One can do the same for the black hole with a spherical horizon (in ‘‘global coordinates’’) which describes the field theory on a sphere. The answer is $E = (N^2\pi^2/8)(3/(2z_H^4) + 3L^2/(8G))$. The limit $z_H \rightarrow \infty$ describes the zero-temperature vacuum. In this limit E is nonzero, and in fact matches the calculation for the zero-point (Casimir) energy of the field theory on a sphere of radius L . The fact that this quantity matches precisely the field theory result at weak coupling [50–52] (unlike, e.g., the free energy which depends on the ’t Hooft coupling) is because it arises from an anomaly.

5.3. Finite Density

Suppose that the boundary theory has a conserved $U(1)$ symmetry, with current J^μ . This means that there should be a massless gauge field A_μ in the bulk. Naturalness suggests the action

$$\Delta S_{\text{bulk}} = -\frac{1}{4g_F^2} \int d^{d+1}x \sqrt{g} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (5.23)$$

The Maxwell equation near the boundary implies the behavior

$$A \stackrel{z \rightarrow 0}{\approx} A^{(0)}(x) + z^{d-2} A^{(1)}(x), \quad (5.24)$$

and in particular for the zeromode of the time component, $A_t \stackrel{z \rightarrow 0}{\approx} \mu + z^{d-2} \rho$. Using the result (4.50), this is the statement that the charge density ρ of the boundary theory is related to the electric field in the bulk:

$$\Pi_{A_t} = \frac{\partial \mathcal{L}}{\partial (\partial_z A_t)} = E_z = A^{(1)} = \rho; \quad (5.25)$$

the subleading behavior encodes the field momentum.

A black hole which describes the equilibrium configuration of the field theory at finite density has the same form of the metric as above:

$$ds^2 = \frac{L^2}{z^2} \left(-f dt^2 + d\vec{x}^2 + \frac{dz^2}{f} \right), \quad (5.26)$$

but with a different emblackening factor $f = 1 - Mz^d + Qz^{2d-2}$, and a nonzero gauge field

$$A = dt \left(\mu + \rho z^{d-2} \right). \quad (5.27)$$

M, Q, ρ can be written in terms of g_F, μ, T ; see, for example, [2].

I will restrict myself to two comments about this solution. First, in the grand canonical ensemble, μ is fixed, and we should think of the z^0 term in A_t as the source, and the coefficient of z^{d-2} , namely, ρ as the response. Changing the boundary conditions on the gauge field can be accomplished as described in Section 4.7, by adding a term of the form $\int_{\partial \text{AdS}} n_\mu A_\nu F^{\mu\nu}$. In this case, the trick *is* the Legendre transformation which takes us to the canonical ensemble, where the ρ -term is the source and μ is the response.

Secondly, this geometry is interesting. At $T \ll \mu$ it describes an example of a ‘‘holographic RG flow’’ between RG fixed points. In particular, the function f now has multiple zeros at real, positive values of z . Let us call the one closest to the boundary z_0 ; this represents a horizon; there is another zero z_1 at a larger value of z , in the inaccessible region behind the horizon. These zeros collide $z_1 \rightarrow z_0$ when we take $T \rightarrow 0$, in which case f has a double-zero, and the black hole is said to be extremal. In this limit, the geometry interpolates between AdS_{d+1} near the boundary $z \sim 0$, and $\text{AdS}_2 \times \mathbb{R}^{d-1}$ near the horizon. The presence of an AdS factor means that this IR region of the geometry is scale invariant. The original scale invariance of the d -dimensional CFT is broken by the chemical potential; so this is an *emergent* scale invariance, like the kind that one might see in a real system. The IR geometry is dual to some nonrelativistic scale-invariant theory. At finite $T \ll \mu$, the IR geometry is a black hole in $\text{AdS}_2 \times \mathbb{R}^{d-1}$, related to this CFT at finite temperature.

