

# Banded Matrices With Banded Inverses

by

Venugopalan Srinivasa Gopala Raghavan

B.Eng (Aero), Nanyang Technological University, Singapore (2009)

Submitted to the School of Engineering

in partial fulfillment of the requirements for the degree of

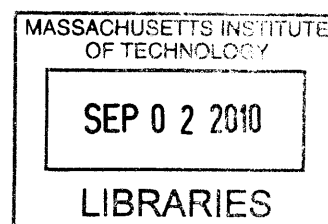
Master of Science in Computation for Design and Optimization

**ARCHIVES**

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2010




© Massachusetts Institute of Technology 2010. All rights reserved.

Author .....

School of Engineering


July 30, 2010

Certified by ..... 

Gilbert Strang

Professor of Mathematics

Thesis Supervisor

Accepted by ..... 

Karen E. Willcox

Associate Professor of Aeronautics and Astronautics

Codirector, Computation for Design and Optimization Program



# **Banded Matrices With Banded Inverses**

by

Venugopalan Srinivasa Gopala Raghavan

Submitted to the School of Engineering

on July 30, 2010 in partial fulfillment of the

requirements for the degree of

Master of Science in Computation for Design and Optimization

## **Abstract**

We discuss the conditions that are necessary for a given banded matrix to have a banded inverse. Although a generic requirement is known from previous studies, we tend to focus on the ranks of the block matrices that are present in the banded matrix. We consider mainly the two factor 2-by-2 block matrix and the three factor 2-by-2 block matrix cases. We prove that the ranks of the blocks in the larger banded matrix need to necessarily conform to a particular order. We show that for other orders, the banded matrix in question may not even be invertible.

We are then concerned with the factorization of the banded matrix into simpler factors. Simpler factors that we consider are those that are purely block diagonal. We show how we can obtain the different factors and develop algorithms and codes to solve for them. We do this for the two factor 2-by-2 and the three factor 2-by-2 matrices. We perform this factorization on both the Toeplitz and non-Toeplitz case for the two factor case, while we do it only for the Toeplitz case in the three factor case.

We then look at extending our results when the banded matrix has elements at its corners. We show that this case is not very different from the ones analyzed before. We end our discussion with the solution for the factors of the circulant case. Appendix A deals with a conjecture about the minimum possible rank of a permutation matrix. Appendices B & C deal with some of the miscellaneous properties that we obtain for larger block matrices and from extending some of the previous work done by Strang in this field.

Thesis Supervisor: Gilbert Strang

Title: Professor of Mathematics



## Acknowledgements

I would like to take this opportunity to thank all the people who have helped me in completing my thesis and also during my journey at MIT.

First and foremost I would like to thank my thesis supervisor Professor Gilbert Strang. He encouraged me to think creatively and differently and was always very willing and patient in explaining many ideas to me which seemed very abstract to me. His suggestions when I got stuck, and feedbacks on a wide variety of issues were truly invaluable and hopefully helped to make me a better researcher. I am truly privileged and indebted to have had the opportunity to work with you, sir.

I would like to thank the administrative staff at MIT for all their help with the formal procedures and who have helped to ensure that I concentrate more on my courses and thesis. Huge thanks to Mrs. Laura Koller without whose patience and timely reminders I would have missed quite a few of the administrative procedures. Thank you!

I would like to thank the MIT IS&T for helping me out innumerable number of times when my computer caused issues and in providing me with needed facilities so that I could concentrate on my work as opposed to troubleshooting. Thank you very much!

I would also like to thank my friends here at MIT and especially at CDO. The support I got from them helped me weather a few storms that I faced. Thanks all!

Last but definitely not the least, I would like to thank my family members who have supported me through thick and thin and whose confidence in me has never ever wavered. Thanks Mom! Thanks Dad! Thanks Venkat! Couldn't have done it without you all! This thesis is dedicated especially to you all!



# Contents

<b>1</b>	<b>Introduction .....</b>	<b>9</b>
1.1	Banded Matrices .....	9
1.2	Notation with Determinants .....	13
<b>2</b>	<b>Banded Matrices – Two factors with 2-by-2 blocks .....</b>	<b>19</b>
2.1	Toeplitz Case .....	19
2.2	Solution Process: Toeplitz Case .....	26
2.3	Sample Problem: Toeplitz Case .....	28
2.4	Non-Toeplitz Case .....	29
2.5	Solution Process: Non-Toeplitz Case .....	30
2.6	Tri-Diagonal Matrices .....	32
<b>3</b>	<b>Banded Matrices – Three factors with 2-by-2 blocks .....</b>	<b>35</b>
3.1	Toeplitz Case .....	35
3.2	Solution Process: Toeplitz Case .....	57
3.3	Sample Problem: Toeplitz Case .....	61
<b>4</b>	<b>Banded Matrices – Three factors with 3-by-3 blocks .....</b>	<b>63</b>
<b>5</b>	<b>Circulant Matrices .....</b>	<b>69</b>
5.1	Two factors with 2-by-2 blocks .....	69
5.2	Solution Process: Toeplitz Case .....	70

5.3	Three factors with 2-by-2 blocks – Toeplitz .....	73
<b>6</b>	<b>Conclusions and Future Work .....</b>	<b>77</b>
<b>7</b>	<b>Appendices .....</b>	<b>79</b>
	Appendix A – Conjecture for a n-by-n matrix formed from the various permutations of n elements .....	79
	Appendix B – Miscellaneous Properties .....	85
	Appendix C – Extension of results obtained in [1] .....	89



# Chapter 1

## Introduction

### 1.1 Banded Matrices

Matrices are an integral part in the effective working of many of the modern day technologies. The compression of pictures and the prediction of flow over a model fighter aircraft under extremely complicated maneuvers are two examples. In the first case, we make use of the fast Fourier transform that is extremely effective in dealing with very large systems of matrices [1]. In the second case we construct extremely large matrices that need to be inverted quickly, in order to get the solution vector such as the velocity of the aircraft and also other control parameters.

In general, inverting a matrix is a memory intensive and operation intensive process. There are a lot of algorithms that are used to compute the inverse of a matrix and consequently to solve for a linear system of equations. We have the common Gaussian elimination method, the LU decomposition and so on. The Gaussian elimination is of complexity  $O(n^3)$ [2] where  $n$  is the size of the square matrix. Please note that the number of operations quoted is for the situation where the matrices are full.

The vast majority of matrices that occur in nature are generally sparse. However, there is no guarantee that a sparse matrix will have a sparse inverse. In general this is not the case. A sparse matrix can have an inverse that is full which in turn can increase the overhead when computing solutions to linear equations. However, there are certain conditions under which a sparse matrix will also have a sparse inverse. It is this kind of scenario that we are interested in.

The simplest possible case of a sparse matrix having a sparse inverse is when the matrix under consideration is purely diagonal. In this case the inverse is simply the inverse of each of the elements on the diagonal. We then move on to the scenario where there are non-zero elements in positions apart from the main diagonal. However we restrict the non-zero elements to a small *band* on either side of the main diagonal, as described in [1]. For the time being we shall consider a matrix that has elements only next to the main diagonal – we call this type of matrix a *tri-diagonal matrix*. A typical tri-diagonal matrix would be the 1D Laplacian finite-difference matrix (K).

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad K^{-1} = \frac{1}{7} \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

While K is indeed banded, its inverse is unfortunately not. In fact we can later try to examine the conditions under which a tri-diagonal matrix's inverse will also be banded.

There are cases when banded matrices also produce banded inverses. These are the cases we are interested in. A typical example of a banded matrix having a banded inverse is shown next

$$L = \begin{bmatrix} 12 & 7 & 20 & 0 & 0 & 0 \\ 60 & 7 & 20 & 0 & 0 & 0 \\ 0 & 82 & 12 & 7 & 20 & 0 \\ 0 & 410 & 60 & 7 & 20 & 0 \\ 0 & 0 & 0 & 82 & 12 & 7 \\ 0 & 0 & 0 & 410 & 60 & 7 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} -1/48 & 1/48 & 0 & 0 & 0 & 0 \\ -15/1556 & 3/1556 & -5/1556 & 5/1556 & 0 & 0 \\ 205/3112 & -41/3112 & 7/6224 & -7/6224 & 0 & 0 \\ 0 & 0 & -15/1556 & 3/1556 & -5/1556 & 5/1556 \\ 0 & 0 & 205/3112 & -41/3112 & 7/6224 & -7/6224 \\ 0 & 0 & 0 & 0 & 5/28 & -1/28 \end{bmatrix}$$

Why are we so interested in banded matrices and their inverses? Why take the time and effort to figure out under what conditions the banded matrix will have a banded inverse also? The reason is that if the inverse is also banded, then the speed of the linear transformation performed on an input vector will also be very fast. By being banded we also reduce the memory requirement when dealing with large systems. Computing inverses both ways will be extremely fast.

We wish to make another point here – while we are definitely interested in finding out under what conditions we can get banded inverses, we are also very interested in trying to figure out if we can factorize the banded matrices into much simpler factors. Ideally we are looking to get them as block diagonal matrices for which the inverse can be obtained very easily. However we will also look at some of the other cases where the elements are present on the minor diagonal and try to figure out if we can extend our methods.

In this work we look at both the forward and reverse directions. In the forward direction, we take the product of simple factors and then examine the structure of the product matrix. We do this to gain a better understanding of what exactly is present in the final product and how it comes about. Once we are done with the forward problem, we try to go in the reverse direction in order to see if we can solve for the individual factors. Please note that the factors we get may not be

the exact same individual factors that were used to construct the product in the first place. However, the product of the factors that we obtain matches the given matrix to within machine precision.

As far as possible we will be looking at both the Toeplitz as well as the non-Toeplitz case. For completeness sake, we furnish here examples of both cases (K=Toeplitz and M=Non-Toeplitz; the repeated elements in the Toeplitz case are highlighted in different colors):

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 43 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 3 & -9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & 5 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 43 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -25 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 9 \end{bmatrix}$$

We will first look at the case where we have only two factors  $F_1$  and  $F_2$ . Each of these is a block diagonal matrix that has full rank 2-by-2 matrices on the diagonal. We will then move on to the case where there are three factors  $F_1$ ,  $F_2$  and  $F_3$ .  $F_1$  and  $F_3$  will share the same structure and all three will be made up of 2-by-2 blocks. For this particular case, we will only consider the Toeplitz form for  $F_1$ ,  $F_2$  and  $F_3$ . In each of the cases, we will also look to obtain the factors. We then move on to the case with three factors  $G_1$ ,  $G_2$  and  $G_3$ , each of which is made up of 3-by-3 blocks. We will however, not be looking at formulae for the different factors because of the complexity of the problem in the three factor 3-by-3 block case. We will also study the effect of having extra non-zero elements that are far away from the centre – the cyclic/circulant cases. Finally, we will also look at the means to solve for them using the codes that we developed for the non-cyclic cases.

