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The goal of much research in relativity is to understand gravitational waves generated by a strong-field dynamical spacetime. Quantities of particular interest for many calculations are the Weyl scalar ψ_4, which is simply related to the flux of gravitational waves far from the source, and the flux of energy carried to distant observers, $\dot{E}$. Conservation laws guarantee that, in asympotically flat spacetimes, $\psi_4 \propto 1/r$ and $\dot{E} \propto 1/r^2$ as $r \to \infty$. Most calculations extract these quantities at some finite extraction radius. An understanding of finite radius corrections to ψ_4 and $\dot{E}$ allows us to more accurately infer their asymptotic values from a computation. In this paper, we show that, if the final state of the system is a black hole, then the leading correction to ψ_4 is $\mathcal{O}(1/r^3)$, and that to the energy flux is $\mathcal{O}(1/r^4)$—not $\mathcal{O}(1/r^2)$ and $\mathcal{O}(1/r^3)$, as one might naively guess. Our argument only relies on the behavior of the curvature scalars for black hole spacetimes. Using black hole perturbation theory, we calculate the corrections to the leading falloff, showing that it is quite easy to correct for finite extraction radius effects.

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I. INTRODUCTION

Extracting radiation from the output of numerical calculations, as well as fluxes of quantities such as energy carried by radiation, is important for many problems in general relativity. Newman and Unti [1] provide an outstanding foundation for understanding analytically the asymptotic behavior of curvature tensors, which determine how gravitational radiation behaves as it propagates far from a radiating source. Perturbation theory also provides an excellent set of tools to help us understand the asymptotic behavior of radiation and fluxes.

Many results on the distant behavior of radiation fields describe how quantities behave in the limit $r \to \infty$. With the exception of characteristic methods (see, for example, Ref. [2]), most numerical calculations extract radiation at some large but finite radius r. Understanding the subleading corrections to the asymptotic behavior of radiative quantities could greatly improve our ability to extract asymptotic fluxes and fields from numerical codes.

Previous work [3] found empirically that the form

$$\dot{E}(r) = \dot{E}_\infty\left(1 + \frac{e_2}{r^2}\right) \quad (1)$$

does an outstanding job describing subleading corrections to the gravitational–wave energy flux. In this paper, we examine this behavior more carefully. In Sec. II, following the formalism developed in Ref. [1], we prove that this form is to be generically expected, and follows from the fact that at finite large radius r, the Weyl curvature scalar describing distant radiation takes the form $\psi_4(r) = \psi_4^\infty(1 + b_2/r^2)$. In Sec. III, we use black hole perturbation theory to calculate the coefficients b_2 and e_2. We conclude Sec. IV by discussing possible applications of this result.

II. TOOLS AND FORMALISM FOR UNDERSTANDING RADIATION FALLOFF

A. Definitions

We begin by defining the quantities which we will need for our analysis. Much of this discussion is adapted from Ref. [1]. We present these general definitions in some detail before specializing to the much simpler black hole case.

Consider a vacuum, asymptotically flat spacetime. Introduce a family of null hypersurfaces, each characterized by a constant parameter u. We take $u = x^0$ as one of the coordinates we will use to describe our geometry. Define

$$l_\alpha = \partial_\alpha u. \quad (2)$$

Since these surfaces are null, the vector l^α is tangent to null geodesics. This vector will be the first leg of a tetrad which we will use to characterize our geometry. Define r as the affine parameter along these geodesics; this will denote another of our coordinates. The remaining coordinates x^k ($k \in 3, 4$) then label the different null geodesics in each constant u hypersurface; they can be taken to be angles.

We define a second null vector n^α by requiring

$$n^\alpha l_\alpha = 1. \quad (3)$$

To complete our tetrad, we next define a pair of unit spacelike vectors ζ^α and ρ^α that are orthogonal to l^α, n^α, and to each other. We then put

$$m^\alpha = (\zeta^\alpha - i\rho^\alpha)/\sqrt{2}, \quad (4)$$

$$\bar{m}^\alpha = (\zeta^\alpha + i\rho^\alpha)/\sqrt{2}. \quad (5)$$

We now use this tetrad to characterize the curvature of our spacetime. Let $C_{\alpha\mu\beta\nu}$ be the Weyl (vacuum) curvature tensor of the spacetime. Define the following 5 complex Weyl projections:

$$\psi_0 = -C_{\alpha\mu\beta\nu} l^\alpha m^\mu l^\beta m^\nu, \tag{6}$$

$$\psi_1 = -C_{\alpha\mu\beta\nu} l^\alpha n^\mu l^\beta m^\nu, \tag{7}$$

$$\psi_2 = -C_{\alpha\mu\beta\nu} l^\alpha m^\mu \bar{m}^\beta n^\nu, \tag{8}$$

$$\psi_3 = -C_{\alpha\mu\beta\nu} l^\alpha n^\mu \bar{m}^\beta n^\nu, \tag{9}$$

$$\psi_4 = -C_{\alpha\mu\beta\nu} n^\alpha \bar{m}^\mu n^\beta \bar{m}^\nu. \tag{10}$$

