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## Non-Abelian two-component fractional quantum Hall states

Maïssam Barkeshli and Xiao-Gang Wen

*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

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A large class of fractional quantum Hall (FQH) states can be classified according to their pattern of zeros, which describes the order of zeros in ground-state wave functions as various clusters of electrons are brought together. The pattern-of-zeros approach can be generalized to systematically classify multilayer/spin-unpolarized FQH states, which has led to the construction of a class of non-Abelian multicomponent FQH states. Here we discuss some of the simplest non-Abelian two-component states that we find and the possibility of their experimental realization in bilayer systems at  $\nu=2/3, 4/5, 4/7, 4/9, 1/4$ , etc.

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There has been an ongoing effort in the condensed-matter community to experimentally realize topological phases of matter whose elementary excitations exhibit non-Abelian statistics.<sup>1-3</sup> While most of the attention on non-Abelian fractional quantum Hall (FQH) states has to date been directed toward single-component two-dimensional electron systems, there is good reason to look closely at two-component systems as well (such as bilayer or spin-unpolarized states).<sup>4-9</sup> Two-component quantum Hall systems allow greater variety and tunability of effective interactions between electrons in the partially filled Landau levels and it is the nature of these effective interactions that ultimately determines the kind of phase that is formed. In this Brief Report, we report on results we have found using a systematic classification of multicomponent FQH states. We will present and discuss some of the simplest non-Abelian two-component FQH states that we find and that occur at experimentally relevant filling fractions. These states may perhaps be realized in situations where the interlayer repulsion is comparable to the intralayer repulsion.

An important unsolved problem in FQH theory is to have a complete, physical, and coherent understanding of how to describe the many different FQH states that may be obtained. Such an understanding will lead to the discovery of additional topological phases of matter and, more importantly, can give us a better overall understanding of which non-Abelian phases are most accessible experimentally. Given the prodigious amount of numerical and experimental effort required in establishing the existence of a non-Abelian FQH state, it is important to have a way to theoretically hone in on the most promising candidates. As a step in this direction, we have constructed a systematic classification of a large class of FQH states, which is based on the pattern of zeros of wave functions. For example, the Laughlin wave function<sup>10</sup> at  $\nu = 1/m$  has an  $m$ th order zero as any two particles are brought together. More generally, we can consider bringing  $a$  particles together by setting  $z_i = \lambda \xi_i + z^{(a)}$  for  $i = 1, \dots, a$  and expanding the wave function in powers of  $\lambda$ ,

$$\Phi(\{z_{ij}\}) = \lambda^{S_a} P(\{\xi_{ij}; z^{(a)}, z_{a+1}, \dots\}) + O(\lambda^{S_a+1}). \quad (1)$$

Note that the full FQH wave function is  $\Psi(\{x_i, y_j\}) = \Phi(\{z_{ij}\}) e^{-\sum_i |z_i|^2 / 4l_B^2}$ , where  $z_i = x_i + iy_i$ ,  $\Phi(\{z_{ij}\})$  is a polynomial in the complex coordinates  $z_i$ , and  $l_B$  is the magnetic length.

The sequence  $\{S_{\vec{a}}\}$  is called the pattern of zeros and serves as a quantitative characterization of a wide class of FQH states.  $\{S_{\vec{a}}\}$  must satisfy certain consistency conditions in order to describe a valid wave function  $\Phi(\{z_{ij}\})$ . Finding all valid sets of  $\{S_{\vec{a}}\}$  that satisfy these consistency conditions then serves as a systematic classification of FQH wave functions. Such an approach first led to a systematic classification of non-Abelian single component quantum Hall states, which includes the known non-Abelian states and many previously unknown ones as well.<sup>11-13</sup> Recently, we have generalized the pattern-of-zeros approach to systematically classify and quantitatively characterize non-Abelian multilayer FQH wave functions; for a complete presentation, see Ref. 14. For  $f$ -component (or  $f$ -layer) states, the pattern of zeros is described by a set of integers  $\{S_{\vec{a}}\}$  indexed by a  $f$ -dimensional vector  $\vec{a} = (a_1, \dots, a_f)$ , where  $S_{\vec{a}}$  is the order of zeros as we bring  $a_l$  electrons together in the  $l$ th layer.

