LECTURE 23

Last time:

- Broadcast channel
- Capacity region for degraded broadcast channel
- Distributed Source Coding,
- Slepian-Wolf Theorem and Random binning

Lecture outline

- Using random binning in broadcast channels
- Marton's region
- Fading Channels

Review

- Degraded broadcast channel

$$
\begin{aligned}
R_2 &\leq I(U;Y_2) \\
R_1 &\leq I(X;Y_1|U)
\end{aligned}
$$

- Distributed source coding
  - Use large enough number of bins to distinguish the typical sequences.

$$
\begin{aligned}
R_1 &> H(U|V) \\
R_2 &> H(V|U) \\
R_1 + R_2 &> H(U,V)
\end{aligned}
$$

Using Random Binning in Channels

Consider a deterministic point-to-point channel $Y = f(X)$ where $f(.)$ is deterministic and one-to-one. We try to transmit R bits per symbol.

- Assign all the possible $\underline{Y}$ sequences into 2^{nR} bins. The receiver will always decode as the bin number upon receiving $\underline{Y}$.
- We want to send $M = i$, need to control $\underline{Y}$ to be in the i^{th} bin. As long as there is a typical $\underline{y}$ in the i^{th} bin, we transmit the corresponding $\underline{x}$.
- To guarantee there is at least one typical sequence in each bin, the number of bins should be much smaller than $|A_\epsilon(Y)|$, $2^{n(H(Y)-\epsilon)}$ suffice.

What about a random channel $P_{Y|X}$?

- Introduce an auxiliary random variable U as the desired output.
- Observing Y, we can determine $2^{nI(U;Y)}$ different $\underline{U}$ sequences.
- Notice we can talk about both the discrete and the continuous cases
- Point-to-point case: $X = U$, trivial

Capacity Region for Deterministic Broadcast Channel

Theorem Fix an input distribution, any rate pair (R_1, R_2) that satisfies

$$\begin{aligned} R_1 &< H(Y_1) \\ R_2 &< H(Y_2) \\ R_1 + R_2 &< H(Y_1, Y_2) \end{aligned}$$

can be achieved.

- Divide the output space $\mathcal{Y}_1^n \times \mathcal{Y}_2^n$ into $2^{nR_1} \times 2^{nR_2}$ bins,
- For an input message $M_1 \times M_2 \in \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{nR_2}\}$, find a typical sequence $\underline{y}_1 \times \underline{y}_2$ in the corresponding bin. This is the desired output
- Transmit $\underline{x}$ to produce $\underline{y}_1, \underline{y}_2$.
- $R_1 < H(Y_1)$ ensures that exists typical $\underline{y}_1$ per bin
- $R_1 + R_2 < H(Y_1, Y_2)$ ensures that exists a typical $\underline{y}_1, \underline{y}_2$ per product bin

General Broadcast Channel

Now if channel has randomness $P_{Y_1,Y_2|X}$.

- Introduce auxiliary random variables U, V.
- For fixed distribution of U, V, from the channel output, the number of distinguishable sequences $\underline{u}$ and $\underline{v}$ are $2^{nI(U;Y_1)}$ and $2^{nI(V;Y_2)}$.

Theorem(Marton 75') The broadcast channel capacity region is given by

$$\begin{aligned} R_1 &\leq I(U;Y_1) \\ R_2 &\leq I(V;Y_2) \\ R_1 + R_2 &\leq I(U;Y_1) + I(V;Y_2) - I(U;V) \end{aligned}$$

for a fixed distribution $P_{U,V,X}$.

Outline of Proof

- generate $2^{nI(U;Y_1)}$ typical sequence $\underline{u}$'s and throw into 2^{nR_1} bins, generate $2^{n(I(V;Y_2)}$ typical $\underline{v}$'s throw into 2^{nR_2} bins.
- upon receiving $\underline{Y}_1$, $\underline{u}$ can be uniquely determined. similar to $\underline{v}$.
- There are $2^{n(I(U;Y_1)+I(V;Y_2))}$ possible $(\underline{u}, \underline{v})$ pairs. Each pair being jointly typical with probability $2^{-nI(U;V)}$.
- If $R_1+R_2 \leq I(U;Y_1)+I(V;Y_2)-I(U;V)$, then there exists a jointly typical pair $(\underline{u}, \underline{v})$ in each product bin. This is the desired received sequences.
- To transmit that bin number, simply transmit $\underline{x}$ that is jointly typical with $(\underline{u}, \underline{v})$.

Wireless Channels

Key difference from AWGN channel Multi-path Fading

Flat fading model

$$Y_i = H_i X_i + W_i$$

where W_i is the AWGN with variance σ_W^2.

- $\{H_i\}$ is a random process, for which the marginal distribution is modelled as **unit variance** Rayleigh or Ricean
- The rate at which $\{H_i\}$ changes over time depends on
  - the speed that receiver moves,
  - the environment
  - the carrier frequency
  - the symbol period

The wireless challenge

- How do we define a capacity for this random channel
  - the assumptions on the availability of CSI
  - the assumptions on the channel time-variation
- How well the different assumptions apply to practical channels
- How do the optimal signaling change with the channel assumptions
- How do these apply when we have a network

Example TCP

First simple example

- Assume H remains constant
- Assume H is perfectly known at the receiver
- Assume the transmitter does not know H, so the input distribution can not depend on H.

The capacity

$$C(H) = \max_{P_X} I(X;Y)$$

where

$$P_{Y|X} = N(HX, \sigma_W^2)$$

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(W) \end{aligned}$$

- To maximize $h(Y)$, let X be Gaussian
- Resulting $Y \sim N(0, |H|^2\sigma_X^2 + \sigma_W^2)$

$$C(H) = \frac{1}{2}\log\left(1 + \frac{|H|^2\sigma_X^2}{\sigma_W^2}\right)$$

The capacity is a function of H

Outage Capacity and Ergodic Capacity

- From the transmitter point of view (H unknown), what is the maximum rate that can be transmitted with 0 probability?
- Shannon capacity for fixed fading channel is 0.
- Considering the distribution of H, $C(H)$ is a random variable.

Definition $a\%$ outage capacity is the data rate that can be supported with $a\%$, i.e.,

$$P(C(H) < R) \leq a\%$$

Definition The ergodic capacity

$$C = E[C(H)]$$

Ergodic Capacity

$$C = E\left[\frac{1}{2}\log\left(1 + |H|^2\frac{P}{N}\right)\right]$$

- To achieve the ergodic capacity, we need to code long enough that the statistics of H start to kick in.
- Do we want channel to be time-varying or static ?
- Interleaving, delay, rate, burstiness trade-off.
- Diversity issues.
- Evaluating the capacity is hard.
- Two asymptotic results
  - as $P/N \to 0$, low SNR

    $$C \to E[|H|^2]\frac{1}{2}\frac{P}{N}$$

    the same limit as the AWGN channel
  - as $P/N \to \infty$, high SNR

    $$C \to \frac{1}{2}E[\log|H|^2] + \frac{1}{2}\log\frac{P}{N}$$

Different from the AWGN channel by a constant.