Reference [1] shows that as we approach the asymptotically flat ($r \to \infty$) regime, these curvature components vary as follows:

$$\psi_0 = \frac{A_0}{r^5} + \mathcal{O}(1/r^6), \tag{11}$$

$$\psi_1 = \frac{A_1}{r^4} + \frac{(4\alpha_{\rm RSC} A_0 - \bar{\xi}^k \partial_k A_0)}{r^5} + \mathcal{O}(1/r^6), \tag{12}$$

$$\psi_2 = \frac{A_2}{r^3} + \frac{(2\alpha_{\rm RSC} A_1 - \bar{\xi}^k \partial_k A_1)}{r^4} + \mathcal{O}(1/r^5), \tag{13}$$

$$\psi_3 = \frac{A_3}{r^2} - \frac{\bar{\xi}^k \partial_k A_2}{r^3} + \mathcal{O}(1/r^4), \tag{14}$$

$$\psi_4 = \frac{A_4}{r} - \frac{(2\alpha_{\rm RSC} A_3 + \bar{\xi}^k \partial_k A_3)}{r^2} + \mathcal{O}(1/r^3). \tag{15}$$

In Eqs. (12)–(15), specifically the index $k \in [3, 4]$, the complex function ξ^k describes the angular components of the tetrad element m^α, and the functions $\alpha_{\rm RSC}$ and $\gamma_{\rm RSC}$ are "Ricci spin coefficients," constructed by certain combinations and projections of the tetrad's covariant derivatives. For more details and discussion of these functions, see Refs. [1,4]. For our purposes, the most important fact to take from Eqs. (11)–(15) is that the leading falloff of ψ_4 is at $\mathcal{O}(1/r)$. The subleading correction at $\mathcal{O}(1/r^2)$ is set by a coefficient that scales with A_3, which controls the behavior of the curvature scalar ψ_3.

B. Perturbed black holes

We now specialize to black holes. We use the Kinnersley tetrad [5], which in Boyer-Lindquist coordinates is given by

$$l^\alpha \doteq \frac{1}{\Delta}(r^2 + a^2, 1, 0, a), \tag{16}$$

$$n^\alpha \doteq \frac{1}{2\Sigma}(r^2 + a^2, -\Delta, 0, a), \tag{17}$$

$$m^\alpha \doteq \frac{1}{\sqrt{2}(r + ia\cos\theta)}(ia\sin\theta, 0, 1, i\csc\theta). \tag{18}$$

For an unperturbed black hole spacetime, $\psi_2 = -M/(r - ia\cos\theta)^3$, and $\psi_n = 0$ for $n \neq 2$. Far from a *perturbed* black hole, ψ_4 is also nonzero, describing the spacetime's outgoing gravitational waves:

$$\psi_4(r \to \infty) = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times) = \frac{1}{2r}(\ddot{H}_+ - i\ddot{H}_\times). \tag{19}$$

The Weyl scalar ψ_0 is also generically nonzero for a perturbed black hole, but we will not need its value in our analysis. Crucially for our argument, we can always put $\Psi_3 = 0$ for our perturbed black hole [6].

Comparing with Eqs. (12)–(15), we read off

$$A_3 = 0, \tag{20}$$

$$A_2 = -M, \tag{21}$$

$$A_4 = \frac{1}{2}(\ddot{H}_+ - i\ddot{H}_\times). \tag{22}$$

Combining these results with Eq. (15), we see that corrections to ψ_4 come in at $\mathcal{O}(1/r^3)$, so that

$$\psi_4 = \frac{A_4}{r}\left(1 + \frac{b_2}{r^2}\right), \tag{23}$$

where b_2 is a complex constant related to the (currently unknown) coefficient of this subleading falloff.

