LECTURE 18

Last time:

- Gaussian Channel Capacity as Sphere Packing
- Parallel Gaussian Channels

Lecture outline

- Waveform Channels
- Wide-band limit
- Colored Gaussian Noise

Review

- Gaussian Channel:

$$Y = X + W$$

$E[X^2] \leq P, E[W^2] = N$

$$C = \frac{1}{2} \log(1 + \frac{P}{N})$$

- Parallel Gaussian Channels

$$Y_i = X_i + W_i$$

$\sum E[X_i^2] = P,\ E[W_i^2] = N_i, i = 1, \ldots, k.$

water-filling.

Waveform Channel: White Gaussian Noise

$$Y(t) = X(t) + W(t)$$

- $W(t)$ is a Gaussian random process. Any set of samples are i.i.d. Gaussian distributed with zero mean and variance $E[W(t)^2] = \frac{N_0}{2}$.
- Autocorrelation function $R(\tau) = E[W(t)W(t-\tau)] = \frac{N_0}{2}\delta(\tau)$.
- Power Spectral Density is flat $\frac{N_0}{2}$ over all frequency
- Do we really have white Gaussian noise?

Front end of the receiver

Goal: provide sufficient statistics for bandlimited transmitted signals.

Assume the $X(t)$ has non-zero frequency components only in $[-W, W]$, can sample at $T_s = \frac{1}{2W}$.

- can be thought as a bank of matched filter matched to

$$\text{sinc}\left(t - \frac{n}{2W}\right)$$

where

$$\text{sinc}(t) = \frac{\sin(2\pi W t)}{2\pi W t}$$

- The front end provide sufficient statistics since
  - the out-of-band noise is independent of the transmitted signals
  - the out-of-band noise is independent of the in-band noise

Discrete-Time Channel Model

- $2W$ samples per second.
- The noise power in each sample is $\frac{N_0}{2}$.
- Average per symbol signal power $P/2W$.

We get back to

$$Y_i = X_i + W_i$$

The overall capacity in (bits/sec)

$$C = 2W\frac{1}{2}\log\left(1 + \frac{P/2W}{N_0/2}\right)$$

$$\boxed{C = W\log\left(1 + \frac{P}{N_0 W}\right)}$$

P has unit (watt), and N_0 (watt/hz).

Admiring Shannon's Formula

$$\boxed{C = W \log\left(1 + \frac{P}{N_0 W}\right)}$$

- **Question 1** Do we always want more bandwidth?
- **Question 2** What can we do with infinite bandwidth?

  $$C \to \frac{P}{N_0} \log_2 e (bits/sec)$$

  Capacity grows linearly with the power (very power efficient).
- The energy required to send 1 bit is $E_b = P/C$ watts $\times 1$ sec,

  $$\frac{E_b}{N_0} = \frac{1}{\log_2 e} = -1.59(dB)$$

- **Question 3** What can we do with infinite power?

  $$\lim_{W \to \infty} \frac{C}{W} \to \log \frac{P}{N_0}$$

  capacity linearly increases with W (spectral efficient)

Discussions

- Waveform channels are simply $2WT$ parallel channels.
- White Gaussian noise allows us to look at the signal space with any chosen basis.
- Differential entropy depends on the coordinate system, but mutual information does not.

$$I(X;Y) = h(Y) - h(Y|X)$$

Key: look at Y and $Y|X$ from the same coordinate system.

Gaussian Channel with Colored Noise

$$Y_i = X_i + W_i, i = 1, \ldots, m$$

where W_i's are not i.i.d., but have a covariance matrix

$$E[\underline{WW}^\dagger] = K_W$$

- Channel with memory, can only treat a block as a super-symbol

$$\underline{Y} = \underline{X} + \underline{W}$$

- Now we do not have the liberty to choose any coordinate system, since otherwise the noise components are correlated.
- Eigen value decomposition

$$K_W = U\Lambda U^\dagger$$

- $\underline{W}$ can be viewed as white Gaussian noise passed through a generating matrix

$$\underline{W} = U\sqrt{\Lambda}\underline{\tilde{W}}$$

  where $\underline{\tilde{W}} \sim N(0, I)$

Choosing the Right Angle

- Left multiply $\underline{Y}$ by U, does not lose any information

$$\begin{aligned} \underline{Y}' &= U\underline{Y} = U\underline{X} + \sqrt{\Lambda}\underline{\tilde{W}} \\ &= \underline{X}' + \sqrt{\Lambda}\underline{\tilde{W}} \end{aligned}$$

Now the elements of the noise vector become independent, and we get back to the parallel Gaussian channel case.

- Water-filling among all the eigen modes.
- $h(\underline{Y})$ and $h(\underline{Y}|\underline{X})$ are computed in the rotated coordinate systems.

K-L Expansion

Theorem K-L expansion:
For a random process with autocorrelation function $R_X(t,s) = E[X(t)X(s)]$. Assume that $R_X(t,s)$ is symmetric, i.e.,

$$R_X(t,s) = R_X(s,t)$$

and

$$\int\int R_X(t,s)dtds < \infty$$

then there exists a complete ortho-normal basis $\{\phi_i(t)\}$ which are the eigen functions of $R_X(t,s)$, i.e.,

$$\int R_X(t,s)\phi_i(s)ds = \lambda_i\phi_i(t)$$

Let

$$X_i = \langle X(t), \phi_i(t)\rangle$$

then

$$X(t) = \sum X_i\phi_i(t)$$

in L^2 norm sense. Furthermore, X_i's are uncorrelated.

$$
\begin{aligned}
E[X_i X_j] &= E\left[\int X(t)\phi_i(t)dt \int X(s)\phi_j(s)ds\right] \\
&= \int\int E[X(t)X(s)]\phi_i(t)\phi_j(s)dtds \\
&= \int \phi_i(t) \int R_X(t,s)\phi_j(s)dsdt \\
&= \lambda_j \int \phi_i(t)\phi_j(t)dt \\
&= \lambda_j \delta_{ij}
\end{aligned}
$$

Key we can find a new basis of the signal space $\{\phi_i(t)\}$ such that the components of $X(t)$ along different base vectors are uncorrelated. For the Gaussian random process, the components can be independent.

- compare to the vector case.

Gaussian WSS Random Process

Now consider the special case that the random process is WSS,

$$R_X(t,s) \rightarrow R_X(\tau)$$

Question what are the eigen functions?

- Any WSS Gaussian random process can be obtained by passing white Gaussian noise through linear time-invariant filters.
- Sinesoid is the eigen function of LTI systems, so it is also the eigen function of WSS random processes.
- infinite time horizon vs. finite time horizon.
- Different frequency components of a Gaussian WSS process are independent, but may have different variance, given by the power spectral density.
- Water-filling over different frequency components.