The entire work has been done using MATLAB R2008a. The source codes used in this thesis can be obtained by email from [sven.mit@gmail.com](mailto:sven.mit@gmail.com)

## 1.2 Notation With Determinants

Consider first a banded matrix that has two block matrices on every two rows. Let us call these block matrices R and S. Previous studies have shown that for banded matrices, the inverse is also banded only if the following holds [1]:

$$\det(M) = \text{monomial}, \quad M = R + Sz, \quad z = \text{arbitrary variable}$$

Let us look closely at the two matrices R and S. Let

$$R = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \text{ and } S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$$

We then have for M:

$$M = \begin{bmatrix} r_1 + s_1z & r_2 + s_2z \\ r_3 + s_3z & r_4 + s_4z \end{bmatrix}$$

which then leads to

$$\det(M) = \begin{vmatrix} r_1 + s_1z & r_2 + s_2z \\ r_3 + s_3z & r_4 + s_4z \end{vmatrix} = (r_1 + s_1z)(r_4 + s_4z) - (r_2 + s_2z)(r_3 + s_3z)$$

It can be seen that the highest degree in the product is  $z^2$ . The different terms can be written as:

$$\det(M) = (r_1r_4 - r_2r_3) + (s_1r_4 - s_2r_3 + s_3r_2 - s_4r_1)z + (s_1s_4 - s_2s_3)z^2$$

This can be re-ordered as follows:

$$\det(M) = \det(R) + (\det(P) + \det(Q))z + (\det(S))z^2$$

Where

$$P = \begin{bmatrix} s_1 & s_2 \\ r_3 & r_4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} r_1 & r_2 \\ s_3 & s_4 \end{bmatrix}$$

Essentially P and Q are matrices that embody the combination principle  $\binom{n}{k}$

We then have:

$$\text{constant term} = \det(R) \quad \text{coefficient of } z^2 = \det(S) \quad \text{coefficient of } z = \det(P) + \det(Q)$$

This method of writing the coefficients of the powers of z in the form of determinants makes it easier to understand which terms vanish and which terms remain. It also opens up the possibility of exploring if there is more than one combination that can ensure that makes the matrix invertible and if so is it banded. The proofs for some of these cases are shown later for which the determinant notation is made use of extensively.

The case that was illustrated was the rather simple case with only two blocks, each of which is a 2-by-2 block. In order to prove the effectiveness of this notation, we look at two other cases – one when we have three 2-by-2 blocks instead of just two (and consequently, three factors) and the other is when we have three 3-by-3 blocks.

Let us first consider the case of three 2-by-2 blocks. Let us denote the blocks by  $R_1$ ,  $S_1$  and  $T_1$ :

$$R_1 = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \quad S_1 = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \quad T_1 = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$$

$$M_1 = R_1 + S_1 z + T_1 z^2$$

$$M_1 = \begin{bmatrix} r_1 + s_1 z + t_1 z^2 & r_2 + s_2 z + t_2 z^2 \\ r_3 + s_3 z + t_3 z^2 & r_4 + s_4 z + t_4 z^2 \end{bmatrix}$$

The order of the determinant of M in this case would be 4 (since  $z^2 \cdot z^2 = z^4$ ). The determinant itself would be:

$$\det(M_1) = (r_1 + s_1z + t_1z^2)(r_4 + s_4z + t_4z^2) - (r_2 + s_2z + t_2z^2)(r_3 + s_3z + t_3z^2)$$

$$\begin{aligned} \det(M_1) = & (r_1r_4 - r_2r_3) + (r_1s_4 - r_2s_3 + r_4s_1 - r_3s_2)z + (s_1s_4 - s_2s_3 + r_1t_4 - r_2t_3 + r_4t_1 - r_3t_2)z^2 \\ & + (s_1t_4 - s_2t_3 + s_4t_1 - s_3t_2)z^3 + (t_1t_4 - t_2t_3)z^4 \end{aligned}$$

Which can once again be written as:

$$\begin{aligned} \det(M_1) = & \det(R_1) + (\det(P_1) + \det(Q_1))z + (\det(S_1) + \det(U_1) + \det(V_1))z^2 \\ & + (\det(W_1) + \det(X_1))z^3 + (\det(T_1))z^4 \end{aligned}$$

Where:

$$P_1 = \begin{bmatrix} r_1 & r_2 \\ s_3 & s_4 \end{bmatrix} \quad Q_1 = \begin{bmatrix} s_1 & s_2 \\ r_3 & r_4 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} r_1 & r_2 \\ t_3 & t_4 \end{bmatrix} \quad V_1 = \begin{bmatrix} t_1 & t_2 \\ r_3 & r_4 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} s_1 & s_2 \\ t_3 & t_4 \end{bmatrix} \quad X_1 = \begin{bmatrix} t_1 & t_2 \\ s_3 & s_4 \end{bmatrix}$$

Now, we can extend it to the case for the 3-by-3 blocks. Let us denote the blocks as  $R_2$ ,  $S_2$  and  $T_2$ , where  $R_2$ ,  $S_2$  and  $T_2$  are given as:

$$R_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad S_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad T_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix}$$

$$M_2 = R_2 + (S_2)z + (T_2)z^2$$

$$M_2 = \begin{bmatrix} r_1 + s_1z + t_1z^2 & r_2 + s_2z + t_2z^2 & r_3 + s_3z + t_3z^2 \\ r_4 + s_4z + t_4z^2 & r_5 + s_5z + t_5z^2 & r_6 + s_6z + t_6z^2 \\ r_7 + s_7z + t_7z^2 & r_8 + s_8z + t_8z^2 & r_9 + s_9z + t_9z^2 \end{bmatrix}$$

The highest power in the determinant of  $M_2$  is 6. The coefficients of the various terms are given by

$$\begin{aligned}
\det(M_2) = & \det(R_2) + (\det(P_2) + \det(Q_2) + \det(U_2))z \\
& + (\det(V_2) + \det(W_2) + \det(X_2) + \det(Y_2) + \det(Z_2) + \det(A_2))z^2 \\
& + (\det(S_2) + \det(B_2) + \det(C_2) + \det(D_2) + \det(E_2) + \det(F_2) + \det(G_2))z^3 \\
& + (\det(H_2) + \det(I_2) + \det(J_2) + \det(K_2) + \det(L_2) + \det(N_2))z^4 \\
& + (\det(O_2) + \det(AA_2) + \det(AB_2))z^5 + (\det(T_2))z^6
\end{aligned}$$

Where:

$$P_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad Q_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_4 & s_5 & s_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad U_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad W_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ t_4 & t_5 & t_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad X_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad Z_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ r_4 & r_5 & r_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad A_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_4 & s_5 & s_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad C_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ r_4 & r_5 & r_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad D_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ t_4 & t_5 & t_6 \\ s_7 & s_8 & s_9 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ t_4 & t_5 & t_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad F_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ s_4 & s_5 & s_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad G_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ r_4 & r_5 & r_6 \\ s_7 & s_8 & s_9 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad I_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ r_4 & r_5 & r_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad J_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad L_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ t_4 & t_5 & t_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad N_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{bmatrix}$$



$$O_2 = \begin{bmatrix} s_1 & s_2 & s_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \quad AA_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ s_7 & s_8 & s_9 \end{bmatrix} \quad AB_2 = \begin{bmatrix} t_1 & t_2 & t_3 \\ s_4 & s_5 & s_6 \\ t_7 & t_8 & t_9 \end{bmatrix}$$

Naturally, this can be extended to any number of matrices of any order. We can see a particular order in each of the matrices – the entries of any one row come from the same row of the matrices  $R_2$ ,  $S_2$  and  $T_2$ . For example, in  $AA_2$ , the last row comes from the last row of the matrix  $S_2$ , while the first two rows come from the first two rows of  $T_2$ .

This notation is very powerful and its power will be seen when we attempt to prove a couple of results involving the ranks of the block matrices.



# Chapter 2

## Banded Matrices – Two factors with 2-by-2 blocks

### 2.1 Toeplitz Case

Let us assume that there are two matrices  $F_1$  and  $F_2$  and each of them is made up of constantly repeating 2-by-2 blocks. The matrix  $F_1$  has blocks only on its diagonal, while the matrix  $F_2$  has its elements shifted one down and one to the right. We then take the product of  $F_1$  and  $F_2$ . The resulting matrix is found to have two blocks per row. The structures of the matrices  $F_1$ ,  $F_2$  and the product  $F_{12}$  are shown next:

$$F_1 = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix} \quad F_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \beta = b_{11}$$

$$F_{12} = \begin{bmatrix} ka_{11} & a_{12}b_{11} & a_{12}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ka_{21} & a_{22}b_{11} & a_{22}b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{11} & a_{12}b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{11} & a_{22}b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{11} & a_{12}b_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{11} & a_{22}b_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{11} & a_{12}b_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{11} & a_{22}b_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{11} \end{bmatrix}$$

We see that the ranks of the repeated matrices in the product are both of rank one. The rank-one matrices are obtained from the product of a column vector with a row vector. We show now that in order to have exactly two matrices for every two rows in the product and have a banded inverse, the ranks of each of them *necessarily need to be unity*.

Let the two matrices present in the product be R and S. Let the elements of the matrices be:

$$R = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$$

Construct M as usual:

$$M = R + Sz$$

This then leads to:

$$\text{constant term} = \det(R) \quad \text{coefficient of } z^2 = \det(S) \quad \text{coefficient of } z = \det(P) + \det(Q)$$

$$Q = \begin{bmatrix} s_1 & s_2 \\ r_3 & r_4 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} r_1 & r_2 \\ s_3 & s_4 \end{bmatrix}$$

**Case i: rank(R) = 2, rank(S) = 2: Not possible**

In this case,

$$\text{constant term} = \det(R) \neq 0, \quad \text{coefficient of } z^2 = \det(S) \neq 0$$

$$\Rightarrow \det(M) = \text{atleast } (\text{constant} + \alpha z^2) \neq \text{monomial}$$

Hence the case of (2, 2) (the first is the rank of R and the second is the rank of S) would definitely not yield a banded inverse.