C. Energy flux

We now relate the curvature scalar ψ_4 to the asymptotic flux of radiation from the source. The energy flux in gravitational waves is given by

$$\dot{E} = \frac{1}{16\pi}\int r^2 d\Omega[(\dot{h}_+)^2 + (\dot{h}_\times)^2]. \tag{24}$$

Using Eq. (19), we can relate this to ψ_4 in the limit $r \to \infty$:

$$\dot{E}^\infty = \frac{1}{4\pi}\lim_{r\to\infty}\int r^2 d\Omega \left|\int dt \psi_4\right|^2. \tag{25}$$

Using Eq. (23), let us now see what this implies about the behavior of $\dot{E}$ when radiation is extracted at some finite radius R. Let us first introduce a modal expansion, writing

$$\psi_4 = \sum_\omega \psi_4^\omega e^{-i\omega t} = \sum_\omega \frac{A_4^\omega}{r}\left(1 + \frac{b_2^\omega}{r^2}\right)e^{-i\omega t}. \tag{26}$$

For simplicity, we have taken the radiation to have a discrete frequency spectrum. The calculation can easily be extended to encompass a continuous spectrum. Combining Eqs. (25) and (26), we find

$$\dot{E}(r) = \frac{1}{4\pi}\sum_\omega \omega^{-2}\int r^2 d\Omega |\psi_4^\omega|^2 \tag{27}$$

$$= \sum_\omega \dot{E}_\infty^\omega\left(1 + \frac{e_2^\omega}{r^2}\right), \tag{28}$$

where

$$\dot{E}_\infty^\omega = \frac{1}{4\pi\omega^2}\int d\Omega |A_4^\omega|^2, \tag{29}$$

$$e_2^\omega = (\dot{E}_\infty^\omega)^{-1} \times \frac{1}{2\pi\omega^2}\int d\Omega |A_4^\omega|^2 ({\rm Re} b_2^\omega). \tag{30}$$

In other words, an $\mathcal{O}(1/r^3)$ correction to ψ_4 produces an $\mathcal{O}(1/r^4)$ correction to $\dot{E}$. We next must understand how to compute the coefficient of this correction. We do so using black hole perturbation theory.

III. SUBLEADING BEHAVIOR VIA PERTURBATION THEORY

Perturbation theory is a powerful tool for calculating ψ_4 and then determining fluxes such as $\dot{E}$. In this section, we use black hole perturbation theory to confirm the general results of Sec. II, and to explicitly compute the magnitude of the subleading contributions to ψ_4 and $\dot{E}$.

Throughout this section, we will assume a frequency-domain decomposition for ψ_4. This assumption means that solutions for ψ_4 separate [7]:

$$\psi_4 = \frac{1}{(r - ia\cos\theta)^4}\sum_\omega R_{lm\omega}(r) S_{lm}(\theta; a\omega) e^{im\phi} e^{-i\omega t}. \tag{31}$$

The function $S_{lm}(\theta; a\omega) \equiv S(\theta)$ is a spin-weighted spheroidal harmonic, and is discussed extensively in Appendix A of Ref. [8]. It satisfies the eigenvalue relation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \bigg[(a\omega)^2\cos^2\theta + 4a\omega\cos\theta - \left(\frac{m^2 - 4m\cos\theta + 4}{\sin^2\theta}\right) + \mathcal{E}\bigg]S = 0. \tag{32}$$

In the $a = 0$ limit, $\mathcal{E} \to l(l+1)$, where l is the usual spherical harmonic index.

The function $R_{lm\omega}(r) \equiv R(r)$ is governed by [7]

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR}{dr}\right) - V(r)R(r) = -\mathcal{T}(r), \tag{33}$$

often called the Teukolsky equation. Here and in what follows, $\Delta = r^2 - 2Mr + a^2$. Detailed discussion of the source $\mathcal{T}(r)$ is given in Refs. [7,8]. For our purpose, it suffices to note that an effective way to solve Eq. (33) is to first find a homogeneous solution, setting the source $\mathcal{T}(r) = 0$. From these solutions, it is fairly simple to build a Green's function, which we integrate over the source to find the particular solution for our problem.