If we put a charged boson in this geometry, we find the holographic superconductor phenomenon ([53–55], for a review see [56]). If we put a charged spinor in this geometry, we can study the two-point functions of the dual fermionic operator, and in particular its spectral density $\sim \text{Im } G^R$. This study was initiated in [57] and developed by [58–60], and one



Figure 18: The geometry of the near-extremal AdS_{d+1} charged black hole for $T \ll \mu$.

finds Fermi surfaces, which sometimes describe nonFermi liquids. For some values of the bulk parameters, the fermion spectral density is that of a “marginal Fermi liquid” introduced [61] to model the strange metal phase of high- T_c superconductors [59].

6. Hydrodynamics and Response Functions

So far we have given evidence that the black hole thermodynamics of the AdS black hole solution is dual to the thermal ensemble of some strongly coupled CFT on \mathbb{R}^{d-1} at large N and large 't Hooft coupling λ . Thus the gauge theory provides the microstates that are being coarse grained by the Bekenstein-Hawking entropy of the black hole S_{BH} . The static black hole describes the field theory in thermodynamic equilibrium.

A few comments on this observation are as follows.

- (i) Perturbing the equilibrium of the *boundary* theory with a kick will result in thermalization—relaxation back to equilibrium.
- (ii) In the *bulk* the response to such a kick is for the energy of the kick to fall into the black hole.

The above two statements are related by the duality [62]. In the long wavelength and small frequency limit both are consistent with the hydrodynamics of a relativistic CFT. Additionally the duality allows one to compute various transport coefficients of the gauge theory at large λ , such as the shear viscosity, and conductivity as we discuss next. We emphasize that these calculations are done at leading order in this mean-field-theory-like description, but they include dissipation.

6.1. Linear Response and Transport Coefficients

As a demonstration of the real-time prescription [30], in the following subsection we will derive the viscosity of a large class of strongly interacting plasmas, made from CFT at finite temperature. First, we quickly recall the formalism of linear response to establish notation.

To see how our system responds to our poking at it, consider the following small perturbation of the field theory action:

$$\Delta S_{\text{QFT}} = \int d^d x \mathcal{O} \phi_0, \quad \phi_0 \text{ small.} \quad (6.1)$$

The response is

$$\delta \langle \mathcal{O} \rangle_{\text{CFT}, T} \stackrel{\phi_0 \rightarrow 0}{=} -G^R(\omega, \mathbf{k}) \phi_0(\omega, \mathbf{k}). \quad (6.2)$$

For simplicity, we are just asking about the diagonal response, perturbing with the operator \mathcal{O} , and measuring the operator \mathcal{O} : $G^R \equiv G_{\mathcal{O}\mathcal{O}}^R$. The subscript on the LHS indicates that we are computing thermal averages at temperature T . In the long-wavelength, low-frequency limit, on very general grounds, this will reduce to the Kubo formula:

$$\delta\langle\mathcal{O}\rangle_{\text{CFT},T} \xrightarrow{k\rightarrow 0, \omega\rightarrow 0} i\omega\chi\phi_0. \quad (6.3)$$

For example, in the case where $\mathcal{O} = j^\mu$ is a conserved current, $\phi_0 = A_\mu$ is the boundary behavior of a bulk gauge field, and the transport coefficient is the conductivity:

$$\delta\langle\vec{J}\rangle_{\text{CFT},T} \xrightarrow{k\rightarrow 0, \omega\rightarrow 0} i\omega\chi(j)\vec{A} = \sigma\vec{E} \text{ (Ohm's law)}. \quad (6.4)$$

In the case where $\mathcal{O} = T_y^x(k_z)$, the source is the boundary value of a metric perturbation δg_x^y , and the transport coefficient is the shear viscosity, $\chi(T_y^x) = \eta$. Do not forget that the order of limits here matters: k must be taken to zero before ω to get the transport coefficient.

6.2. Holographic Calculation of Transport Coefficients

Now we discuss the bulk calculation of these quantities. We will follow the discussion of [40], but we should emphasize that this calculation has a long history [63–66]. We can consider a very general bulk metric:

$$ds^2 = g_{tt}dt^2 + g_{zz}dz^2 + g_{ij}dx^i dx^j, \quad (6.5)$$

which satisfies two conditions.