In general, the number of integers  $S_{\vec{a}}$  that need to be specified is infinite in the thermodynamic limit. However, some wave functions can be specified by much less data; the Laughlin wave function is fully specified by  $S_2$  and by the fact that there are no off-particle zeros. The Moore-Read Pfaffian state<sup>1</sup> is fully specified by  $S_2, S_3$ , and the fact that after combining every pair of electrons in the Pfaffian wave function into bound states, the induced effective wave function for the bound states becomes a Laughlin wave function which has no off-particle zeros. Such a two-cluster structure in the Pfaffian state is the reason why  $S_2$  and  $S_3$  can already fully specify the state. More generally, we believe that gapped ideal FQH wave functions have a  $n$ -cluster structure: after combining every  $n$  cluster of electrons into bound states, the induced effective wave function for the bound states becomes a Laughlin wave function with no off-particle zeros. For such  $n$ -cluster states, one only needs to specify  $S_a$  for  $a \leq n$  to fully characterize the states. The  $Z_k$  parafermion states,<sup>15</sup> for instance, have  $n=k$ . The value of  $n$  serves to gauge the complexity of a FQH state. For a fixed  $\nu$ , as  $n$  increases, the number of topologically distinct quasiparticles, the ground-state degeneracy on higher genus surfaces and the complexity of interactions necessary to realize the state all increase. This suggests that the energy gap typically decreases with increasing  $n$ . Wave functions that do not obey a cluster condition can be thought of as having infinite  $n$  and are not expected to correspond to gapped phases. This intu-

ition also comes from the conformal field theory (CFT) approach to FQH wave functions; infinite  $n$  corresponds to an irrational conformal field theory, which does not yield a finite number of quasiparticles and a finite ground-state degeneracy on the torus. In the  $f$ -layer case, the cluster structure is characterized by an  $f \times f$  invertible matrix: there are  $f$  different kinds of clusters that can be characterized by vectors  $\vec{n}_I$ ,  $I=1, \dots, f$ . The cluster  $\vec{n}_I$  contains  $(\vec{n}_I)_J$  electrons in the  $J$ th layer ( $J=1, \dots, f$ ). When all of the electrons combine into these bound states, the resulting wave function has a Laughlin-Halperin form with no off-particle zeros.  $S_{\vec{a}}$  needs to be specified only for  $\vec{a}$  lying in the unit cell of the lattice spanned by  $\{\vec{n}_I\}$ . In this case we may use the volume of this unit cell as one measure of the complexity of a multilayer FQH state and as a guide to the stability and size of the energy gap of a FQH state.

One of the most crucial results of the pattern-of-zeros classification is that it gives us a broad perspective over a large class of FQH states. So we can determine, e.g. using the cluster structure, which states are the simplest non-Abelian generalizations of Halperin's wave functions and therefore which non-Abelian bilayer states are the most promising candidates to be realized experimentally.

In the following, we will limit ourselves to describing results for which the bilayer system is symmetric between the two layers, which is usually (but not always) the case in experiments. The simplest FQH states in this case are the Halperin  $(m, m, n)$  states,<sup>16</sup>

$$\Phi_{(m,m,n)} = \prod_{i < j} (z_i - z_j)^m \prod_{i < j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^n, \quad (2)$$

which describe incompressible and Abelian FQH states at  $\nu = \frac{2}{m+n}$ . Such a state has the simplest cluster structure described by  $(\vec{n}_1, \vec{n}_2)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . However these Abelian FQH states can only explain incompressible states at  $\nu = 2/p$ , where  $p = m+n$  is an integer. Experiments have also seen incompressible states in two-component systems at other filling fractions such as  $\nu = 4/5, 4/7, 6/7$ , etc.<sup>7-9</sup> The proposed states for these filling fractions are either two independent single-layer phases each of which is in a hierarchy state at  $\nu = 2/5, 2/7, 3/7$ , respectively, or some more complicated bilayer hierarchy (e.g., composite fermion) state. If the interlayer repulsion is comparable to the intralayer repulsion, the existence of two independent single-layer phases is not a viable possibility. In such a situation, it is unknown what incompressible state would form, if any. Our pattern-of-zeros classification yields non-Abelian states that, in addition to the bilayer composite fermion states, should be seriously considered under these circumstances.