We show the potential $V(r)$ in the appendix. It depends on the eigenvalue $\mathcal{E}$ via the parameter $\lambda = \mathcal{E} - 2am\omega + a^2\omega^2 - 2$. An important property of $V(r)$ is that it is long-ranged: as $r \to \infty$, $V(r) \to r^2$. This makes computing $R(r)$ for large r difficult. An excellent way to circumvent this difficulty is to first solve the Sasaki-Nakamura equation [9],

$$\frac{d^2X}{dr_*^2} - F(r)\frac{dX}{dr_*} - U(r)X = 0, \tag{34}$$

where $r_*(r)$ is the "tortoise coordinate,"

$$r_* = r + \frac{2Mr_+}{r_+ - r_-}\ln\left(\frac{r - r_+}{2M}\right) - \frac{2Mr_-}{r_+ - r_-}\ln\left(\frac{r - r_-}{2M}\right). \tag{35}$$

The potentials $F(r)$ and $U(r)$ are also shown in the appendix. Their key property is that, unlike the Teukolsky equation's $V(r)$, they are short ranged: As $r \to \infty$, $F \to (\text{constant})/r^2$ and $U \to -\omega^2 + (\text{constant})/r^2$. The solutions $X(r)$ thus approach plane waves in the asympotically flat region, $X(r \to \infty) \to e^{\pm i\omega r_*}$. Teukolsky equation solutions can be then be built from Sasaki-Nakamura equation solutions by the transformation

$$R = \frac{1}{\eta}\left[\left(\alpha + \frac{\partial_r\beta}{\Delta}\right)\chi - \frac{\beta}{\Delta}\frac{d\chi}{dr}\right], \tag{36}$$

where $\chi = X\Delta/\sqrt{r^2 + a^2}$. The functions $\alpha(r)$, $\beta(r)$, and $\eta(r)$ are listed in the appendix.

A more accurate asymptotic form of $X(r)$ is

$$X(r) = A^{\rm out}P^{\rm out}(r)e^{i\omega r_*} + A^{\rm in}P^{\rm in}(r)e^{-i\omega r_*}, \tag{37}$$

where

$$P^{\rm in/out}(r) = 1 + \frac{p_1^{\rm in/out}}{\omega r} + \frac{p_2^{\rm in/out}}{(\omega r)^2} + \frac{p_3^{\rm in/out}}{(\omega r)^3}. \tag{38}$$

The coefficients appearing in this expansion are given by

$$p_1^{\rm in} = -\frac{i}{2}(\lambda + 2 + 2am\omega), \tag{39}$$

$$p_2^{\rm in} = -\frac{1}{8}[(\lambda+2)^2 - (\lambda+2)(2 - 4am\omega) - 4[am\omega + 3iM\omega - am\omega(am\omega + 2iM\omega)]], \tag{40}$$

$$p_3^{\rm in} = -\frac{i}{6}[4am\omega + p_2^{\rm in}(\lambda - 4 + 2am\omega + 8iM\omega) + 12(M\omega)^2 - 2p_1^{\rm in}\lambda M\omega - (a\omega)^2(\lambda - 3 + m^2 + 2am\omega)]; \tag{41}$$

and

$$p_1^{\rm out} = \bar{p}_1^{\rm in} + \frac{\omega c_1}{c_0}, \tag{42}$$

$$p_2^{\rm out} = \bar{p}_2^{\rm in} + \frac{1}{c_0}\left[\omega^2 c_2 - \omega c_1\left(p_1^{\rm in} + \frac{i}{2}\right)\right], \tag{43}$$

$$p_3^{\rm out} = \bar{p}_3^{\rm in} + \frac{1}{c_0}\bigg[\omega^3 c_3 - \omega^2 c_2(p_1^{\rm in} + i) + \omega c_1\left[\bar{p}_2^{\rm in} + \frac{ip_1^{\rm in}}{2} - \frac{1}{2} + 2iM\omega(a\omega m - 1)\right]\bigg]. \tag{44}$$

The coefficients c_0, c_1, c_2, and c_3 appear in the definition of the function $\eta(r)$, and are given in the appendix; overbar denotes complex conjugate.

The condition that radiation be purely outgoing far from the black hole picks out a solution of the form

$$X = X_\infty P^{\rm out}(r)e^{i\omega r_*} \tag{45}$$

as $r \to \infty$. Performing the transformation (36), we find that the Teukolsky solution $R(r)$ can be written

$$R(r) = r^3 Z_\infty Q^{\rm out}(r)e^{i\omega r_*} \tag{46}$$

for $r \to \infty$, where $Z_\infty = -4X_\infty\omega^2/c_0$, and where

$$Q^{\rm out}(r) = 1 + \frac{q_1}{\omega r} + \frac{q_2}{(\omega r)^2} + \frac{q_3}{(\omega r)^3} + \ldots. \tag{47}$$

The coefficients $q_{1,2,3}$ are given in the appendix.