- (i) Near $z \rightarrow 0$, it approaches AdS, or some other asymptotics for which we know how to construct a holographic GKPW-like prescription. Other examples include systems which in the UV are described by nonrelativistic CFTs, as described in [16–18].
- (ii) The geometry has a horizon at some $z = z_H$, near which the metric coefficients take the form $g_{tt} \approx 2\kappa(z_H - z)$, $g_{zz} \approx 1/2\kappa(z_H - z)$. Such a thing is called a Rindler horizon, and means that the Euclidean time coordinate must have period $1/T = 2\pi/\kappa$. T is therefore the temperature at which we have put the field theory.

In this spacetime, consider the bulk action:

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{g} \frac{\partial_A \phi \partial^A \phi}{q(z)}. \quad (6.6)$$

Several points are noteworthy here. First, the quantity $q(z)$ is some effective coupling of the mode, which can depend on the radial direction; such dependence can arise, for example from the profiles of some background fields to which the mode is coupled. In addition, we have assumed that the field ϕ is massless and does not mix with other modes (at least in the kinematical regime of interest); this will be important for the calculation below. An important

example of such a ϕ is precisely the metric fluctuation $\delta g_x^y(k_z)$ which computes the shear viscosity. In the case of Einstein gravity, the coefficient is $1/q_{\text{Einstein}} = 1/16\pi G_N$. For other theories of gravity with higher-curvature corrections, q can take some other forms [40]. We will continue to leave q general and refer to the operator to which ϕ couples as \mathcal{O} .

Recall that the general formula for the expectation value of an operator in terms of the behavior of its dual bulk field is

$$\langle \mathcal{O}(x^\mu) \rangle_{\text{QFT}} = \lim_{z \rightarrow 0} \Pi_\phi(z, x^\mu), \quad (6.7)$$

where $\Pi_\phi \equiv \partial \mathcal{L}_{\text{bulk}} / \partial(\partial_z \phi)$ is the field momentum (with the radial direction thought of as time). We have specialized the formula (4.50) to the case of a massless field. The fact that (6.7), evaluated on the infalling solution for ϕ , correctly computes the retarded response was demonstrated in [41]. Equation (6.7) means that the transport coefficient is

$$\chi = \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \lim_{z \rightarrow 0} \left(\frac{\Pi_\phi(z, k_\mu)}{i\omega \phi(z, k_\mu)} \right). \quad (6.8)$$

We will calculate this in two steps. First we find its value at the black hole horizon, and then we propagate it to the boundary using the equation of motion.

By assumption (2) about the metric, the horizon at $z = z_H$ is a regular singular point of the wave equation, near which solutions behave as

$$\phi(z) \approx (z_H - z)^{\pm i\omega/4\pi T} \quad (6.9)$$

(the exponents are determined by plugging in a power-law ansatz into the wave equation and Taylor expanding). Since the time-dependence of the solution is of the form $e^{-i\omega t}$ (recall (4.16)), these two solutions describe waves which fall into (–) or come out of (+) the black hole horizon as time passes. To compute the retarded Green's function, we pick the ingoing solution [30]. This says that near the horizon, the field momentum is

$$\Pi_\phi(z, k) = \frac{\sqrt{|g|}}{q(z)} g^{zz} \partial_z \phi \Big|_{z \rightarrow z_H} \approx \frac{1}{q(z_H)} \sqrt{\left| \frac{g}{g_{zz} g_{tt}} \right|} i\omega \phi(z, k) \Big|_{z=z_H}. \quad (6.10)$$

The outgoing solution would give a minus sign in front here.