**Case ii: rank(R) = 1, rank(S) = 2: Not possible**

The next option would be to look at the case when one rank is 1 and the other rank is 2. Let us assume for simplicity that the rank of R is one and the rank of S is two. We are currently looking at the ordered pair (1, 2) of the ranks. This then leads to:

$$\text{constant term} = \det(R) = 0, \quad \text{coefficient of } z^2 = \det(S) \neq 0$$

Clearly, if we want the inverse of the banded matrix to be banded, we need that the coefficient of  $z$  be zero. (We already have a non-zero term in the coefficient of  $z^2$ , as S is of full rank):

$$\text{coefficient of } z = 0 = \det(P) + \det(Q)$$

The matrix R is of rank one. So, we can re-write the elements of R as:

$$r_2 = pr_1, r_3 = kr_1 \Rightarrow r_4 = kpr_1$$

So then we have:

$$Q = \begin{bmatrix} s_1 & s_2 \\ kr_1 & kpr_1 \end{bmatrix} \text{ and } P = \begin{bmatrix} r_1 & pr_1 \\ s_3 & s_4 \end{bmatrix}$$

$$\det(P) = r_1 s_4 - pr_1 s_3, \quad \det(Q) = kpr_1 s_1 - kr_1 s_2$$

$$\det(P) + \det(Q) = r_1 s_4 - pr_1 s_3 + kpr_1 s_1 - kr_1 s_2 = 0$$

$r_1$  definitely cannot be zero. So we can cancel it from all the terms in the equation. We then get:

$$s_4 - ps_3 + kps_1 - ks_2 = 0$$

$$s_4 - ks_2 = p(s_3 - ks_1)$$

Now in the big matrix we have the matrices R and S side-by-side. Using the first row of R to perform elimination leads to:

$$H = \begin{bmatrix} x_1 & s_1 & s_2 & 0 & 0 & 0 \\ x_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & s_2 & 0 \\ 0 & kr_1 & kpr_1 & s_3 & s_4 & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & kr_1 & kpr_1 & s_3 \end{bmatrix}$$

$$\Rightarrow H_i = \begin{bmatrix} x_1 & s_1 & s_2 & 0 & 0 & 0 \\ x_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & s_2 & 0 \\ 0 & 0 & 0 & (s_3 - ks_1) & (s_4 - ks_2) & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & 0 & 0 & (s_3 - ks_1) \end{bmatrix}$$

$$\Rightarrow H_{ii} = \begin{bmatrix} x_1 & s_1 & s_2 & 0 & 0 & 0 \\ x_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & s_2 & 0 \\ 0 & 0 & 0 & (s_3 - ks_1) & p(s_3 - ks_1) & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & 0 & 0 & (s_3 - ks_1) \end{bmatrix}$$

$$\Rightarrow H_{iii} = \begin{bmatrix} x_1 & s_1 & s_2 & 0 & 0 & 0 \\ x_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & s_2 & 0 \\ 0 & 0 & 0 & (s_3 - ks_1) & p(s_3 - ks_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & (s_3 - ks_1) \end{bmatrix}$$

$$\Rightarrow H_{iv} = \begin{bmatrix} x_1 & s_1 & s_2 & 0 & 0 & 0 \\ x_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & s_2 & 0 \\ 0 & 0 & 0 & (s_3 - ks_1) & p(s_3 - ks_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly we see that there is a zero row in  $H_{iv}$ . This means that any matrix that is made up of R and S with ranks 1 and 2 respectively (or vice-versa) will not be invertible (if we impose that the matrix have a banded inverse also)

By a similar argument, we can prove that the same holds (i.e. the matrix will not be invertible) if R is of full rank and S has rank one.

**Case iii: rank(R) = 1, rank(S) = 1: Possible**

We write R and S as:

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} \text{ and } S = \begin{bmatrix} s_1 & qs_1 \\ ls_1 & lqs_1 \end{bmatrix}$$

Then we have:

$$\text{constant term} = \det(R) = 0, \quad \text{coefficient of } z^2 = \det(S) = 0$$

$$\text{coefficient of } z = \det(P) + \det(Q)$$

$$P = \begin{bmatrix} r_1 & pr_1 \\ ls_1 & lqs_1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} s_1 & qs_1 \\ kr_1 & kpr_1 \end{bmatrix}$$

$$\det(P) = \begin{vmatrix} r_1 & pr_1 \\ ls_1 & lqs_1 \end{vmatrix} = r_1 s_1 (lq - lp), \quad \det(Q) = \begin{vmatrix} s_1 & qs_1 \\ kr_1 & kpr_1 \end{vmatrix} = r_1 s_1 (kp - kq)$$

$$\det(P) + \det(Q) = r_1 s_1 (lq - lp) + r_1 s_1 (kp - kq) = r_1 s_1 (l - k)(q - p)$$

Now, we need to take care to ensure that the coefficient of z is not zero. For this, examine where it does become zero.

$$\text{coefficient of } z = 0 = r_1 s_1 (l - k)(q - p) = 0$$

Clearly,  $s_1$  and  $r_1$  are not zero. The other possibilities are:

$$l = k \text{ or } q = p$$

Consider first that  $l = k$ :

$$\text{mat} = \begin{bmatrix} x_1 & s_1 & qs_1 & 0 & 0 & 0 \\ x_2 & ls_1 & lqs_1 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & qs_1 & 0 \\ 0 & lr_1 & lpr_1 & ls_1 & lqs_1 & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & lr_1 & lpr_1 & ls_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 & s_1 & qs_1 & 0 & 0 & 0 \\ x_2 & ls_1 & lqs_1 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & qs_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are a couple of zero rows which make the matrix singular. A similar argument applies for the case when  $p=q$ .

$$\text{mat} = \begin{bmatrix} x_1 & s_1 & ps_1 & 0 & 0 & 0 \\ x_2 & ls_1 & lps_1 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & ps_1 & 0 \\ 0 & kr_1 & kpr_1 & ls_1 & lps_1 & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & kr_1 & kpr_1 & ks_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 & s_1 & 0 & 0 & 0 & 0 \\ x_2 & ls_1 & 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & s_1 & 0 & 0 \\ 0 & kr_1 & 0 & ls_1 & 0 & 0 \\ 0 & 0 & 0 & r_1 & 0 & s_1 \\ 0 & 0 & 0 & kr_1 & 0 & ks_1 \end{bmatrix}$$

Thus as long as the ratios of the two matrices R and S are not the same then, the coefficient of z is non-zero and in that case it is possible to get an banded inverse matrix.

It can be shown that as long as the two conditions are not met, the reduced row echelon form of the big matrix is the identity matrix. This implies that the matrix indeed is of full rank. And by construction, it satisfies the conditions necessary to have a banded inverse.

$$\text{mat} = \begin{bmatrix} x_1 & s_1 & qs_1 & 0 & 0 & 0 \\ x_2 & ls_1 & lqs_1 & 0 & 0 & 0 \\ 0 & r_1 & pr_1 & s_1 & qs_1 & 0 \\ 0 & kr_1 & kpr_1 & ls_1 & lqs_1 & 0 \\ 0 & 0 & 0 & r_1 & pr_1 & s_1 \\ 0 & 0 & 0 & kr_1 & kpr_1 & ls_1 \end{bmatrix} \Rightarrow \text{rref}(\text{mat}) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



**Remark:** Please note that if there exist factors  $F_1$  and  $F_2$  that upon multiplication give matrices  $R$  and  $S$ , then we would have:

$$R = \begin{bmatrix} a_{11}b_{21} & a_{11}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} \end{bmatrix} \Rightarrow k = \frac{a_{21}}{a_{11}} \text{ and } p = \frac{b_{22}}{b_{21}}$$

$$S = \begin{bmatrix} a_{12}b_{11} & a_{12}b_{12} \\ a_{22}b_{11} & a_{22}b_{12} \end{bmatrix} \Rightarrow l = \frac{a_{22}}{a_{12}} \text{ and } q = \frac{b_{12}}{b_{11}}$$

So if we have

i)  $l = k$  it means:

$$\frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}}$$

$$\Rightarrow a_{22}a_{11} - a_{12}a_{21} = 0$$

$$\text{But: } a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(a) = a_{22}a_{11} - a_{12}a_{21}$$

$$\Rightarrow \det(a) = 0$$

This would mean that the matrix  $F_1$  is not invertible in the first place. But we are looking for factors  $F_1$  and  $F_2$  such that they are invertible. Hence we would need to ensure that  $l$  should not be the same as  $k$

ii)  $p = q$  means:

$$\frac{b_{12}}{b_{11}} = \frac{b_{22}}{b_{21}} \Rightarrow b_{22}b_{11} - b_{12}b_{21} = 0$$

$$\Rightarrow \det(b) = 0$$

In this case,  $F_2$  would be non-invertible and we would be stuck with the same scenario as in i).

Thus we need to ensure that both  $l$  does not equal  $k$  and  $p$  does not equal  $q$ .

Next we look at the solution process for the Toeplitz case.

## 2.2 Solution Process: Toeplitz Case

We now look at the solution process for obtaining the 2-by-2 blocks in the factors, for both the Toeplitz and the non-Toeplitz case. We will first consider the Toeplitz case and then extend the same to the non-Toeplitz case. Let us call the two factors as  $F_1$  and  $F_2$ . Now we need to obtain the blocks that are present in each of  $F_1$  and  $F_2$ .

Denote the two rank one matrices in the product as 'x' and 'y'; we call the first two elements of the first column of  $F_{12}$  as the vector 'w'. Also, let us call the first and second columns of the matrix  $a$  to be  $a_1$  and  $a_2$ . We call the rows of the matrix 'b' to be  $r'_1$  and  $r'_2$ . Then we have the following relations:

$$a_1 r'_2 = x, \quad a_1 r'_1 = y, \quad k a_1 = w, \quad a_2 b_{11} = y_1$$

where  $y = [\underline{y_1} \quad \underline{y_2}]$

To solve this set of equations, we need to set some of the entries of the matrices 'a' and 'b'. The simplest would be to set 'b<sub>11</sub>' to 1 and choose a random value for the variable 'k'. We then get:

$$a_1 = \frac{w}{k} \text{ and } a_2 = y_1$$

(i.e.) the second column of the matrix  $a$  is the same as the first column of the matrix 'y' in the product.

Now that the vectors  $a_1$  and  $a_2$  are completely known, these can be used in order to solve for the other unknowns –  $r'_1$  and  $r'_2$ . This is accomplished as follows:

$$a_1 r'_2 = x$$

Multiply both sides with  $a_1'$

$$(a_1'a_1)r_2' = a_1'x$$

$$r_2' = \frac{(a_1'x)}{(a_1'a_1)}$$

Similarly

$$r_1' = \frac{(a_2'y)}{(a_2'a_2)}$$

We can check for consistency in our solutions by looking at the solution for  $r_1$ . We had set 'b<sub>11</sub>' to be 1. Since  $r_1' = [b_{11} \quad b_{12}]$ , we need that the first entry of  $r_1$  be 1. We have:

$$r_1^T = \frac{(a_2'[y_1 \quad y_2])}{(a_2'a_2)} \Rightarrow r_1^T = \left[ \frac{(a_2'y_1)}{(a_2'a_2)} \quad \frac{(a_2'y_2)}{(a_2'a_2)} \right]$$

But we have,  $a_2 = y_1$

$$r_1^T = \left[ \frac{(y_1'y_1)}{(y_1'y_1)} \quad \frac{(y_1'y_2)}{(y_1'y_1)} \right] \Rightarrow r_1^T = \left[ 1 \quad \frac{(y_1'y_2)}{(y_1'y_1)} \right]$$

We see that the first element of  $r_1$  is indeed 1 and hence the solution we have obtained is consistent.