For example, we find wave functions describing non-Abelian states at  $\nu = \frac{2}{m+n}$ , at which there are also Halperin  $(m, m, n)$  wave functions; the non-Abelian versions though have higher order zeros as particles from the different layers approach each other, indicating that they may obtain if interlayer Coulomb interactions are comparable to intralayer interactions. We also find interlayer-correlated non-Abelian states at  $\nu = 4/p$ , with  $p$  odd (e.g.,  $4/5, 4/7, 4/9$ ). These non-Abelian FQH phases may be more favorable than their Abe-

TABLE I. Quasiparticle minimal charges  $q_{min}$  and the corresponding scaling dimensions  $h$  for the non-Abelian bilayer states described in the given equations. The interedge quasiparticle tunneling  $I$ - $V$  curve has a form  $I \propto V^{4h-1}$  at  $T=0$ . In the scaling dimension, the first term comes from the non-Abelian part, the second term from the total density fluctuations [the  $U(1)$  part], and the third term from the relative density fluctuations of the two layers [also the  $U(1)$  part].

$\nu$		Charge $q_{min}$	Scaling dimension $h$
$2/3 _{\text{inter}}$	Equation (3)	1/3	$\frac{1}{16} + \frac{1}{12} + 0$
$2/3 _{\text{intra}}$	Equation (4)	1/6	$\frac{1}{16} + \frac{1}{48} + \frac{1}{16}$
4/5	Equation (5)	1/5	$\frac{1}{10} + \frac{1}{40} + \frac{1}{24}$
4/7	Equation (6)	1/7	$\frac{1}{10} + \frac{1}{56} + \frac{1}{8}$
4/9	Equation (6)	1/9	$\frac{1}{10} + \frac{1}{72} + \frac{1}{56}$
1/4	Equation (7)	1/8	$\frac{1}{16} + \frac{1}{32} + 0$

lian counterparts in regimes where a gapped bilayer FQH phase exists and where interlayer repulsion is also strong.

The first example that we discuss is the FQH plateau seen at  $\nu = 2/3$  in bilayer systems, for which experiments have already observed a phase transition between two FQH states.<sup>17</sup> The bilayer state at this filling fraction that is usually considered is the (3,3,0) Halperin state, which consists of two independent  $1/3$  Laughlin states in each layer. Another possible bilayer state is the Halperin (1,1,2) state but this wave function appears somewhat unrealistic since the order of zeros is larger when particles from different layers approach each other than particles from the same layer. The simplest non-Abelian bilayer states that we find appear to be more realistic; one is the following interlayer Pfaffian state:

$$\Psi_{2/3|_{\text{inter}}} = \text{Pf} \left( \frac{1}{x_i - x_j} \right) \Phi_{(2,2,1)}(\{z_i, w_i\}). \quad (3)$$

Here,  $x_i$  refers to the coordinates of all of the electrons. This interlayer Pfaffian state may be expected if the system is intrinsically bilayer but for which there is also strong interlayer repulsion. Then, instead of forming the (3,3,0) state, something like the (2,2,1) state would be more favorable. However the (2,2,1) state violates Fermi statistics, so we can think of adding the Pfaffian factor in order to convert it to a valid fermion wave function. Another non-Abelian bilayer state is the following state:

$$\Psi_{2/3|_{\text{intra}}} = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \text{Pf} \left( \frac{1}{w_i - w_j} \right) \Phi_{(2,2,1)}(\{z_i, w_i\}), \quad (4)$$

which has even stronger interlayer correlation. The  $\Psi_{2/3|_{\text{inter}}}$  state has  $2\frac{1}{2}$  edge modes (i.e., central charge  $c = 2\frac{1}{2}$ ) while the  $\Psi_{2/3|_{\text{intra}}}$  state has three edge modes.<sup>18</sup> If we use the number of edge modes to gauge the complexity of a FQH state, then the  $\Psi_{2/3|_{\text{intra}}}$  state is slightly more complicated than the  $\Psi_{2/3|_{\text{inter}}}$  state. For the cluster structure,  $\Psi_{2/3|_{\text{inter}}}$  has  $(\vec{n}_1, \vec{n}_2)^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$  and a minimal charge  $q_{min} = \nu/2$  while  $\Psi_{2/3|_{\text{intra}}}$  has  $(\vec{n}_1, \vec{n}_2)^T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  and a  $q_{min} = \nu/4$  (see Table I).