To propagate this to the boundary, we use the bulk equations of motion, which relate $\partial_z \Pi_\phi \sim \partial_z^2 \phi$ to ϕ . It is not hard to show that in the limit $k \rightarrow 0, \omega \rightarrow 0$,

$$\partial_z \Pi_\phi(z, k \rightarrow 0, \omega \rightarrow 0) = 0. \quad (6.11)$$

A similar statement holds for the denominator of Green's function, $\omega \phi$. This means that

$$\frac{\Pi}{\omega \phi} \Big|_{z=0} = \frac{\Pi}{\omega \phi} \Big|_{z=z_H}, \quad (6.12)$$

from which we learn that

$$\chi = \frac{1}{q(z_H)} \sqrt{\left| \frac{g}{g_{zz}g_{tt}} \right|}. \quad (6.13)$$

Here it was important that the bulk field was massless; this fails, for example, for the mode which computes the bulk viscosity.

Let us apply this discussion to the case where χ is the shear viscosity η , defined in the previous subsection. The shear viscosity is dimensionful (it comes in some units called “poise”); a dimensionless measure of the quality of a liquid is its ratio with the entropy density, which is also something we know how to compute. The entropy density of our system is related to the “area-density” a_{BH} of the black hole:

$$s = \frac{a_{\text{BH}}}{4G_N} = \frac{1}{4G_N} \sqrt{\left| \frac{g}{g_{zz}g_{tt}} \right|}. \quad (6.14)$$

We see therefore that

$$\frac{\eta}{s} = \frac{1}{q(z_H)}. \quad (6.15)$$

In the special case of Einstein gravity in the bulk this gives the celebrated KSS value [65]:

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (6.16)$$

This value is much smaller than that of common liquids. The substances which come the closest [67] are cold atoms at unitarity ($\eta/s \sim 1/2$ [68, 69]) and the fireball at RHIC ($\eta/s \sim 0.16$ [70, 71]). This computation of the shear viscosity of a strongly interacting plasma seems to have been quite valuable to people trying to interpret the experiments at RHIC.²⁹

It seems not to be a lower bound. For example, in some particular higher-curvature gravity theory called Gauss-Bonnet gravity, where the black-hole solution is known, the parameter q is related to a coefficient of a higher-derivative term λ_{GB} , and one finds [72, 73]

$$\frac{\eta}{s} = \frac{1}{q(z_H)} = \frac{1}{4\pi} > \frac{16}{25}. \quad (6.17)$$

The inequality on the right arises from demanding causality of the boundary theory, which fails if λ_{GB} is too large. It is not clear that GB gravity has a sensible UV completion, but other theories where the KSS value is violated by a small amount do [74].

A nice thing about the ratio η/s is that the number of degrees of freedom (N^2) cancels out. Attempts to find other such observables, for example, related to charge transport, include [75].

For lack of time, I have spoken here only about the extreme long-wavelength limit of the response functions. The frequency and momentum dependence is also very revealing; see, for example, [76, 77].

7. Concluding Remarks

7.1. Remarks on Other Observables

Besides correlation functions and thermodynamic potentials, a number of other observables of interest can be computed easily using the correspondence. Some, like expectation values of Wilson loops, are relatively specific to gauge theories.

A very ubiquitous observable, which is notoriously hard to compute otherwise, is the entanglement entropy. If we divide the hilbert space of the QFT into

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \quad (7.1)$$

and declare ourselves ignorant of \mathcal{H}_B , we have an ensemble described by the density matrix:

$$\rho_A = \text{tr}_{\mathcal{H}_B} \rho, \quad (7.2)$$

where ρ is the density matrix of the whole system, which let's take to be the one associated with the ground state $\rho = |\Omega\rangle\langle\Omega|$. Then the entanglement entropy between A and B is the von Neumann entropy of ρ_A :

$$S = -\text{tr}_A \rho_A \ln \rho_A. \quad (7.3)$$

In the special case where the subdivision (7.1) is done by cutting out a region of space B bounded by some $d - 2$ -dimensional surface Σ , there exists a proposal for how to calculate the associated entanglement entropy S_Σ using the holographic dual [78] (for a recent review, see [79]). The idea is extremely simple: just find the surface M ($d - 1$ -dimensional), ending on the boundary at Σ , which extremizes its area. The formula for S_Σ is then