The blocks 'a' and 'b' can now be written as

$$a = [a_1 \quad a_2], \quad b = \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix}$$

The factor 'k' would go into the matrix  $F_2$  as was seen in its structure. The method mentioned here is exactly the one that is used by the code written to compute the factors.

So far we have discussed the methods we use when the shift in position is in the second factor  $F_2$ . However, it is quite possible that the shift is not in  $F_2$  but instead is in  $F_1$ . Let us call the new factors as  $f_1$  and  $f_2$ . In this case, a simple tweaking of the code `solver2_2x2` is all that is needed. Instead of using the product  $f_{12}$  and solving for the two factors, we solve for the product

$$F_{34} = f'_{12}$$

The structure of  $F_{34}$  will be the same as  $F_{12}$  that we worked with previously. We can then use the same code to solve for  $F_3$  and  $F_4$  (for the product  $F_{34}$ ). Then we have

$$\begin{aligned} F_{34} &= F_3 F_4 \text{ and } F_{34} = f'_{12} \\ \Rightarrow f_{12} &= F_{34}' = (F_3 F_4)' = F_4' F_3' \\ &\Rightarrow f_{12} = f_1 f_2 = F_4' F_3' \\ &\Rightarrow f_1 = F_4' \text{ and } f_2 = F_3' \end{aligned}$$

Thus we have solved for the factors in a slightly different case by a simple transformation.

### 2.3 Sample Problem: Toeplitz Case

Let us now apply the solution technique we have just discussed and apply it to the well-known four wavelet coefficients presented by Daubechies [3]. We compare the results obtained with those stated by Strang [4]. We write the problem out in the same way as it has been done in [4].

$$W = \begin{bmatrix} \dots & 0 & 0 \\ W_1 & W_2 & 0 \\ 0 & W_1 & W_2 \\ 0 & 0 & \dots \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 + \sqrt{3} & 3 + \sqrt{3} \\ 1 - \sqrt{3} & -3 + \sqrt{3} \end{bmatrix}, \quad W_2 = \begin{bmatrix} 3 - \sqrt{3} & 1 - \sqrt{3} \\ 3 + \sqrt{3} & -1 - \sqrt{3} \end{bmatrix}$$

The solution we obtain from the codes that were written is:

$$F_1 = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & 1 & 2.4286 & 0 \\ 0 & -0.2679 & 9.0636 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.5221 & -0.3014 & 0 & 0 \\ 2.7321 & 4.7321 & 0 & 0 \\ 0 & 0 & 0.5221 & \dots \\ 0 & 0 & \vdots & \ddots \end{bmatrix}$$

The solution obtained by Strang [4] is:

$$B = \begin{bmatrix} k_2 & 0 & 0 & 0 \\ 0 & 1 + \sqrt{3} & -1 + \sqrt{3} & 0 \\ 0 & 1 - \sqrt{3} & 1 + \sqrt{3} & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} = \begin{bmatrix} k_2 & 0 & 0 & 0 \\ 0 & 2.7321 & 0.7321 & 0 \\ 0 & -0.7321 & 2.7321 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

$$C = \begin{bmatrix} \sqrt{3} & -1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & \dots \\ 0 & 0 & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1.7321 & -1 & 0 & 0 \\ 1 & 1.7321 & 0 & 0 \\ 0 & 0 & 1.7321 & \dots \\ 0 & 0 & \vdots & \ddots \end{bmatrix}$$

Clearly, it can be seen that the factors obtained from the two different methods are not equivalent. This does not mean that one of the results is wrong and the other is right. It simply means that the factors are non-unique. Depending on which variables are given values, the individual factors in themselves will change. However, the product remains the same and hence the product of the inverses is the same as the inverse of the matrix  $W$ . This is what is necessary and hence this is all that is of concern.

## 2.4 Non-Toeplitz Case

The major difference in this case is that the blocks on the diagonal are varying. In such a case, we need to determine each of the blocks in each of the factors. The properties concerning the ranks of the matrices in the product  $F_{12}$  are exactly the same as the Toeplitz case. The matrices in the product should be singular. The argument for the Toeplitz case is valid for the non-Toeplitz case also, when the blocks are no longer constant along the diagonals of the two factors  $F_1$  and

F<sub>2</sub>. We shall now look at solving the interesting problem of finding out all the different blocks on the diagonals of each of the factors.

## 2.5 Solution Process: Non-Toeplitz Case

For the time being we shall assume that the shift is in the second factor F<sub>2</sub> and the product is F<sub>12</sub>. (Solving for the case where F<sub>1</sub> has the shift is the same as the one in the Toeplitz case - we simply solve using the transpose instead of the original product F<sub>12</sub>). The structures for F<sub>1</sub> and F<sub>2</sub> are shown next along with the structure F<sub>12</sub>.

$$F_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{11} & \alpha_{11} \\ 0 & 0 & 0 & 0 & \alpha_{11} & \alpha_{11} \end{bmatrix}, \quad F_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 & 0 & 0 \\ 0 & b_{21} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & 0 & B_{21} & B_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_{11} \end{bmatrix}$$

$$F_{12} = \begin{bmatrix} a_{11}k & a_{12}b_{11} & a_{12}b_{12} & 0 & 0 & 0 \\ a_{21}k & a_{22}b_{11} & a_{22}b_{12} & 0 & 0 & 0 \\ 0 & A_{11}b_{21} & A_{11}b_{22} & A_{12}B_{11} & A_{12}B_{12} & 0 \\ 0 & A_{21}b_{21} & A_{21}b_{22} & A_{22}B_{11} & A_{22}B_{12} & 0 \\ 0 & 0 & 0 & \alpha_{11}B_{21} & \alpha_{11}B_{22} & \alpha_{12}\beta_{11} \\ 0 & 0 & 0 & \alpha_{21}B_{21} & \alpha_{21}B_{22} & \alpha_{22}\beta_{11} \end{bmatrix}$$

Define vector 'w' that consists of the first two elements of the first column of the matrix F<sub>12</sub>. Similarly, we define vector 's' to consist of the last two elements of the last column of F<sub>12</sub>. The matrices that are to the right side of the diagonal element are denoted by 'x<sub>i</sub>' and the matrices to the left of the diagonal element are denoted as 'y<sub>i</sub>'. Thus the matrix x<sub>1</sub> starts from the first row while the matrix y<sub>1</sub> begins only from the third row.

$$w = \begin{bmatrix} a_{11}k \\ a_{21}k \end{bmatrix} = \underline{a_1}k, \quad s = \begin{bmatrix} \alpha_{12}\beta_{11} \\ \alpha_{22}\beta_{11} \end{bmatrix} = \underline{\alpha_2}\beta_{11}$$

$$x_1 = \begin{bmatrix} a_{12}b_{11} & a_{12}b_{12} \\ a_{22}b_{11} & a_{22}b_{12} \end{bmatrix}, \quad x_2 = \begin{bmatrix} A_{12}B_{11} & A_{12}B_{12} \\ A_{22}B_{11} & A_{22}B_{12} \end{bmatrix} \text{ and so on ...}$$

$$y_1 = \begin{bmatrix} A_{11}b_{21} & A_{11}b_{22} \\ A_{21}b_{21} & A_{21}b_{22} \end{bmatrix}, \quad y_2 = \begin{bmatrix} \alpha_{11}B_{21} & \alpha_{11}B_{22} \\ \alpha_{21}B_{21} & \alpha_{21}B_{22} \end{bmatrix} \text{ and so on ...}$$

In a bid to reduce the number of unknowns, we set the first element of every matrix to be 1 and solve for the rest of the elements of that matrix. By doing this essentially we end up with a situation where for each of the  $x_i$ , the first column is the same as the second column of each of the blocks in  $F_1$ . Similarly, in each of the  $y_i$ , the first row is the same as the second row of each of the blocks in  $F_2$ . This can be understood better by looking it as:

$$p_j q_j' = x_i, \quad \text{where } p_i \text{ is known } \Rightarrow \text{solve for } q_i$$

$$u_j v_j' = y_i, \quad \text{where } v_i \text{ is known } \Rightarrow \text{solve for } u_i$$

Once this is understood, the solution process becomes straightforward and is given by:

$$p_j q_j' = x_i$$

$$(p_j' p_j) q_j' = (p_j' x_i)$$

$$q_j' = \frac{(p_j' x_i)}{(p_j' p_j)}$$

And in a similar manner, we have:

$$u_j = \frac{(y_i v_j)}{(v_j' v_j)}$$

In order to get the element  $k$  that is present in the matrix  $F_1$ , we simply use:

$$k = \frac{w_1}{a_{11}} \text{ and } a_{21} = \frac{w_2}{k}$$

Similarly the second column of the last block of  $F_1$  is given as:  $\underline{\alpha_2} = s$

Now we have all the blocks and hence we can construct the big matrices  $F_1$  and  $F_2$  using these smaller blocks.

## 2.6 Tri-diagonal Matrices

We will now try to apply what we have derived so far to tri-diagonal matrices. Let us consider a generic tri-diagonal matrix. We then have

$$K = \begin{bmatrix} c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & c_2 & d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & c_3 & d_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & c_4 & d_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & c_5 & d_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & c_6 & d_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 & c_7 & d_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_7 & c_8 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} b_2 & c_3 \\ 0 & b_3 \end{bmatrix}, \quad S_1 = \begin{bmatrix} d_3 & 0 \\ c_4 & d_4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} b_4 & c_5 \\ 0 & b_5 \end{bmatrix}, \quad S_2 = \begin{bmatrix} d_5 & 0 \\ c_6 & d_6 \end{bmatrix}$$

Define:

$$M_1 = R_1 + S_1 z \Rightarrow \det(M_1) = \det(R_1) + (\det(P_1) + \det(Q_1))z + (\det(S_1))z^2$$

$$\det(M_1) = b_2 b_3 + (b_2 d_4 + b_3 d_3 - c_4 c_3)z + (d_3 d_4)z^2$$

Need:  $\det(R_1) = 0$  and  $\det(S_1) = 0$

$$\Rightarrow \det(R_1) = b_2 b_3 = 0 \text{ and } \det(S_1) = d_3 d_4 = 0$$

$$\Rightarrow b_2 = 0 \text{ or } b_3 = 0 \text{ and } d_3 = 0 \text{ or } d_4 = 0, \quad \Rightarrow \begin{cases} b_2 = 0, & d_3 = 0 \\ b_2 = 0, & d_4 = 0 \\ b_3 = 0, & d_3 = 0 \\ b_3 = 0, & d_4 = 0 \end{cases}$$

Similarly we get:



$$b_4 = 0 \text{ or } b_5 = 0 \text{ and } d_5 = 0 \text{ or } d_6 = 0, \quad \Rightarrow \begin{cases} b_4 = 0, & d_5 = 0 \\ b_4 = 0, & d_6 = 0 \\ b_5 = 0, & d_5 = 0 \\ b_5 = 0, & d_6 = 0 \end{cases}$$

We thus have a total of 16 possible conditions based on what we have derived so far that states that the tri-diagonal matrix  $K$  would have a banded inverse. However, studies done on tri-diagonal matrices [5] impose the requirement that for a banded inverse, there can be no two consecutive non-zero entries. This strict requirement is part of the 16 possible conditions. Thus we find that what we have developed is actually a weaker statement, because the general statement did not begin with blocks  $R_1$  and  $S_1$ .