TABLE II. Proposed explanations for incompressible states at experimentally relevant filling fractions,  $\nu=2/3$ ,  $4/5$ ,  $4/7$ , and  $1/4$ , in two-component FQH systems. The bilayer composite fermion state  $(\nu_1, \nu_2|m)$  (Ref. 25) refers to the state  $\Pi_{i,j}(z_i - w_j)^m \Phi_{\nu_1}(\{z_i\}) \Phi_{\nu_2}(\{w_j\})$ , where  $\Phi_\nu$  is a single-layer composite fermion state at filling fraction  $\nu$ . For  $(2/3, 2/3|m)$ , we have taken the single-layer  $2/3$  state to be the particle-hole conjugate of the Laughlin state.  $n_R+n_L$  indicates that there are  $n_R$  right-moving edge modes and  $n_L$  left-moving edge modes.

$\nu$	Proposed states	Edge modes	Shift $S$
2/3	(3,3,0)	2	3
	(1,1,2)	2	1
	$2/3 _{\text{inter}}$ [see Eq. (3)]	$2\frac{1}{2}$	3
	$2/3 _{\text{intra}}$ [see Eq. (4)]	3	3
	$Z_4$ parafermion	3	3
	P-H conjugate of $\nu=1/3$	$1_R+1_L$	0
4/5	(2/5, 2/5 0)	4	4
	$su(3)_2/u(1)^2$ [see Eq. (5)]	$2\frac{6}{5}$	3
	(2/3, 2/3 1)	$2_R+2_L$	0
4/7	(2/7, 2/7 0)	4	2
	$su(3)_2/u(1)^2$ [see Eq. (6)]	$2\frac{6}{5}$	3
	(2/5, 2/5 1)	4	4
	(2/3, 2/3 2)	$1_R+3_L$	0
1/4	(5,5,3)	2	5
	(7,7,1)	2	7
	Interlayer Pfaffian [see Eq. (7)]	$2\frac{1}{2}$	7
	Single-layer Pfaffian	$1\frac{1}{2}$	5

This also suggests  $\Psi_{2/3|_{\text{intra}}}$  to be more complicated than  $\Psi_{2/3|_{\text{inter}}}$ .

The interlayer Pfaffian state  $\Psi_{2/3|_{\text{inter}}}$  has in fact been already constructed as a possible non-Abelian spin-singlet state.<sup>19</sup> Here, we stress that, according to our systematic classification, the non-Abelian states  $\Psi_{2/3|_{\text{inter}}}$  and  $\Psi_{2/3|_{\text{intra}}}$  are among the simplest of all non-Abelian bilayer states, which indicates that they may be experimentally viable and deserve further consideration.

Experiments have also observed a spin-unpolarized to spin-polarized phase transition in single-layer samples at  $\nu=2/3$ .<sup>4</sup> One candidate for the spin-unpolarized state is the (1,1,2) state which has only two edge modes. However, the (1,1,2) state has very different orders of intralayer and interlayer zeros. Thus the spin-singlet interlayer Pfaffian state  $\Psi_{2/3|_{\text{inter}}}$  may be more favorable than the (1,1,2) state if the electron repulsion is spin independent. Another main candidate for the spin-unpolarized state is a spin-singlet composite fermion state introduced in Ref. 20, which probably has the same topological order as the (1,1,2) state. For the single-component (or spin-polarized) phase, the candidate states are the particle-hole conjugate of the  $1/3$  Laughlin state and the non-Abelian  $Z_4$  parafermion state.