Now, we use this solution to examine the flux of energy a finite distance from the black hole. Using Eq. (27), we find that

$$\dot{E}(r) = \sum_\omega \dot{E}^\infty_\omega |Q^{\rm out}_\omega|^2, \tag{48}$$

where

$$\dot{E}^\infty_\omega = \frac{|Z_\infty|^2}{4\pi\omega^2}, \tag{49}$$

$$|Q^{\rm out}_\omega|^2 = 1 + \frac{6a\omega(a\omega - m) - \lambda}{2\omega^2 r^2} + \frac{M(\lambda - 1)}{\omega^2 r^3}. \tag{50}$$

Notice that the leading correction to the energy flux appears at $\mathcal{O}(1/r^2)$, in agreement with Eq. (28). Comparing with Eq. (1), we find that the coefficient e_2, which labels the $1/r^2$ falloff, is

$$e_2 = \frac{6a\omega(a\omega - m) - \lambda}{2\omega^2}. \tag{51}$$

Recall that λ is related to the spheroidal harmonic eigenvalue $\mathcal{E}$; cf. Eq. (32) and following discussion. For Schwarzschild, this correction is particularly simple:

$$e_2(a = 0) = -\frac{l(l+1) - 2}{2\omega^2}, \tag{52}$$

where l is the spherical harmonic index associated with the mode under consideration.

IV. DISCUSSION

In this analysis, we have demonstrated that whenever one extracts radiation and radiative fluxes at a finite large radius, the subleading correction to these quantities is at an order $\mathcal{O}(1/r^2)$ beyond the leading asymptotic behavior. Hence, the correction to the curvature scalar ψ_4 is at $\mathcal{O}(1/r^3)$, and to the energy flux is at $\mathcal{O}(1/r^4)$.

Using black hole perturbation theory, we have shown it is not difficult to calculate the coefficient of the subleading falloff, at least for a plane wave. The results we have found are consistent with the results shown in Table VI of Ref. [3]. In that paper, a time-domain code was used to examine radiation from circular orbits. The time-domain code does not separate the angular behavior, and so many values of l are included in the analysis simultaneously. The radiation tends to be dominated by $l = m$, with important but decreasing contributions from $l = m + 1$, $l = m + 2$, etc. Our expectation for the Schwarzschild radiation is thus likely to be close to the prediction from Eq. (52) for $l = m$, skewed somewhat by contributions from $l = m + 1$.

Let us test that prediction. Consider first the results for $m = 2$. If we assume that the waves presented in Ref. [3] for this case are dominated by radiation in the $l = 2$ and $l = 3$ modes, then we expect e_2 to be between

$$e_2(a = 0, l = m = 2) = -2\omega^{-2} \tag{53}$$

and

$$e_2(a = 0, l = m + 1 = 3) = -5\omega^{-2}. \tag{54}$$

Table VI of Ref. [3] shows

$$e_2(a = 0, m = 2) = -2.59\omega^{-2}, \tag{55}$$

in reasonably good agreement with the intuition provided by our plane-wave expansion. Table VI also provides Schwarzschild data for $m = 3$; if those data are dominated by $l = 3$ and $l = 4$, we expect e_2 to be between

$$e_2(a = 0, l = m = 3) = -5\omega^{-2} \tag{56}$$

and

$$e_2(a = 0, l = m + 1 = 4) = -9\omega^{-2}. \tag{57}$$

Table VI of Ref. [3] shows

$$e_2(a = 0, m = 2) = -6.20\omega^{-2}, \tag{58}$$

again agreeing reasonably well with the plane-wave expansion. By computing the eigenvalues of the spheroidal harmonics for nonzero spin, one can likewise show that the Kerr values in Table VI agree reasonably well with the expectation of our plane-wave expansion.

Bear in mind that the numerical magnitude of the correction we derived strictly applies only for plane-wave expansions. As such, although we can provide good *post facto* justification of the coefficients of the subleading falloff, it would be difficult to predict those coefficients in advance. To do so, we would need to know the weighting of the different l modes which contribute to the radiation. Our only purpose in analyzing the coefficients shown in Ref. [3] is to show that the results presented there are consistent with our results here.

For many calculations, it will not be worthwhile to decompose the angular distribution of the waves, and thus to compute the subleading falloff in the manner shown here. It should be emphasized that the radial behavior of the falloff is *independent* of the l modes which contribute to the waves. As such, it would not be difficult to extract the radiation at several radii and simply fit the coefficient. That is what was done in Refs. [3,10]. Implementing such a multiradius fit should make it possible to more accurately extract the asymptotic radiation computed by numerical analysis, potentially reducing errors in such calculations by several percent.

