LECTURE 19

Last time:

- Waveform Channels
- Gaussian Channel with Colored Noise

Lecture outline

- Gaussian Channel with Feedback

Review

- Parallel Gaussian Channels

$$Y_i = X_i + W_i$$

$\sum E[X_i^2] = P$, $E[\underline{W W}^\dagger] = \Lambda$.

water-filling.

- Gaussian Channel with colored noise

$$\underline{Y} = \underline{X} + \underline{W}$$

with $E[\underline{W W}^\dagger] = K_W$.

water-filling over eigen-modes

- Waveform Gaussian channel with colored noise

$$Y(t) = X(t) + W(t)$$

where $W(t)$ is WSS Gaussian noise with $R_W(\tau)$.

Eigen-modes are the different frequency components.

Define Feedback Capacity

Consider

$$\underline{Y}^n = \underline{X}^n + \underline{W}^n$$

A data rate R is achievable if there exists $(2^{nR}, n)$ codes, consists of mapping

$$x_i(M, \underline{Y}^{i-1}), \quad M = 1, \ldots, 2^{nR}$$

such that $P_e^{(n)} \to 0$, subject to the power constraint.

- Information capacity

$$\max \frac{1}{n} I(M; \underline{Y}^n)$$

  maximize over all possible inputs with feedback.

Be careful

$$\frac{1}{n} I(\underline{X}^n; \underline{Y}^n) = \frac{1}{n}(h(\underline{X}^n) - h(\underline{X}^n | \underline{Y}^n))$$

set $\underline{X}^n = (0, W_1, W_2, \ldots, W_{n-1})$, the mutual information is infinity. Compare your notes for the discrete case.

Gaussian Channel with Feedback

- Recall for memoryless channels, feedback does not increase the capacity
- Gaussian channel with colored noise has memory, so does feedback increase the capacity?

Idea observing the noise at time 1 helps to estimate noise at time 2, and therefore reduces the power of the "unknown" part of the noise, this certainly helps to communicate at time 2.

——**Not Quite**

Even without feedback, the receiver can also estimate the noise at time 2, and subtract the estimate from Y_2. The capacity gain comes from the "color" instead of the feedback.

The Gain of Feedback

- Want to compare with colored noise channel (water-filling) without feedback.
- Just get rid of a part of the noise is not good enough.
- The key is "maximize over all possible inputs with feedback".

Consider

$$\begin{aligned}
& I(M;\underline{Y}^n) \\
= & \sum_{i=1}^{n} I(M;Y_i|\underline{Y}^{i-1}) \\
= & \sum [h(Y_i|\underline{Y}^{i-1}) - h(Y_i|\underline{Y}^{i-1}, M)] \\
= & \sum [h(Y_i|\underline{Y}^{i-1}) - h(Y_i|\underline{Y}^{i-1}, M, \underline{X}^i, \underline{W}^{i-1})] \\
= & \sum [h(Y_i|\underline{Y}^{i-1}) - h(W_i|\underline{Y}^{i-1}, M, \underline{X}^i, \underline{W}^{i-1})] \\
= & \sum [h(Y_i|\underline{Y}^{i-1}) - h(W_i|\underline{W}^{i-1})] \\
= & h(\underline{Y}^n) - h(\underline{W}^n)
\end{aligned}$$

Now we just need to find $\underline{X}^n$ to maximize $h(\underline{Y}^n)$.

Without Feedback

- Without loss of generality, we will only consider Gaussian inputs.
- $\underline{X}^n$ and $\underline{W}^n$ are independent, so

$$K_Y = K_X + K_W$$

- The capacity

$$C_n = \max \frac{1}{2n} \log \frac{|K_X + K_W|}{|K_W|}$$

where the maximization is over all K_X : $\frac{1}{n}\text{trace} K_X \le P$.

- Solution, water-filling in eigen-modes.

With Feedback

- $\underline{X}^n$ and $\underline{W}^n$ can be dependent

$$C_{n,FB} = \max \frac{1}{2n} \frac{|K_{X+W}|}{|K_W|}$$

where the maximization is over all K_X subject to the same power constraint as before, but $\underline{X}$ can be of the form

$$X_i = \sum_{j=1}^{i-1} b_{ij} W_j + V_i$$

$$K_Y = (B+I)K_W(B+I)^\dagger + K_V$$

where B is strictly lower triangular, $\underline{V}$ is independent of $\underline{W}$

- The case without feedback corresponding to $B = 0$.

- Instead of getting rid of the noise, the distribution of $\underline{X}$ is correlated with the noise, and use noise to increase $h(\underline{Y})$.