With so many different possibilities for the  $\nu=2/3$  FQH state in bilayer systems, which one is actually realized in a

particular sample? Two dimensionless quantities may be important. The first one is  $\alpha \equiv V_{\text{inter}}/V_{\text{intra}}$ , where  $V_{\text{inter}}$  is the potential for interlayer repulsion and  $V_{\text{intra}}$  is the potential for intralayer repulsion. The second one is  $\gamma \equiv t/V_{\text{intra}}$ , where  $t$  is the interlayer hopping amplitude. When  $\alpha \sim 0$  and  $\gamma \sim 0$ , the (3,3,0) state will be realized. If we keep  $\gamma \sim 0$  and increase  $\alpha$ , the interlayer non-Abelian Pfaffian states  $\Psi_{2/3|_{\text{inter}}}$  or  $\Psi_{2/3|_{\text{intra}}}$  may be realized. In the limit  $\alpha \sim 0$  and  $\gamma \gg 1$ , the single-layer  $\nu=2/3$  states are realized.

A particularly interesting case is the FQH plateau observed in two-component systems at  $\nu=4/5$ . There are few proposed explanations for two-component states at this filling fraction. The main proposal is that the incompressible state is described by two independent single layer systems, each in a  $2/5$ -hierarchy state. This is a reasonable possibility, considering the fact that experiments on bilayer and wide single-layer quantum wells see incompressible states at  $\nu=2/3$ ,  $4/5$ , and  $6/7$  simultaneously.<sup>7</sup> This is twice the main sequence that one sees in single-layer samples,  $1/3$ ,  $2/5$ , and  $3/7$ , respectively, which indicates that perhaps each layer is forming its own independent FQH state. However, when the interlayer repulsion between the two layers is increased while the interlayer tunneling remains small, then the system will undergo a phase transition into either an incompressible state or a compressible one.

If the system goes into a new incompressible state, then one possibility for such a state is the following  $\nu=4/5$  non-Abelian bilayer state:

$$\Psi(\{z_i, w_i\}) = \Phi_{sc}(\{z_i, w_i\}) \Phi_{(2,2,1/2)}(\{z_i, w_i\}), \quad (5)$$

where  $\Phi_{sc} = \langle \Pi_i \psi_1(z_i) \psi_2(w_i) \rangle$  is a correlation function in the  $su(3)_2/u(1)^2$  parafermion CFT (Ref. 21), and  $\psi_1$  and  $\psi_2$  are Majorana fermions with scaling dimension  $1/2$ . Some explicit expressions for such correlation functions were discussed in Ref. 22. This is another one of the simplest non-Abelian bilayer states that we find in our systematic classification of multilayer FQH states. It is closely related to the non-Abelian spin-singlet states at  $\nu = \frac{4}{4k+3}$  that were proposed in Ref. 23 ( $k$  is an odd or even integer for fermionic or bosonic FQH states, respectively).

The other major possibility for an incompressible state at  $\nu=4/5$  is that the system forms a bilayer hierarchy state, with interlayer correlations, which would be described by a  $4 \times 4$   $K$  matrix<sup>24</sup> and would have four edge modes. An example is the  $(2/3, 2/3|1)$  bilayer composite fermion state.<sup>25</sup> The primary question then is whether it is more favorable for the system to form an Abelian hierarchy state or a non-Abelian state. The  $su(3)_2/u(1)^2$  non-Abelian state, having only  $2\frac{6}{5}$  edge modes, is simpler than the  $(2/3, 2/3|1)$  state. Thus, the  $su(3)_2/u(1)^2$  non-Abelian state may be more likely to appear. All of the states based on  $su(3)_2/u(1)^2$  have a cluster structure  $(\vec{n}_1, \vec{n}_2)^T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ , and a minimal charge  $q_{\text{min}} = \nu/4$  (see Table I).