# Chapter 3

## Banded Matrices – Three factors with 2-by-2 blocks

### 3.1 Toeplitz Case

So far we have analyzed the case where there are only two 2-by-2 blocks in the product. In this chapter we take the next step which would be to increase that number to three. We would then have three factors  $F_1$ ,  $F_2$  and  $F_3$ . For the time being, we shall concentrate on the case where  $F_2$  has a shift in its elements. Please note that throughout this section, the structure of  $F_3$  is the same as  $F_1$ .

For the case of  $F_{123}$ , it was seen that the ranks of the three matrices (R, S and T) in the product were one, two and one (when  $F_{123}$  is obtained from the product of factors). Once again like the case for the two block case, this can be proved. The proof is what is discussed next. There are six potential cases that need to be considered: (1,1,1), (1,2,1), (1,1,2), (2,1,2), (2,2,1) and (2,2,2). The other scenarios are simply different ways of looking at the above 6 cases.

Let us now define the matrices R, S and T that we will be using in the proofs:

$$R = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \quad T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$$

Construct

$$M = R + Sz + Tz^2$$

In the current scenario, the highest power in the determinant will be  $z^4$ . The coefficients of the different terms are given by:

$$\text{constant term} = \det(R) \quad \text{coefficient of } z^4 = \det(T) \quad \text{coefficient of } z = \det(P) + \det(Q)$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) \quad \text{coefficient of } z^3 = \det(W) + \det(X)$$

Where:

$$P = \begin{bmatrix} r_1 & r_2 \\ s_3 & s_4 \end{bmatrix} \quad Q = \begin{bmatrix} s_1 & s_2 \\ r_3 & r_4 \end{bmatrix} \quad U = \begin{bmatrix} r_1 & r_2 \\ t_3 & t_4 \end{bmatrix}$$

$$V = \begin{bmatrix} t_1 & t_2 \\ r_3 & r_4 \end{bmatrix} \quad W = \begin{bmatrix} s_1 & s_2 \\ t_3 & t_4 \end{bmatrix} \quad X = \begin{bmatrix} t_1 & t_2 \\ s_3 & s_4 \end{bmatrix}$$

We have six cases to examine, out of which only two cases succeed. However, if we are looking to be able to factorize the banded matrix, then there is only one case that works (case vi).

**Case i: Ranks are (2, 2, 2)**

In this case it is obvious that there will be two non-zero terms in the determinant – the constant term and the coefficient of  $z^4$ . As a result, the resulting inverse will not be banded (because the determinant is not a monomial)

**Case ii: Ranks are (1, 1, 1)**

In this case it is pretty clear to see that the constant term and the coefficient of  $z^4$  will both vanish. We now re-write R, S and T as:

$$r_2 = pr_1, \quad r_3 = kr_1, \quad r_4 = kpr_1$$

$$s_2 = qs_1, \quad s_3 = ls_1, \quad s_4 = lqs_1$$

$$t_2 = ut_1, \quad t_3 = mt_1, \quad t_4 = mut_1$$

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & qs_1 \\ ls_1 & lqs_1 \end{bmatrix} \quad T = \begin{bmatrix} t_1 & ut_1 \\ mt_1 & mut_1 \end{bmatrix}$$

Then we have:

$$\det(P) = \begin{vmatrix} r_1 & pr_1 \\ ls_1 & lqs_1 \end{vmatrix} = r_1 s_1 l(q - p), \quad \det(Q) = \begin{vmatrix} s_1 & qs_1 \\ kr_1 & kpr_1 \end{vmatrix} = r_1 s_1 k(p - q)$$

$$\det(U) = \begin{vmatrix} r_1 & pr_1 \\ mt_1 & mut_1 \end{vmatrix} = r_1 t_1 m(u - p), \quad \det(V) = \begin{vmatrix} t_1 & ut_1 \\ kr_1 & kpr_1 \end{vmatrix} = r_1 t_1 k(p - u)$$

$$\det(W) = \begin{vmatrix} s_1 & qs_1 \\ mt_1 & mut_1 \end{vmatrix} = s_1 t_1 m(u - q), \quad \det(X) = \begin{vmatrix} t_1 & ut_1 \\ ls_1 & lqs_1 \end{vmatrix} = s_1 t_1 l(q - u)$$

Which then leads to:

$$\text{constant term} = \det(R) = 0 \quad \text{coefficient of } z^4 = \det(T) = 0$$

$$\text{coefficient of } z = \det(P) + \det(Q) = r_1 s_1 (q - p)(l - k)$$

$$\text{coefficient of } z^3 = \det(W) + \det(X) = s_1 t_1 (u - q)(l - m)$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = r_1 t_1 (p - u)(k - m)$$

If this is to be banded, then exactly one of the terms should be non-zero. **Assume that the coefficient of  $z$  is non-zero.** Then we need:

$$\text{coefficient of } z^2 = r_1 t_1 (p - u)(k - m) = 0$$

$$\text{coefficient of } z^3 = s_1 t_1 (u - q)(l - m) = 0$$

Which implies:

$$p = u \text{ or } k = m, \quad u = q \text{ or } l = m$$

There are 4 sub-cases which would need to be considered

i)  $p = u \text{ and } u = q \Rightarrow p = q$

$$\Rightarrow \text{coefficient of } z^2 = r_1 t_1 (p - u)(k - m) = 0$$

$$\Rightarrow \text{coefficient of } z^3 = s_1 t_1 (u - q)(l - m) = 0$$

$$\Rightarrow \text{coefficient of } z = r_1 s_1 (q - p)(l - k) = 0$$

This means that for this particular scenario, the matrix is not invertible. Thus this case is unfavorable.

ii)  $k = m \text{ and } l = m \Rightarrow l = k$

$$\Rightarrow \text{coefficient of } z^2 = r_1 t_1 (p - u)(k - m) = 0$$

$$\Rightarrow \text{coefficient of } z^3 = s_1 t_1 (u - q)(l - m) = 0$$

$$\Rightarrow \text{coefficient of } z = r_1 s_1 (q - p)(l - k) = 0$$

The matrix is not invertible. As a result having  $k = m$  and  $l = m$  is not favorable.

iii)  $p = u \text{ and } l = m$

Please note that  $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$  is an arbitrary matrix of rank either one or two. Ideally,

$x$  would come from the end effects and be of full rank.



The matrix has a zero column which can be clearly seen. Thus it is not of full rank. Hence  $p = u$  and  $l = m$  also does not allow the big matrix to be invertible.

iv)  $u = q$  and  $k = m$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & t_1 & ut_1 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & mt_1 & mut_1 & 0 & 0 & 0 & 0 \\ r_1 & pr_1 & s_1 & qs_1 & t_1 & ut_1 & 0 & 0 \\ kr_1 & kpr_1 & ls_1 & lqs_1 & mt_1 & mut_1 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & qs_1 & t_1 & ut_1 \\ 0 & 0 & kr_1 & kpr_1 & ls_1 & lqs_1 & mt_1 & mut_1 \\ 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & qs_1 \\ 0 & 0 & 0 & 0 & kr_1 & kpr_1 & ls_1 & lqs_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & t_1 & qt_1 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & kt_1 & kqt_1 & 0 & 0 & 0 & 0 \\ r_1 & pr_1 & s_1 & qs_1 & t_1 & qt_1 & 0 & 0 \\ kr_1 & kpr_1 & ls_1 & lqs_1 & kt_1 & kqt_1 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & qs_1 & t_1 & qt_1 \\ 0 & 0 & kr_1 & kpr_1 & ls_1 & lqs_1 & kt_1 & kqt_1 \\ 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & qs_1 \\ 0 & 0 & 0 & 0 & kr_1 & kpr_1 & ls_1 & lqs_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & t_1 & 0 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & kt_1 & 0 & 0 & 0 & 0 & 0 \\ r_1 & pr_1 & s_1 & 0 & t_1 & 0 & 0 & 0 \\ kr_1 & kpr_1 & ls_1 & 0 & kt_1 & 0 & 0 & 0 \\ 0 & 0 & r_1 & (p-q)r_1 & s_1 & 0 & t_1 & 0 \\ 0 & 0 & kr_1 & k(p-q)r_1 & ls_1 & 0 & kt_1 & 0 \\ 0 & 0 & 0 & 0 & r_1 & (p-q)r_1 & s_1 & 0 \\ 0 & 0 & 0 & 0 & kr_1 & k(p-q)r_1 & ls_1 & 0 \end{bmatrix}$$

The last column is a zero column meaning that the matrix is singular. Hence  $u = q$  and  $k = m$  also does not allow the big matrix to be invertible.

Let us now look at the other two scenarios – where the coefficient of  $z^2$  and  $z^3$  are respectively non-zero. Each of these scenarios results in:



*Coefficient of  $z^3$  is non-zero:*

$$\text{coefficient of } z^3 = s_1 t_1 (u - q)(l - m) \neq 0$$

$$\text{coefficient of } z^2 = r_1 t_1 (p - u)(k - m) = 0$$

$$\text{coefficient of } z = r_1 s_1 (q - p)(l - k) = 0$$

$$\Rightarrow u = p \text{ or } k = m, \quad q = p \text{ or } l = k$$

The cases for  $z^3$  that we need to consider are very similar to what we did for  $z$ . Thus we can expect that the matrix will not be invertible if we have only the coefficient of  $z^3$  to be non-zero.