Example

Let $n = 2$,

$$K_W = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

With feedback

$$\begin{aligned} \underline{X} &= \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \underline{W} + \underline{V} \\ &= B\underline{W} + \underline{V} \end{aligned}$$

Power constraint

$$\text{trace}(BK_W B^\dagger + K_V) \leq 4$$

Goal: maximize

$$\det(K_Y) = \det((B + I)K_W(B + I)^\dagger + K_V)$$

Questions:

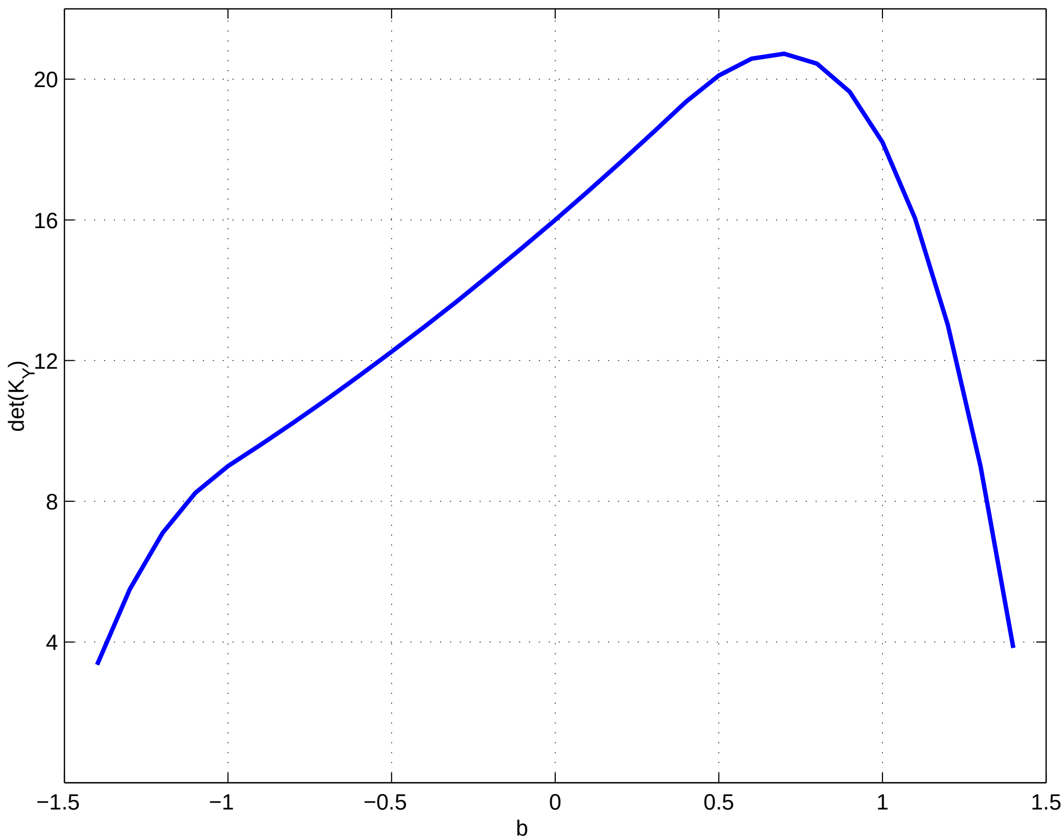
- Should b be positive or negative?
- Should we use all the power in b and let $K_V = 0$?
- What happens when n increases? What should the entries in B look like?
- Exactly how much gain can we obtain by having feedback?

Most Important Fight against the noise and try to cancel it is ''usually'' not a very good idea.

Important We will use some other techniques to derive an upper bound of the capacity gain from feedback.

Less Important The upper bound can be explicitly computed.

Numerical Results for the Example



Upper Bound of the Gain of Feedback

Lemma 1 Any $\underline{X}$ and $\underline{W}$ (maybe dependent)

$$K_{X+W} + K_{X-W} = 2K_X + 2K_W$$

Proof

$$\begin{aligned} K_{X+W} &= E[(\underline{X} + \underline{W})(\underline{X} + \underline{W})^{\dagger}] \\ &= K_X + K_W + K_{XW} + K_{WX} \\ K_{X-W} &= K_X + K_W - K_{XW} - K_{WX} \end{aligned}$$

Lemma 2 For A and B as positive definite matrices, if $A - B$ is also positive definite, than $|A| \geq |B|$.

Lemma 3 $|K_{X+W}| \leq 2^n |K_X + K_W|$

Proof

$$|K_{X+W}| \leq |2K_X + 2K_W|$$

Upper Bound of the Gain from Feedback

Relax the constraint of X for the feedback case to be only the power constraint.

$$
\begin{aligned}
C_{n,FB} &\leq \max_{K_X} \frac{1}{2n} \log \frac{|K_{X+W}|}{|K_W|} \\
&\leq \max_{K_X} \frac{1}{2n} \log \frac{2^n |K_X + K_W|}{|K_W|} \\
&= C_n + \frac{1}{2} (\text{ bits per channel use})
\end{aligned}
$$