LECTURE 20

Last time:

- Waveform Channels
- Gaussian Channel with Colored Noise
- Gaussian Channel with Feedback

Lecture outline

- Multiple Access Channel
- Coding Theorem and Converse
- Successive Cancellation

Review

- Parallel Gaussian Channels, water-filling
  - Homework problem 1.
- Gaussian channel with feedback,

$$C_{n,FB} = \frac{1}{2n} \max \log \frac{|K_Y|}{|K_W|}$$

  Transmit signals that are correlated with the noise.
- When do we have to worry about the operational meaning of "information capacity"?
  - point-to-point channel
  - multi-terminal networks.

Multiple Access Channel

Definition: a multiple access channel is described by

$$\left(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, P_{Y|X_1,X_2}\right)$$

a $(2^{nR_1}, 2^{nR_2}, n)$ code is a sequence of encoding functions

$$\mathcal{W}_1 = \{1, 2, \ldots, 2^{nR_1}\} \;\to \mathcal{X}_1^n$$
$$\mathcal{W}_2 = \{1, 2, \ldots, 2^{nR_2}\} \;\to \mathcal{X}_2^n$$

and a decoding rule

$$\mathcal{Y}^n \to \mathcal{W}_1 \times \mathcal{W}_2$$

Motivation

- lack of coordination: X_1 and X_2 are independent
- interference suppression
- multiple users share the medium

The Performance Measures

- Claim an error if any of the users data is not correctly received.

$$P_e^{(n)} = E[P(g(\underline{Y}) \neq (w_1, w_2) \\ |(w_1, w_2)\text{transmitted})]$$

- (R_1, R_2) is achievable if there exist $(2^{nR_1}, 2^{nR_2}, n)$ codes with $P_e^{(n)} \to 0$.

Definition The *capacity region* is the closure of the set of (R_1, R_2)'s that are achievable.

why closure?

Theorem The capacity region is the closure of the convex hull of all rate (R_1, R_2) satisfying

$$\begin{aligned} R_1 &< I(X_1; Y|X_2) \\ R_2 &< I(X_2; Y|X_1) \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned}$$

for some distribution $P_{X_1} \times P_{X_2}$.

Examples

Example 1: independent channels

Example 2: binary multiplier channel

$$Y = X_1 X_2$$

- set $X_1 = 1$, we can achieve $R_2 = 1$
- Even if X_1, X_2 cooperate, sum rate is bounded by 1
- Time sharing

Example 3:

$$Y = X_1 + X_2$$

- Maximum individual rate is $R_i = 1$.
- Let X_1 transmit at the maximum rate, X_2 has a binary erasure channel with erasure probability $1/2$.
- Successive cancellation
- Is this the optimum?

Example 4 Gaussian Multiple Access Channel

$$Y = X_1 + X_2 + W$$

Coding Theorem

- Encoding, fix an input distribution $P(X_1, X_2) = P(X_1)P(X_2)$.
  - Generate two random codebooks with i.i.d. entries, of size 2^{nR_1} and 2^{nR_2} codewords, each codeword has n symbols.
  - Each encoder choose independently a codeword to transmit, according to the incoming data W_i.
- Decoding,
  - if there exists a unique pair of codewords $(\underline{x}_1(i), \underline{x}_2(j))$ that is joint typical with $\underline{y}$, decode as (i, j).
  - otherwise claim an error.

Notation Define event

$E_{ij} = \{(\underline{x}_1(j), \underline{x}_2(j), \underline{y})$ are jointly typical$\}$

Probability of Error

By symmetry, w.o.l.g. assume $(1,1)$ is transmitted.

$$
\begin{aligned}
P_e^{(n)} &= P(E_{1,1}^c \bigcup \quad \cup_{(i,j)\neq(1,1)} E_{i,j}) \\
&\leq P(E_{1,1}^c) + \sum_{i=1, j\neq 1} P(E_{1,j}) \\
&\quad + \sum_{i\neq 1, j=1} P(E_{i,1}) + \sum_{i\neq 1, j\neq 1} P(E_{i,j})
\end{aligned}
$$

Three different types of error.

