LECTURE 21

Last time:

- Multiple Access Channel
- Coding Theorem and Converse
- Successive Cancellation

Lecture outline

- Gaussian multiple access channel
- Discussions

Review

- Multiple Access channels
  - Time Sharing
  - Successive Cancellation
- Capacity Region

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2) \\ R_2 &\leq I(X_2; Y|X_1) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) \end{aligned}$$

Gaussian Multiple Access Channel

$$Y = X_1 + X_2 + W$$

with Gaussian noise $W \sim N(0, \sigma_W^2)$, power constraint for individual users P_1, P_2.

Let

$$C(x) = \frac{1}{2}\log(1+x)$$

Capacity region

$$\begin{aligned} R_1 &\leq C(P_1/\sigma_W^2) \\ R_2 &\leq C(P_2/\sigma_W^2) \\ R_1 + R_2 &\leq C((P_1+P_2)/\sigma_W^2) \end{aligned}$$

Upper bound achieved by using $X_1 \sim N(0, P_1), X_2 \sim N(0, P_2)$.

Discussions

- The achievable sum capacity is exactly the as the single user capacity with $P_1 + P_2$. Recall the greedy view of the Gaussian channel capacity.

Question 1 Is this always true for any channel?

In Gaussian channel, the optimal input distribution does not depend on the noise level.

Question 2 Recall the jammer problem. Difference between coexistence and adversary.

Question 3 If I have infinite number of users, each transmit at power constraint P, will the interference be so strong that nothing can be transmitted?

Interference

- From an individual user point of view, other users signal is independent, and is equivalent as the noise.
  - the interference is independent
  - the interference is white over the signal space

Question 4 If the transmitter 1 knows exactly the data of user 2, and vice versa, can we do better?

  - Shall we try cancel the interference?
  - Shall we try avoid the interference?

- Coherence combining: the difference between one user with power P and two users each has power $P/2$.

Question 5 Comparing to single user with total power constraint, do we lose anything when divide it into two independent sub-users?

X-DMA: multiple access in high dimensional space

Consider

$$\underline{Y} = \underline{h}_1 X_1 + \underline{h}_2 X_2 + \underline{W}$$

where all the vectors are 2-dimensional.

- The dimensionality can be thought as multiple samples over time
- Use a bandwidth larger than $1/T_s$ to accommodate multiple users.
- TDMA

$$\underline{h}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{h}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- FDMA

$$\underline{h}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underline{h}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Goal Design $\underline{h}_1, \underline{h}_2, X_1, X_2$ to maximize the overall and individual throughput, with simple encoding and decoding, and fairness.

To avoid interference

Consider a slightly different channel

$$\underline{Y} = \underline{X}_1 + \underline{h}_2 X_2 + \underline{W}$$

where $\underline{h}_2$ is fixed and known, $X_2 \sim N(0, P_2)$, how should user 1 design $\underline{X}_1$ to maximize his own throughput? (assuming successive cancellation)

- Depends on the order of decoding.
- If user 1 is decoded first, water-filling

Conclusion For certain power constraints, restricting $\underline{X}_1$ to be in the direction orthogonal to $\underline{h}_2$ is optimal.

- Practical reasons for orthogonal signaling
- Number of degrees of freedom

Orthogonal Multiple Access

Question 6 Can we always divide the users to be transmitting in orthogonal sub-spaces, say, different frequency bands or different time slots, to avoid multiple access?

Consider user 1 and 2 with power constraint P_1 and P_2, divide them into two disjoint frequency bands of bandwidth W_1 and W_2, let the background noise to be white Gaussian noise with psd. N.

$$
\begin{aligned}
R_1 &= \frac{W_1}{2} \log\left(1 + \frac{P_1}{W_1 N}\right) \\
R_2 &= \frac{W_2}{2} \log\left(1 + \frac{P_2}{W_2 N}\right)
\end{aligned}
$$

If instead we can joint allocate $P_1 + P_2$ over the entire bandwidth $W_1 + W_2$,

$$R_1 + R_2 = \frac{W_1 + W_2}{2} \log \left(1 + \frac{P_1 + P_2}{N(W_1 + W_2)}\right)$$

- The frequency allocation is optimal only if W_i is proportional to P_i.

How about TDMA? CDMA?

Non-Orthogonal Signatures

Consider

$$\underline{Y} = \underline{h}_1 X_1 + \underline{h}_2 X_2 + \underline{W}$$

where $\|\underline{h}_i\|^2 = 1$, $E[X_1^2] = P_1$, $E[X_2^2] = P2, E[W^2] = \sigma^2$.

- $\underline{h}_1$ and $\underline{h}_2$ are not orthogonal.

$$\begin{aligned}
R_2 &\leq I(X_2; Y|X_1) \\
&= I(X_2; \underline{h}_2 X_2 + \underline{W}|X_1) \\
&= h(\underline{h}_2 X_2 + \underline{W}) - h(\underline{W}) \\
&= \frac{1}{2}\log\frac{|K_{W'}|}{|K_W|} \qquad (K_{W'} = P_2 \underline{h}_2 \underline{h}_2^\dagger + \sigma_2 I) \\
&= \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma^2}\|\underline{h}_2\|^2\right)
\end{aligned}$$

- Be careful when writing differential entropies.
- In white Gaussian noise, projecting to $\underline{h}_2$ is optimal.

Now decode user 1 first, $\underline{h}_2 X_2 + \underline{W}$ as noise.

$$\begin{aligned} R_1 + R_2 &\leq I(X_1, X_2; Y) \\ &= \frac{1}{2} \log \frac{|K_Y|}{|K_W|} \end{aligned}$$

- Should be achieve
$$R_1 = I(X_1; Y) = \frac{1}{2} \log(|K_Y| / |K_{W'}|)$$
given that the second user transmit at the maximum power and rate.

Idea: All the information about X_1 is on the direction of $\underline{h}_1$, so project $\underline{Y}$ onto $\underline{h}_1$.

- After projection, the SNR is $P_1 / (P_2 \langle \underline{h}_1, \underline{h}_2 \rangle^2 + \sigma^2)$.

Another Idea Reduce back to the white noise problem: whitening

$$\begin{aligned} K_{W'} &= P_2 \underline{h}_2 \underline{h}_2^\dagger + \sigma^2 I \\ &\triangleq BB^\dagger \end{aligned}$$

- Multiply $\underline{Y}$ by B^{-1} does not lose anything:

$$I(X_1; \underline{Y}) = I(X_1; B^{-1}\underline{Y})$$

- Now a new channel

$$X_1 \to \underline{Y}' = B^{-1}\underline{Y} = B^{-1}\underline{h}_1 X_1 + \underline{Z}$$

where $\underline{Z}$ is white with unit variance entries.

Now project $\underline{Y}'$ into $B^{-1}\underline{h}_1$.

$$X_1 \to \underline{h}_1^\dagger (B^{-1})^\dagger \underline{Y}' = \underline{h}_1^\dagger K_{W'}^{-1} \underline{Y}$$

Achievable rate

$$\begin{aligned} R_1 &= \frac{1}{2} \log \frac{\underline{h}_1^\dagger K_{W'}^{-1} K_Y K_{W'}^{-1} \underline{h}_1}{\underline{h}_1^\dagger K_{W'}^{-1} \underline{h}_1} \\ &= \frac{1}{2} \log(|K_Y| / |K_{W'}|) \end{aligned}$$

Summary

- Multiple access channels with high dimensional signal space allow different techniques to avoid and suppress interference.
- Restricting individual users in single dimensional subspace, even orthogonal, is not optimal in general, but simple in practice.
- Sufficient statistics and MMSE are useful.