Similar discussions hold also for FQH states at  $\nu=4/7$  and  $\nu=4/9$ . An incompressible state has been observed at  $\nu=4/7$  in wide quantum wells,<sup>7</sup> but not at  $\nu=4/9$ . On the other hand, phase transitions have been observed at these filling fractions in single layer systems, purportedly between

spin-polarized and spin-unpolarized states.<sup>8</sup> This suggests an incompressible state at  $\nu=4/9$  may also be observed in bilayer or wide single-layer quantum wells if the system can be made clean enough and the interlayer repulsion be made comparable to the intralayer repulsion while keeping the interlayer tunneling small. Among the simplest non-Abelian bilayer states that we find through the pattern-of-zeros classification is the non-Abelian spin-singlet state at  $\nu=4/7$ , which was already proposed in Ref. 23 and a close relative at  $\nu=4/9$ ,

$$\begin{aligned} \Phi_{sc}(\{z_i, w_i\})\Phi_{(2,2,3/2)}(\{z_i, w_i\}) \quad \nu=4/7, \\ \Phi_{sc}(\{z_i, w_i\})\Phi_{(4,4,1/2)}(\{z_i, w_i\}) \quad \nu=4/9. \end{aligned} \quad (6)$$

As before,  $\Phi_{sc} = \langle \Pi_i \psi_1(z_i) \psi_2(w_i) \rangle$  is a correlation function in the  $su(3)_2/u(1)^2$  parafermion CFT.

Recently, an incompressible state was found at  $\nu=1/4$  and it is unclear what phase this corresponds to and even whether it is a single-layer or bilayer phase.<sup>26</sup> Some possibilities that have recently been considered<sup>27</sup> are the (5,5,3) and (7,7,1) Halperin states and the  $\nu=1/4$  single-layer Pfaffian. The pattern-of-zeros construction yields many other alternative possibilities, perhaps the most physical (and simplest) of which is the following interlayer Pfaffian,

$$\Psi(\{z_i, w_i\}) = \text{Pf} \left( \frac{1}{x_i - x_j} \right) \Phi_{(6,6,2)}(\{z_i, w_i\}). \quad (7)$$

A useful tool for identifying FQH states in numerical studies of exact diagonalization on finite systems on a sphere is to look at what values of the shift,  $S = \nu^{-1}N_e - N_\Phi$ , a ground state with zero total angular momentum is found.<sup>28</sup> This then limits the possibilities of which topological phase is realized in the system to those that have that particular value of the

shift. Similarly, in numerical studies of multilayer systems, one can look for the different sets  $(N_1, \dots, N_f; N_\Phi^1, \dots, N_\Phi^f)$  that yield a ground state with zero total angular momentum. Here  $N_I$  and  $N_\Phi^I$  are the number of particles and number of flux quanta, respectively, in the  $I$ th layer ( $I=1, \dots, f$ ). Each topological phase will have its own list of  $(N_1, \dots, N_f; N_\Phi^1, \dots, N_\Phi^f)$  that let it fill the sphere; analyzing this can be a useful way of determining which topological phase is obtained numerically. In Ref. 13, we have found conditions that  $\tilde{N}$  and  $\tilde{N}_\Phi$  should satisfy for the FQH state to fill the sphere. For the states presented here,  $N_1$  and  $N_2$  must be even, and they determine  $N_\Phi^1$  and  $N_\Phi^2$  through  $\binom{N_\Phi^1 + S}{N_\Phi^1 + S} = M \binom{N_1}{N_2}$ , where  $S$  is the shift on the sphere and  $M$  is a matrix. For these states, which are of the form  $\Phi = \Phi_{sc} \Phi_{(m,m,l)}$ ,  $M = \binom{m}{l} \binom{l}{m}$ . The values of the shifts are listed in Table II.

Finally, we briefly comment on the relation between the pattern-of-zeros approach and other approaches recently developed involving either thin torus limits of FQH systems or the clustering structure of FQH wave functions.<sup>29-32</sup> In the case of single-component FQH wave functions, these approaches yield similar data that characterize FQH states. However the spirit is quite different, and the pattern-of-zeros approach is unique in that it is a general attempt at systematically classifying all possible ideal wave functions. Furthermore, the multilayer pattern-of-zeros characterization presented here does not have any parallel in the other approaches. Thin torus limits of multicomponent FQH wave functions have been taken<sup>33</sup> but the data that characterizes them is quite different than the structure we obtain in the pattern-of-zeros approach.

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