In general numerical spacetimes, it may be more complicated to take advantage of this result. The key ingredient to making the falloff work as we have discussed is to choose a tetrad such that the Weyl scalar $\Psi_3 = 0$. As long as one can perform a null rotation to put the spacetime into such a "transverse" tetrad [6,11], one should find that subleading corrections to the flux of radiation fall off as $1/r^3$. It may be challenging to implement this rotation for the general case, but the improvement in accuracy could make it worthwhile.

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APPENDIX: FUNCTIONS FROM BLACK HOLE PERTURBATION THEORY

In this appendix, we present various functions which arise in black hole perturbation theory that we need for our analysis. The functions $\eta(r)$, $\alpha(r)$, and $\beta(r)$ which appear in the transformation law (36) are given by

$$\eta(r) = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4, \tag{A1}$$

$$\alpha(r) = -\frac{iK(r)\beta(r)}{\Delta^2} + 3i\frac{dK}{dr} + 6\frac{\Delta}{r^2} + \lambda. \tag{A2}$$

$$\beta(r) = 2\Delta[r - M - 2\Delta/r - iK(r)]. \tag{A3}$$

These functions in turn depend on the coefficients

$$c_0 = -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m), \tag{A4}$$

$$c_1 = 8ia[3a\omega - \lambda(a\omega - m)], \tag{A5}$$

$$c_2 = -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \tag{A6}$$

$$c_3 = 24ia^3(a\omega - m) - 24Ma^2, \tag{A7}$$

$$c_4 = 12a^4, \tag{A8}$$

and the function

$$K(r) = (r^2 + a^2)\omega - ma. \tag{A9}$$

Recall that $\lambda = \mathcal{E} - 2am\omega + a^2\omega^2 - 2$, where $\mathcal{E}$ is the eigenvalue of the spheroidal harmonic.

The potential $V(r)$ appearing in the Teukolsky Eq. (33) is given by

$$V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda. \tag{A10}$$

The potentials $F(r)$ and $U(r)$ appearing in the Sasaki-Nakamura Eq. (34) are

$$F(r) = \frac{d\eta/dr}{\eta}\frac{\Delta}{r^2 + a^2}, \tag{A11}$$

$$U(r) = \frac{\Delta U_1(r)}{(r^2 + a^2)^2} + \frac{\Delta dG/dr}{r^2 + a^2} - F(r)G(r) + G(r)^2, \tag{A12}$$

where

$$U_1(r) = V(r) + \frac{\Delta^2}{\beta}\left[\frac{d}{dr}\left(2\alpha + \frac{d\beta/dr}{\Delta}\right) - \frac{d\eta/dr}{\eta}\left(\alpha + \frac{d\beta/dr}{\Delta}\right)\right]. \tag{A13}$$

The coefficients $p_{1,2,3}^{\rm in/out}$ defined in Eqs. (39)–(44) are found by requiring that the solution (37) satisfy the Sasaki-Nakamura equation in each order in $1/r$. After transforming to the Teukolsky equation solution $R(r)$, the different orders in $1/r$ are labeled by the coefficients $q_{1,2,3}$ defined in Eq. (47):

$$q_1 = p_1^{\rm out} - i - c_1\omega/c_0, \tag{A14}$$

$$\begin{aligned} q_2 = -\frac{1}{4c_0^2}[&-4c_1^2\omega^2 + 4c_0\omega[(p_1^{\rm out} - i)c_1 + c_2\omega] \\ &+ c_0^2[2 + 2ip_1^{\rm out} - 4p_2^{\rm out} + \lambda + 6am\omega \\ &- 12iM\omega - 6a^2\omega^2]], \end{aligned} \tag{A15}$$

$$\begin{aligned} q_3 = \frac{1}{4c_0^3}[&4c_0c_1\omega^2[(p_1^{\rm out} - i)c_1 + 2c_2\omega - 4c_1^3\omega^3] \\ &+ c_0^2\omega[-4\omega[(p_1^{\rm out} - i)c_2 + c_3\omega] \\ &+ c_1[2 + 2ip_1^{\rm out} - 4p_2^{\rm out} + \lambda + 6am\omega - 12iM\omega \\ &- 6a^2\omega^2]] + c_0^3[4p_3^{\rm out} + 2\omega[M(5 + \lambda) - 5ia^2\omega] \\ &- p_1^{\rm out}[\lambda - 2\omega(3a^2\omega - 3am + 4iM)]]]. \end{aligned} \tag{A16}$$

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