*Coefficient of  $z^2$  is non-zero:*

$$\text{coefficient of } z^2 = r_1 t_1 (p - u)(k - m) \neq 0$$

$$\text{coefficient of } z = r_1 s_1 (q - p)(l - k) = 0$$

$$\text{coefficient of } z^3 = s_1 t_1 (u - q)(l - m) = 0$$

$$\Rightarrow q = p \text{ or } k = l, \quad u = q \text{ or } l = m$$

The situation where the coefficient of  $z^2$  is non-zero has one particular case that ensures that the matrix has a banded inverse. This is the case where  $p = q$  and  $l = m$ . However, although this case gives a banded inverse, the matrix cannot be decomposed into factors. We show very quickly why this is so:

$$s = \begin{bmatrix} s_1 & ps_1 \\ ls_1 & lps_1 \end{bmatrix} = \begin{bmatrix} a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \end{bmatrix}$$

$$\begin{aligned} \det(S) = 0 &= (a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21})(a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22}) \\ &\quad - (a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21})(a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22}) = 0 \end{aligned}$$

$$\Rightarrow (\det(a))b_{11}b_{22}(\det(c)) = 0$$

$$\Rightarrow \det(a) = 0 \text{ or } \det(c) = 0 \text{ or } b_{11}b_{22} = 0$$

In each of the cases, the factors will not be invertible. Thus although *this case is invertible and produces a banded inverse, it cannot be factorized*. Hence for our purposes of being factorizable, it is not useful.

**Case iii: Ranks are (1, 1, 2)**

The constant term vanishes in this particular case. We now re-write R, S and T as:

$$r_2 = pr_1, \quad r_3 = kr_1, \quad r_4 = kpr_1$$

$$s_2 = qs_1, \quad s_3 = ls_1, \quad s_4 = lqs_1$$

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & qs_1 \\ ls_1 & lqs_1 \end{bmatrix} \quad T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$$

Then we have:

$$\det(P) = \begin{vmatrix} r_1 & pr_1 \\ ls_1 & lqs_1 \end{vmatrix} = r_1s_1l(q - p), \quad \det(Q) = \begin{vmatrix} s_1 & qs_1 \\ kr_1 & kpr_1 \end{vmatrix} = r_1s_1k(p - q)$$

$$\det(U) = \begin{vmatrix} r_1 & pr_1 \\ t_3 & t_4 \end{vmatrix} = r_1(t_4 - pt_3), \quad \det(V) = \begin{vmatrix} t_1 & t_2 \\ kr_1 & kpr_1 \end{vmatrix} = kr_1(pt_1 - t_2)$$

$$\det(W) = \begin{vmatrix} s_1 & qs_1 \\ t_3 & t_4 \end{vmatrix} = s_1(t_4 - qt_3), \quad \det(X) = \begin{vmatrix} t_1 & t_2 \\ ls_1 & lqs_1 \end{vmatrix} = ls_1(qt_1 - t_2)$$

Which then leads to:

$$\text{constant term} = \det(R) = 0 \quad \text{coefficient of } z^4 = \det(T) \neq 0$$

$$\text{coefficient of } z = \det(P) + \det(Q) = r_1s_1(q - p)(l - k)$$

$$\text{coefficient of } z^3 = \det(W) + \det(X) = s_1[(t_4 - lt_2) + q(lt_1 - t_3)]$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = r_1[(t_4 - kt_2) + p(kt_1 - t_3)]$$

Because  $\det(T) \neq 0$ , we need that the other coefficients all disappear. This then leads to:

$$\text{coefficient of } z = \det(P) + \det(Q) = r_1 s_1 (q - p)(l - k) = 0$$

$$\Rightarrow \boxed{q = p \text{ or } l = k}$$

$$\text{coefficient of } z^3 = \det(W) + \det(X) = s_1[(t_4 - lt_2) + q(lt_1 - t_3)] = 0$$

$$\Rightarrow \boxed{(t_4 - lt_2) = q(t_3 - lt_1)}$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = r_1[(t_4 - kt_2) + p(kt_1 - t_3)] = 0$$

$$\Rightarrow \boxed{(t_4 - kt_2) = p(t_3 - kt_1)}$$

$$\Rightarrow (k - l)t_2 = (q - p)t_3 + (kp - ql)t_1$$

If  $q = p$

$$\Rightarrow (k - l)t_2 = (k - l)pt_1$$

$$\Rightarrow t_2 = pt_1$$

$$\text{Now: } (t_4 - lt_2) = q(t_3 - lt_1)$$

$$\Rightarrow (t_4 - plt_1) = q(t_3 - lt_1)$$

$$\Rightarrow (t_4 - plt_1) = p(t_3 - lt_1), (\because p = q)$$

$$\Rightarrow (t_4 - plt_1) = (pt_3 - plt_1), \quad \Rightarrow t_4 = pt_3$$

$$\Rightarrow T = \begin{bmatrix} t_1 & pt_1 \\ t_3 & pt_3 \end{bmatrix}$$

$$\Rightarrow \text{rank}(T) = 1$$

Which violates the fact that we have T to be of full rank.

If  $l = k$

$$\Rightarrow 0 = (q - p)t_3 + (p - q)lt_1$$

$$\Rightarrow t_3 = lt_1$$

$$\text{Now: } (t_4 - lt_2) = q(t_3 - lt_1)$$

$$\Rightarrow (t_4 - lt_2) = q(lt_1 - lt_1)$$

$$\Rightarrow (t_4 - lt_2) = 0, \quad \Rightarrow t_4 = lt_2$$

$$\Rightarrow T = \begin{bmatrix} t_1 & t_2 \\ lt_1 & lt_2 \end{bmatrix}$$

$$\Rightarrow \text{rank}(T) = 1$$

Which violates the fact that we have T to be of full rank. So this case is not useful in terms of being able to get a banded matrix.

#### **Case iv: Ranks are (1, 2, 2)**

Once again the constant term vanishes in this case. We now re-write R as:

$$r_2 = pr_1, \quad r_3 = kr_1, \quad r_4 = kpr_1$$

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \quad T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$$

Then we have:

$$\det(P) = \begin{vmatrix} r_1 & pr_1 \\ s_3 & s_4 \end{vmatrix} = r_1(s_4 - ps_3), \quad \det(Q) = \begin{vmatrix} s_1 & s_2 \\ kr_1 & kpr_1 \end{vmatrix} = kr_1(ps_1 - s_2)$$

$$\det(U) = \begin{vmatrix} r_1 & pr_1 \\ t_3 & t_4 \end{vmatrix} = r_1(t_4 - pt_3), \quad \det(V) = \begin{vmatrix} t_1 & t_2 \\ kr_1 & kpr_1 \end{vmatrix} = kr_1(pt_1 - t_2)$$

$$\det(W) = \begin{vmatrix} s_1 & s_2 \\ t_3 & t_4 \end{vmatrix} = s_1t_4 - s_2t_3, \quad \det(X) = \begin{vmatrix} t_1 & t_2 \\ s_3 & s_4 \end{vmatrix} = s_4t_1 - s_3t_2$$

Which then leads to:

$$\text{constant term} = \det(R) = 0 \quad \text{coefficient of } z^4 = \det(T) \neq 0$$

$$\text{coefficient of } z = \det(P) + \det(Q) = r_1(s_4 - ps_3) + kr_1(ps_1 - s_2)$$

$$\text{coefficient of } z^3 = \det(W) + \det(X) = s_1t_4 - s_2t_3 + s_4t_1 - s_3t_2$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = r_1(t_4 - pt_3) + kr_1(pt_1 - t_2) + (s_1s_4 - s_2s_3)$$

Need:

$$\text{coefficient of } z = r_1(s_4 - ps_3) + kr_1(ps_1 - s_2) = 0$$

$$\Rightarrow (s_4 - ps_3) + k(ps_1 - s_2) = 0$$

$$\Rightarrow (s_4 - ks_2) = p(s_3 - ks_1)$$

$$\Rightarrow s_4 = ps_3 + ks_2 - pks_1$$

$$\text{coefficient of } z^2 = r_1(t_4 - pt_3) + kr_1(pt_1 - t_2) + (s_1s_4 - s_2s_3) = 0$$

$$\text{coefficient of } z^3 = s_1t_4 - s_2t_3 + s_4t_1 - s_3t_2 = 0$$

We then have in the big matrix (we look at the matrix from the middle rows to the end):

$$\begin{array}{cccccccc}
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\
\vdots & kr_1 & kpr_1 & s_3 & s_4 & t_3 & t_4 & 0 & 0 \\
\vdots & 0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 \\
\vdots & 0 & 0 & kr_1 & kpr_1 & s_3 & s_4 & t_3 & t_4 \\
\vdots & 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\
\vdots & 0 & 0 & 0 & 0 & kr_1 & kpr_1 & s_3 & s_4
\end{array}$$

$$\Rightarrow \begin{array}{cccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\
0 & 0 & s_3 - ks_1 & s_4 - ks_2 & t_3 - kt_1 & t_4 - kt_2 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 \\
0 & 0 & 0 & 0 & s_3 - ks_1 & s_4 - ks_2 & t_3 - kt_1 & t_4 - kt_2 \\
0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\
0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & s_4 - ks_2
\end{array}$$

$$\Rightarrow \begin{array}{cccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\
0 & 0 & s_3 - ks_1 & p(s_3 - ks_1) & t_3 - kt_1 & t_4 - kt_2 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 \\
0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\
0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & p(s_3 - ks_1)
\end{array}$$

$$x_1 = (t_3 - kt_1) - \left(\frac{s_3 - ks_1}{r_1}\right)s_1, \quad x_2 = (t_4 - kt_2) - \left(\frac{s_3 - ks_1}{r_1}\right)s_2$$

$$\Rightarrow \begin{array}{cccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\
0 & 0 & s_3 - ks_1 & p(s_3 - ks_1) & t_3 - kt_1 & t_4 - kt_2 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 - pt_1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 - px_1 \\
0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 - ps_1 \\
0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & 0
\end{array}$$

$$\text{Now: } x_2 - px_1 = \left( (t_4 - kt_2) - \left(\frac{s_3 - ks_1}{r_1}\right)s_2 \right) - p \left( (t_3 - kt_1) - \left(\frac{s_3 - ks_1}{r_1}\right)s_1 \right)$$

$$\Rightarrow x_2 - px_1 = \frac{1}{r_1} [(r_1(t_4 - kt_2) - (s_3 - ks_1)s_2) - (r_1p(t_3 - kt_1) - p(s_3 - ks_1)s_1)]$$

$$\Rightarrow x_2 - px_1 = \frac{1}{r_1} [(r_1(t_4 - kt_2) - (s_3 - ks_1)s_2) - (r_1p(t_3 - kt_1) - (s_4 - ks_2)s_1)]$$

$$\Rightarrow x_2 - px_1 = \frac{1}{r_1} [(r_1(t_4 - pt_3) + r_1k(pt_1 - t_2) - s_3s_2 + ks_1s_2 + s_4s_1 - ks_1s_2)]$$

$$\Rightarrow x_2 - px_1 = \frac{1}{r_1} [(s_4s_1 - s_3s_2 + r_1(t_4 - pt_3) + r_1k(pt_1 - t_2))]$$

$$\Rightarrow x_2 - px_1 = \frac{1}{r_1} [\text{coefficient of } z^2]$$

But we need the coefficient of  $z^2 = 0$

$$\Rightarrow x_2 - px_1 = 0$$

$$\Rightarrow \begin{array}{cccccccc} r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\ 0 & 0 & s_3 - ks_1 & p(s_3 - ks_1) & t_3 - kt_1 & t_4 - kt_2 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 - pt_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 - ps_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & 0 \end{array}$$

$$\Rightarrow \begin{array}{cccccccc} r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 & 0 & 0 \\ 0 & 0 & s_3 - ks_1 & p(s_3 - ks_1) & t_3 - kt_1 & t_4 - kt_2 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & t_2 - pt_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 - ps_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The matrix thus has a zero row making it singular and hence of no further interest.