Recall joint AEP

- the typical set according to the joint distribution has size $2^{nH(X_1,X_2,Y)}$.
- if X_1, X_2, Y are independently drawn from their marginal distributions, a typical outcome has probability $2^{-n(H(X_1)+H(X_2)+H(Y))}$.

Now for $i \neq 1, j \neq 1$,

$$\begin{aligned}
P(E_{i,j}) &= P((\underline{x}_1(i), \underline{x}_2(j), \underline{y}) \in A_\epsilon^{(n)}) \\
&\leq |A_\epsilon^{(n)}| 2^{-n(H(X_1)+H(X_2)+H(Y)-\epsilon)} \\
&\leq 2^{n(H(X_1,X_2,Y)-H(X_1)-H(X_2)-H(Y)-2\epsilon} \\
&= 2^{-n(I(X_1,X_2;Y)-2\epsilon)}
\end{aligned}$$

- As long as $R_1 + R_2 < I(X_1, X_2; Y)$, can drive the last type of error probability to 0.
- Does this meet the upper bound when cooperation is allowed?

$$P(E_{i,1}) \; = \; P((\underline{x}_1(i), \underline{x}_2(1), \underline{y}) \in A_\epsilon^{(n)})$$

What is the probability that when $\underline{x}_1$ is independently drawn from the marginal, and $\underline{x}_2, \underline{y}$ is drawn from the joint distribution, and the three end up typical according to the joint distribution?

$$\begin{aligned} P(E_{i,1}) &\leq |A_\epsilon^{(n)}| 2^{-n(H(X_1)-\epsilon)} 2^{-n(H(X_2,Y)-\epsilon)} \\ &\leq 2^{n(I(X_1;Y|X_2)-3\epsilon)} \end{aligned}$$

To see this

$$\begin{aligned} & H(X_1, X_2, Y) - [H(X_1) + H(X_2, Y)] \\ = & \; H(X_1|X_2, Y) - H(X_1) \\ = & \; -I(X_1; X_2, Y) \\ = & \; -I(X_1; Y|X_2) \end{aligned}$$

To drive this type of error probability to 0, need

$$R_1 < I(X_1; Y|X_2)$$

Discussions

- Converse of the coding theorem, trivial, read the book.
  - Two different upper bounds of achievable rate, 1) the sum rate is bounded by the point-to-point channel with product marginal distribution. 2) individual rate is bounded by point-to-point channel with a genie revealing the other user's data
  - These upper bounds can be achieved with joint typicality decoding.
- A typical capacity region, dominating rate pairs

$$(I(X_1;Y|X_2), I(X_2;Y))$$
$$(I(X_1;Y), I(X_2;Y|X_1))$$

- Successive cancellation is optimal in information theoretical sense.
- Bias in successive cancellation schemes
- Time sharing to achieve any point on the dominating face of the capacity region

Gaussian Multiple Access Channel

$$Y = X_1 + X_2 + W$$

with Gaussian noise $W \sim N(0, \sigma_W^2)$, power constraint for individual users P_1, P_2.

Let

$$C(x) = \frac{1}{2}\log(1 + x)$$

Capacity region

$$\begin{aligned} R_1 &\leq C(P_1/\sigma_W^2) \\ R_2 &\leq C(P_2/\sigma_W^2) \\ R_1 + R_2 &\leq C((P_1 + P_2)/\sigma_W^2) \end{aligned}$$

Upper bound achieved by using $X_1 \sim N(0, P_1), X_2 \sim N(0, P_2)$.

Discussions

- The achievable sum capacity is exactly the as the single user capacity with $P_1 + P_2$. Recall the greedy view of the Gaussian channel capacity.

Question 1 Is this always true for any channel?

In Gaussian channel, the optimal input distribution does not depend on the noise level.

Question 2 Recall the jammer problem. Difference between coexistence and adversary.

Question 3 If I have infinite number of users, each transmit at power constraint P, will the interference be so strong that nothing can be transmitted?

Question 4 If the transmitter 1 knows exactly the data of user 2, and vice versa, what can we do? Shall we try cancel the interference? Shall we try avoid the interference?

Question 5 Comparing to single user with total power constraint, do we lose anything when divide it into two independent sub-users?

Question 6 Can we divide the users to be transmitting in orthogonal sub-spaces, say, different frequency bands or different time slots, to avoid multiple access?