$$S_\Sigma = \text{extremum}_{\partial M = \Sigma} \frac{\text{area}(M)}{4G_N}, \quad (7.4)$$

very reminiscent of the Bekenstein formula for the entropy of a black hole. This formula passes many checks. For example, it gives the correct universal behavior $S \sim (c/3) \ln L$ (L is the length of the region B) in a $1 + 1$ -dimensional CFT of central charge c . Like the Casimir energy, this match between weak and strong coupling is precise because it is determined by the conformal anomaly. In higher dimensions, the holographic prescription gives a prediction for which terms in the expansion of $S(L)$ in powers of L are universal. There even exists a heuristic derivation [80].³⁰

7.2. Remarks on the Role of Supersymmetry

Supersymmetry has played important roles in the historical development of the AdS/CFT correspondence.

- (i) It constrains the form of the interactions, meaning that there are fewer possible candidates for the dual (e.g., the maximal AdS supergravity theory in five dimensions is unique).

- (ii) A supersymmetric theory has many observables which are independent of the coupling. These so-called BPS quantities allow for many quantitative checks of a proposed dual pair.
- (iii) Supersymmetry can stabilize a *line* of exact fixed points (e.g., in the $\mathcal{N} = 4$ SYM), rendering the coupling constant a dimensionless parameter which interpolates between the weakly coupled field theory description and the gravity regime.

However, it has played no role in our discussion. Some people believe that supersymmetry may be necessary for the construction of a consistent theory of quantum gravity. But it seems more likely to me that the formulation of specific examples of the duality without supersymmetry is a (perhaps hard) technical problem, not one of principle.

7.3. Lessons for How to Use AdS/CFT

Critical exponents depend on “Landscape Issues”. By this I mean just that they depend on the values of the couplings in the bulk action (in particular, the masses of bulk fields), which are specified only by some UV completion, that is, by available string vacua. For each possible bulk coupling, it is very much an open question which values arise in a consistent theory of quantum gravity. This situation—that the critical exponents are UV-sensitive quantities—is a rather unfamiliar one!

At least in examples we know, thermodynamics is not very sensitive to strong coupling. In both the $\mathcal{N} = 4$ SYM, and from lattice QCD, we find for, for example, the free energy a relation of the form

$$F_{\text{strong}} \sim \frac{3}{4} F_{\text{weak}}. \quad (7.5)$$

Real-time dynamics and transport *are* very sensitive to the strength of the interactions. For example,

$$\left(\frac{\eta}{s}\right)_{\text{weak}} \sim \frac{1}{g^4 \ln g} \gg \left(\frac{\eta}{s}\right)_{\text{strong}} \sim \frac{1}{4\pi}. \quad (7.6)$$

Not only are these observables sensitive to strong coupling, but they are very natural things to compute using the holographic technology. In particular, although it is a classical description, it automatically includes dissipation. Ordinary techniques seem to require the existence of a description of what is being transported in terms of quasiparticles, so that the Boltzmann equation can be used. Since we know that such a description need not exist, this is a very good opportunity for the machinery described above to be useful.

I would like to close with a final optimistic philosophical comment. The following gedanken experiment was proposed by Weisskopf [81]: take a bunch of theoretical physicists and lock them away from birth so that they are never exposed to any substance in the liquid phase. Will they predict the existence of the liquid phase from first principles? Weisskopf thinks not, because its existence depends on the fact that the constituents interact strongly with each other. The same statement applies to any state of matter which depends for its existence on strong interactions, such as confinement, and fractional quantum hall phases.^{31, 32}


I think it is a defensible claim [82] that if we did not know the IR physics of QCD (e.g., if we did not happen to be made out of color-neutral boundstates of quarks and gluons) before the discovery of AdS/CFT, we would have predicted color confinement by finding its dual geometry [19, 45, 46]. Our ability to imagine the possible behavior of a bunch of stuff has been limited by our dependence on our weak coupling tools and on experimenters to actually assemble the stuff. It is exciting that we now have another tool, which allows us to ask these questions in a way which involves such simple geometrical pictures. Perhaps there are even states of matter that we can describe this way which have already been seen, but which have not yet been understood.