### Case v: Ranks are (2, 1, 2)

In this case we have both R and T to be of full rank. As a result,

$$M = R + Sz + Tz^2$$

$$\text{constant term} = \det(R) \neq 0 \quad \text{coefficient of } z^4 = \det(T) \neq 0$$

Hence the inverse in this case is not going to be banded owing to the fact that  $\det(M)$  is not a monomial.

**Case vi: Ranks are (1, 2, 1)**

In this case we have both R and T to be of rank-1 while S is of full rank. We then have:

$$r_2 = pr_1, \quad r_3 = kr_1, \quad r_4 = kpr_1$$

$$t_2 = ut_1, \quad t_3 = mt_1, \quad t_4 = mut_1$$

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} \quad S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \quad T = \begin{bmatrix} t_1 & ut_1 \\ mt_1 & mut_1 \end{bmatrix}$$

Then we have:

$$\det(P) = \begin{vmatrix} r_1 & pr_1 \\ s_3 & s_4 \end{vmatrix} = r_1(s_4 - ps_3), \quad \det(Q) = \begin{vmatrix} s_1 & s_2 \\ kr_1 & kpr_1 \end{vmatrix} = kr_1(ps_1 - s_2)$$

$$\det(U) = \begin{vmatrix} r_1 & pr_1 \\ mt_1 & mut_1 \end{vmatrix} = r_1t_1m(u - p), \quad \det(V) = \begin{vmatrix} t_1 & ut_1 \\ kr_1 & kpr_1 \end{vmatrix} = kr_1t_1(p - u)$$

$$\det(W) = \begin{vmatrix} s_1 & s_2 \\ mt_1 & mut_1 \end{vmatrix} = mt_1(us_1 - s_2), \quad \det(X) = \begin{vmatrix} t_1 & ut_1 \\ s_3 & s_4 \end{vmatrix} = t_1(s_4 - us_3)$$

Which then leads to:

$$\text{constant term} = \det(R) = 0 \quad \text{coefficient of } z^4 = \det(T) = 0$$

$$\text{coefficient of } z = \det(P) + \det(Q) = r_1(s_4 - ps_3) + kr_1(ps_1 - s_2)$$

$$\text{coefficient of } z^3 = \det(W) + \det(X) = t_1(s_4 - us_3) + mt_1(us_1 - s_2)$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = (s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p)$$

Let us assume that the *coefficient of z is the only non-zero entry*. Then we need that the coefficient of  $z^2$  and  $z^3$  both be zero.

$$\text{coefficient of } z^3 = 0 = \det(W) + \det(X) = t_1(s_4 - us_3) + mt_1(us_1 - s_2)$$



$$\Rightarrow (s_4 - us_3) + m(us_1 - s_2) = 0$$

$$\Rightarrow (s_4 - ms_2) = u(s_3 - ms_1)$$

$$\Rightarrow s_4 = ms_2 + us_3 - mus_1$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = (s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p)$$

$$(s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p) = 0$$

$$(s_1(ms_2 + us_3 - mus_1) - s_2s_3) + r_1t_1(m - k)(u - p) = 0$$

$$(ms_2s_1 + us_3s_1 - mus_1^2 - s_2s_3) + r_1t_1(m - k)(u - p) = 0$$

$$ms_1(s_2 - us_1) + s_3(us_1 - s_2) + r_1t_1(m - k)(u - p) = 0$$

$$(s_2 - us_1)(ms_1 - s_3) + r_1t_1(m - k)(u - p) = 0$$

Looking at the big matrix (we look from the middle of the matrix to the end) leads to:

$$\begin{array}{cccccccc} r_1 & pr_1 & s_1 & s_2 & t_1 & ut_1 & 0 & 0 \\ kr_1 & kpr_1 & s_3 & s_4 & mt_1 & mut_1 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 & t_1 & ut_1 \\ 0 & 0 & kr_1 & kpr_1 & s_3 & s_4 & mt_1 & mut_1 \\ 0 & 0 & 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\ 0 & 0 & 0 & 0 & kr_1 & kpr_1 & s_3 & s_4 \end{array}$$

$$\begin{array}{cccccccc} r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\ kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 - us_1 & t_1 & 0 \\ 0 & 0 & kr_1 & kpr_1 & s_3 & m(s_2 - us_1) & mt_1 & 0 \\ 0 & 0 & 0 & 0 & r_1 & (p - u)r_1 & s_1 & s_2 - us_1 \\ 0 & 0 & 0 & 0 & kr_1 & k(p - u)r_1 & s_3 & m(s_2 - us_1) \end{array}$$

$$\begin{array}{cccccccc} r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\ kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & (s_2 - us_1) & t_1 & 0 \\ 0 & 0 & 0 & 0 & s_3 - ks_1 & (m - k)(s_2 - us_1) & (m - k)t_1 & 0 \\ 0 & 0 & 0 & 0 & r_1 & (p - u)r_1 & s_1 & (s_2 - us_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & (m - k)(s_2 - us_1) \end{array}$$

$$\begin{array}{ccccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\
kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & s_1 - \frac{s_3 - ks_1}{(m-k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_3 - ks_1 & (m-k)(s_2 - us_1) & (m-k)t_1 & 0 \\
0 & 0 & 0 & 0 & r_1 & (p-u)r_1 & s_1 - \frac{s_3 - ks_1}{(m-k)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_3 - ks_1 & (m-k)(s_2 - us_1)
\end{array}$$

$$\begin{array}{ccccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\
kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & \frac{ms_1 - s_3}{(m-k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_3 - ks_1 & (m-k)(s_2 - us_1) & (m-k)t_1 & 0 \\
0 & 0 & 0 & 0 & r_1 & (p-u)r_1 & \frac{ms_1 - s_3}{(m-k)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (m-k)(s_2 - us_1)
\end{array}$$

$$\begin{array}{ccccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\
kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & \frac{ms_1 - s_3}{(m-k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_3 - ks_1 & (m-k)(s_2 - us_1) & 0 & 0 \\
0 & 0 & 0 & 0 & r_1 & (p-u)r_1 & \frac{ms_1 - s_3}{(m-k)} - \frac{t_1}{(s_2 - us_1)}(p-u)r_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (m-k)(s_2 - us_1)
\end{array}$$

$$\frac{(ms_1 - s_3)}{(m-k)} - \frac{t_1}{(s_2 - us_1)}(p-u)r_1 = \frac{(ms_1 - s_3)(s_2 - us_1) + (m-k)(u-p)r_1 t_1}{(m-k)(s_2 - us_1)}$$

But

$$\text{coefficient of } z^2 = (s_1 s_4 - s_2 s_3) + r_1 t_1 (m-k)(u-p) = 0$$

$$\Rightarrow (s_2 - us_1)(ms_1 - s_3) + r_1 t_1 (m-k)(u-p) = 0$$

$$\Rightarrow \frac{(ms_1 - s_3)}{(m-k)} - \frac{t_1}{(s_2 - us_1)}(p-u)r_1 = (ms_1 - s_3)(s_2 - us_1) + (m-k)(u-p)r_1 t_1 = 0$$

$$\begin{array}{cccccccc}
r_1 & pr_1 & s_1 & s_2 & t_1 & 0 & 0 & 0 \\
kr_1 & kpr_1 & s_3 & s_4 & mt_1 & 0 & 0 & 0 \\
0 & 0 & r_1 & pr_1 & \frac{ms_1 - s_3}{(m - k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_3 - ks_1 & (m - k)(s_2 - us_1) & 0 & 0 \\
0 & 0 & 0 & 0 & r_1 & (p - u)r_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (m - k)(s_2 - us_1)
\end{array}$$

In which there is a column of zeros. **Hence it is not possible to get an invertible banded matrix such that the only non-zero term in the determinant comes from the 'z' term** for the (1, 2, 1) case.

By symmetry, we can say that the same holds true for the  $z^3$  term.

Finally we assume that the **coefficient of  $z^2$  is the only non-zero entry**. Then we need that the coefficient of  $z$  and  $z^3$  both be zero

$$\text{coefficient of } z^3 = 0 = \det(W) + \det(X) = t_1(s_4 - us_3) + mt_1(us_1 - s_2)$$

$$\Rightarrow t_1(s_4 - us_3) + mt_1(us_1 - s_2) = 0$$

$$\Rightarrow (s_4 - ms_2) = u(s_3 - ms_1)$$

$$\Rightarrow s_4 = ms_2 + us_3 - mus_1$$

$$\text{coefficient of } z = 0 = \det(P) + \det(Q) = r_1(s_4 - ps_3) + kr_1(ps_1 - s_2)$$

$$\Rightarrow r_1(s_4 - ps_3) + kr_1(ps_1 - s_2) = 0$$

$$\Rightarrow (s_4 - ks_2) = p(s_3 - ks_1)$$

$$\Rightarrow s_4 = ks_2 + ps_3 - pks_1$$

$$\Rightarrow ms_2 + us_3 - mus_1 = ks_2 + ps_3 - pks_1$$

$$(u - p)s_3 = (k - m)s_2 + (mu - kp)s_1$$

$$\text{If } (u - p) = 0, (k - m) = 0$$

$$\Rightarrow s_4 = ks_2 + ps_3 - pks_1 \equiv s_4 = ms_2 + us_3 - mus_1$$

$$\Rightarrow \text{coefficient of } z^3 = \text{coefficient of } z = 0$$

$$\text{coefficient of } z^2 = (s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p) = (s_1s_4 - s_2s_3) = \det(S)$$

$$\text{mat} = \begin{bmatrix} x_1 & x_2 & t_1 & pt_1 & 0 & 0 \\ x_3 & x_4 & kt_1 & kpt_1 & 0 & 0 \\ r_1 & pr_1 & s_1 & s_2 & t_1 & pt_1 \\ kr_1 & kpr_1 & s_3 & s_4 & kt_1 & kpt_1 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\ 0 & 0 & kr_1 & kpr_1 & s_3 & s_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & t_1 & pt_1 & 0 & 0 \\ x_3 & x_4 & kt_1 & kpt_1 & 0 & 0 \\ r_1 & pr_1 & s_1 & s_2 & t_1 & pt_1 \\ 0 & 0 & (s_3 - ks_1) & (s_4 - ks_2) & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\ 0 & 0 & 0 & 0 & (s_3 - ks_1) & (s_4 - ks_2) \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & t_1 & pt_1 & 0 & 0 \\ x_3 & x_4 & kt_1 & kpt_1 & 0 & 0 \\ r_1 & pr_1 & s_1 & s_2 & t_1 & pt_1 \\ 0 & 0 & (s_3 - ks_1) & p(s_3 - ks_1) & 0 & 0 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\ 0 & 0 & 0 & 0 & (s_3 - ks_1) & p(s_3 - ks_1) \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & t_1 & 0 & 0 & 0 \\ x_3 & x_4 & kt_1 & 0 & 0 & 0 \\ r_1 & pr_1 & s_1 & (s_2 - ps_1) & t_1 & 0 \\ 0 & 0 & (s_3 - ks_1) & 0 & 0 & 0 \\ 0 & 0 & r_1 & 0 & s_1 & (s_2 - ps_1) \\ 0 & 0 & 0 & 0 & (s_3 - ks_1) & 0 \end{bmatrix} \Rightarrow \text{rref}(\text{mat}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This situation is indeed invertible and the inverse is banded.