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Endnotes

1. Matthew Fisher raises the point that there are systems (ones with topological order) where it is possible to create an information-carrying excitation which does not change the energy. I am not sure exactly how to defend Bekenstein's argument from this. I think that an important point must be that the effects of such excitations are not completely local (which is why they would be good for quantum computing). A related issue about which more work has been done is the *species problem*: if there are many species of fields in the bulk, information can be carried by the species label, without any cost in energy. There are two-points which save Bekenstein from this: (1) if there are a large number of species of fields, their fluctuations renormalize the Newton constant (to make gravity weaker) and weaken the bound. (2) Being able to encode information in the species label implies that there is some continuous global symmetry. It is believed that theories of quantum gravity do not have continuous global symmetries (roughly because virtual black holes can eat the charge and therefore generate symmetry-breaking operators in the effective action, see, e.g., [84, page 12]).
2. The criterion "different" may require some effort to check. This principle is sometimes also called "Conservation of Difficulty".

3. It turns out that this metric also has conformal invariance. So scale and Poincaré symmetry implies conformal invariance, at least when there is a gravity dual. This is believed to be true more generally [85], but there is no proof for $d > 1 + 1$. Without Poincaré invariance, scale invariance definitely does *not* imply conformal invariance; indeed there are scale-invariant metrics without Poincaré symmetry, which do not have special conformal symmetry [16].
4. For verifying statements like this, it can be helpful to use Mathematica or some such thing.
5. An example of this is the relationship (2.12) between the Newton constant in the bulk and the *number* of species in the field theory, which we will find in the next subsection.
6. Rational CFTs in two dimensions do not count because they fail our other criterion for a simple gravity dual: in the case of a 2d CFT, the central charge of the Virasoro algebra, c , is a good measure of “ N^2 ”, the number of degrees of freedom per point. But rational CFTs have c of order unity and therefore can only be dual to very quantum mechanical theories of gravity. But this is the right idea. Joe Polchinski has referred to the general strategy being applied here as “the Bootstrap for condensed matter physics”. The connection with the bootstrap in its CFT incarnation [86] is made quite direct in [14].
7. Eva Silverstein and Shamit Kachru have emphasized that this special property of these field theories is a version of the “cosmological constant problem;” that is, it is dual to the specialness of having a small cosmological constant in the bulk. At least in the absence of supersymmetry, there is some tuning that needs to be done in the landscape of string vacua to choose these vacua with a small vacuum energy and hence a large AdS radius. Here is a joke about this: when experimentalists look at some material and see lots of complicated crossovers, they will tend to throw it away; if they see instead some simple beautiful power laws, as would happen in a system with few low-dimension operators, they will keep it. Perhaps these selection effects are dual to each other.
8. Note that I am not saying here that the configuration of the elementary fields in the path integral necessarily has some simple description at the saddle point. Thanks to Larry Yaffe for emphasizing this point.
9. The standard pedagogical source for this material is [87], available from the KEK KISS server.
10. Recently, there has been an explosion of literature on a case where the number of degrees of freedom per point should go like $N^{3/2}$ [88, 89].
11. Note that the important distinction between these models and those of the previous subsection is not the difference in groups ($U(N)$ versus $O(N)$), but rather the difference in representation in which the fields transform: here the fields transform in the adjoint representation rather than the fundamental.
12. Had we been considering $SU(N)$, the result would be $\langle \Phi_b^a \Phi_c^d \rangle \propto \delta_c^a \delta_b^d - \delta_b^c \delta_a^d / N^2 =$


. This difference can be ignored at leading order in the $1/N$ expansion.
13. Please do not be confused by multiple uses of the word “vertex”. There are interaction vertices of various kinds in the Feynman diagrams and these correspond to vertices in the triangulation only in the first formulation.

14. The following two paragraphs may be skipped by the reader who does not want to hear about string theory.
15. From the point of view of the worldsheet, these operators create closed-string excitations, such as the graviton.
16. It is clear that the 't Hooft limit is not the only way to achieve such a situation, but I am using the language specific to it because it is the one I understand.
17. *Monovacuiist* (n): one who believes that a theory of quantum gravity should have a unique groundstate (in spite of the fact that we know many examples of much simpler systems which have many groundstates, and in spite of all the evidence to the contrary (e.g., [90, 91])).
18. We will use an underline to denote fields which solve the equations of motion.
19. This $z \rightarrow \infty$ far IR region of the geometry is called the “Poincaré horizon”. A few words of clarification may be useful regarding this terminology. The form (2.5) of the AdS metric that we have been discussing is not geodesically complete. If we follow all the geodesics to their logical conclusions, the geometry we find is called “global AdS”; it has constant- z spatial sections which are $d - 1$ -spheres. The coordinates we have been using in (2.5) cover a subset of this geometry called the “Poincaré patch”. The Poincaré horizon at $z \rightarrow \infty$ represents a boundary of this coordinate patch.
20. This is not yet the complete prescription for computing retarded functions; the other ingredient will appear in Section 4.4.
21. If you do not like functional derivatives, you can see (4.38) by calculating

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2) \rangle_c^\epsilon = \left(\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} W[\phi_0(x) = \lambda_1 e^{ik_1 x} + \lambda_2 e^{ik_2 x}] \right) \Big|_{\lambda_1 = \lambda_2 = 0}.$$

22. The correctness of this normalization of the two-point function can be verified by computing a *three*-point function $\langle J_\mu \mathcal{O}^\dagger \mathcal{O} \rangle$ (where J is a conserved current under which \mathcal{O} is charged) and using the Ward identity for J -conservation to relate this to $\langle \mathcal{O}^\dagger \mathcal{O} \rangle$ [34].
23. This formula is not correct on the support of the source ϕ_0 . If one wants to use this formula to compute expectation values with a nonzero source (and not just correlation functions at finite separation), the terms proportional to the source must be included and renormalized. For cautionary examples and their correct treatment, please see [92, 93]. Thanks to Kostas Skenderis for emphasizing this point.
24. An important practical remark: in general, the bulk system will be nonlinear—a finite perturbation of ϕ_0 will source other bulk fields, such as the metric. Actually finding the resulting $\phi(z, x)$ in (4.47) may be quite complicated. The prescription for extracting the response, given that solution, is always as described here.
25. More precisely, as we saw in the previous subsection, it is better to use a regulated bulk-to-boundary propagator which approaches a delta function at the regulated boundary: $K_\epsilon^\Delta(\epsilon, x; x') = \epsilon^{\Delta-} \delta(x, x')$.
26. In this subsection, we work in units of the AdS radius, that is, $L = 1$.

27. If there were more than one saddle point geometry with the required asymptotics, we would need to sum over them. In fact there are examples where there are multiple saddle points which even have different topology in the bulk, which do exchange dominance as a function of temperature [45, 94]. In this example, this behavior matches a known phase transition in the dual gauge theory. This is therefore strong evidence that quantum gravity involves summing over topologies.
28. Without this term, integration by parts in the Einstein-Hilbert term to get the EOM produces some boundary terms proportional to variations of derivatives of the metric:

$$\delta S_{\text{EH}} = \text{EOM} + \int_{\partial\text{AdS}} \gamma^{\mu\nu} n \cdot \partial \delta \gamma_{\mu\nu},$$

which is incompatible with imposing a Dirichlet condition on the metric.

29. For reviews of applications of holographic duality to RHIC, please see [95, 96].
30. It should be noted that, at the moment at least, there is some confusion about the case of two disconnected regions; see [97].
31. Son points out that even some phenomena that only involve weak coupling, such as the BCS mechanism, took a long time to figure out, even after the relevant experiments.
32. I would like to have more things to add to this list. What am I forgetting? Is our ignorance that complete?

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