If  $(u - p) = 0, (k - m) \neq 0$

$$\Rightarrow 0 = (k - m)s_2 + p(m - k)s_1$$

$$\Rightarrow s_2 = ps_1$$

$$\Rightarrow s_4 = ks_2 + ps_3 - ps_1 \Rightarrow s_4 = ps_3$$

$$S = \begin{bmatrix} s_1 & ps_1 \\ s_3 & ps_3 \end{bmatrix} \Rightarrow \text{rank}(S) = 1$$

If  $(u - p) \neq 0$

$$s_3 = \frac{(k - m)s_2 + (mu - kp)s_1}{(u - p)}$$

$$\text{coefficient of } z^2 = \det(S) + \det(U) + \det(V) = (s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p)$$

$$\Rightarrow s_1s_4 - s_2s_3 = s_1(ms_2 + us_3 - mus_1) - s_2s_3$$

$$\Rightarrow s_1s_4 - s_2s_3 = (ms_1 - s_3)(s_2 - us_1)$$

$$\Rightarrow (s_1s_4 - s_2s_3) + r_1t_1(m - k)(u - p) = (ms_1 - s_3)(s_2 - us_1) + r_1t_1(m - k)(u - p)$$

$$(ms_1 - s_3) = ms_1 - \frac{1}{(u - p)} [(k - m)s_2 + (mu - kp)s_1] = \frac{(m - k)(s_2 - ps_1)}{(u - p)}$$

$$(ms_1 - s_3)(s_2 - us_1) = \frac{(m - k)(s_2 - ps_1)(s_2 - us_1)}{(u - p)}$$

We then have:

$$(ms_1 - s_3)(s_2 - us_1) + r_1t_1(m - k)(u - p) = \frac{(m - k)(s_2 - ps_1)(s_2 - us_1)}{(u - p)} + r_1t_1(m - k)(u - p)$$

$$(ms_1 - s_3)(s_2 - us_1) + r_1t_1(m - k)(u - p) = (m - k) \left[ \frac{(s_2 - ps_1)(s_2 - us_1)}{(u - p)} + r_1t_1(u - p) \right]$$

Let us examine when this is zero:

$$\begin{aligned}
& (ms_1 - s_3)(s_2 - us_1) + r_1 t_1 (m - k)(u - p) = 0 \\
& \Rightarrow (m - k) \left[ \frac{(s_2 - ps_1)(s_2 - us_1)}{(u - p)} + r_1 t_1 (u - p) \right] = 0 \\
& \Rightarrow (m - k) = 0 \text{ or } \left[ \frac{(s_2 - ps_1)(s_2 - us_1)}{(u - p)} + r_1 t_1 (u - p) \right] = 0 \\
& \Rightarrow (m - k) = 0 \text{ or } [(s_2 - ps_1)(s_2 - us_1) + r_1 t_1 (u - p)^2] = 0 \\
& \Rightarrow m = k \text{ or } -\frac{(s_2 - ps_1)(s_2 - us_1)}{r_1 (u - p)^2} = t_1
\end{aligned}$$

Let  $m = k$ :

$$\begin{aligned}
& (s_4 - ks_2) = p(s_3 - ks_1), \quad (s_4 - ms_2) = u(s_3 - ms_1) \\
& \Rightarrow (m - k)s_2 = (p - u)s_3 + (mu - pk)s_1 \\
& \Rightarrow 0 = (p - u)s_3 + k(u - p)s_1 \\
& \Rightarrow 0 = (p - u)(s_3 - ks_1) \\
& \Rightarrow (s_3 - ks_1) = 0, \quad \because \text{we assume } (p - u) \neq 0 \\
& \Rightarrow s_3 = ks_1
\end{aligned}$$

If  $s_3 = ks_1 \Rightarrow$

$$s_4 = ks_2 + ps_3 - kps_1 \Rightarrow s_4 = ks_2 + kps_1 - kps_1 = ks_2$$

$$\Rightarrow S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ ks_1 & ks_2 \end{bmatrix}$$

$$\Rightarrow \text{rank}(S) = 1$$

But the case we are dealing with has  $\text{rank}(S) = 2$ . Thus it is a contradiction if  $m=k$  and

$$s_3 = ks_1$$

For the other possibility, it can only be opined that the matrix will be invertible as long as

$$-\frac{(s_2 - ps_1)(s_2 - us_1)}{r_1(u - p)^2} \neq t_1$$

Essentially we can show that the reduced row echelon form for the big matrix with the current rank order is the identity matrix which means that there are no zero rows or columns (as long as the above condition holds and the conditions on the coefficients of  $z$  and  $z^3$  hold).

$$\text{mat} = \begin{bmatrix} x_1 & x_2 & t_1 & ut_1 & 0 & 0 \\ x_3 & x_4 & mt_1 & mut_1 & 0 & 0 \\ r_1 & pr_1 & s_1 & s_2 & t_1 & ut_1 \\ kr_1 & kpr_1 & s_3 & s_4 & mt_1 & mut_1 \\ 0 & 0 & r_1 & pr_1 & s_1 & s_2 \\ 0 & 0 & kr_1 & kpr_1 & s_3 & s_4 \end{bmatrix} \Rightarrow \text{rref}(\text{mat}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This proves that the matrix in question is invertible under certain conditions.

**Remark:** Another way to look at it is to assume that the current matrix is obtained from the product of  $F_1$ ,  $F_2$  and  $F_3$  having block diagonal matrices  $a$ ,  $b$  and  $c$  respectively. In that case:

$$R = \begin{bmatrix} r_1 & pr_1 \\ kr_1 & kpr_1 \end{bmatrix} = \begin{bmatrix} a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} \end{bmatrix} \Rightarrow r_1 = a_{11}b_{21}c_{21}, \quad p = \frac{c_{22}}{c_{21}}, \quad k = \frac{a_{21}}{a_{11}}$$

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} \\ a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} t_1 & ut_1 \\ mt_1 & mut_1 \end{bmatrix} = \begin{bmatrix} a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \end{bmatrix} \Rightarrow t_1 = a_{12}b_{12}c_{11}, \quad u = \frac{c_{12}}{c_{11}}, \quad m = \frac{a_{22}}{a_{12}}$$

$$m = k \Rightarrow \frac{a_{22}}{a_{12}} = \frac{a_{21}}{a_{11}} \Rightarrow \det(a) = 0, \quad p = u \Rightarrow \frac{c_{22}}{c_{21}} = \frac{c_{12}}{c_{11}} \Rightarrow \det(c) = 0$$

Which would not be acceptable because each of the factors is invertible and that means that each of matrices a, b and c are invertible.

Now consider:

$$\begin{aligned}
 & -\frac{(s_2 - ps_1)(s_2 - us_1)}{r_1(u-p)^2} = t_1 \\
 LHS & := -\frac{(s_2 - ps_1)(s_2 - us_1)}{r_1(u-p)^2} \\
 (s_2 - ps_1) & = -\left(\frac{a_{11}b_{22}}{c_{21}}\right) \det(c), \quad (s_2 - us_1) = \left(\frac{a_{12}b_{11}}{c_{11}}\right) \det(c) \\
 (u-p) & = \frac{c_{12}}{c_{11}} - \frac{c_{22}}{c_{21}} = -\frac{\det(c)}{c_{11}c_{21}} \Rightarrow (u-p)^2 = \left(\frac{\det(c)}{c_{11}c_{21}}\right)^2 \\
 \Rightarrow -\frac{(s_2 - ps_1)(s_2 - us_1)}{r_1(u-p)^2} & = \frac{\left(\frac{a_{11}b_{22}}{c_{21}}\right)\left(\frac{a_{12}b_{11}}{c_{11}}\right)(\det(c))^2}{a_{11}b_{21}c_{21}\left(\frac{\det(c)}{c_{11}c_{21}}\right)^2} = \frac{a_{12}b_{11}b_{22}c_{11}}{b_{21}} \\
 RHS & := t_1 = a_{12}b_{12}c_{11} \\
 \Rightarrow \frac{a_{12}b_{11}b_{22}c_{11}}{b_{21}} & = a_{12}b_{12}c_{11} \Rightarrow b_{11}b_{22} = b_{12}b_{21} \\
 \Rightarrow \det(b) & = 0
 \end{aligned}$$

But this would again mean that one of the factors would not be invertible which is not what we want.

**Remark:** So far we have only dealt with the scenario where  $F_2$  has the shift in its elements. In order to be complete, we need to look at the scenario where  $F_1$  and  $F_3$  have the shifts instead. In this situation too, we get results very similar to what we have already discussed. The ordered sets



of ranks - (2, 2, 2), (2, 1, 2), (1, 1, 2), (1, 2, 2) – for the matrices R, S and T yield singular matrices. The lone exceptions are the cases (1, 1, 1) and (1, 2, 1). The former cannot be factorized and both lead to banded inverses only when the coefficient of  $z^2$  is not zero.

### 3.2 Solution Process: Toeplitz Case

Now that we are done with proving that the ranks for the three factor case need to necessarily be (1, 2, 1) (in order to be able to factorize the matrix), we proceed to solve for the factors. There are two cases to consider here – one when there is only one matrix that has been shifted ( $F_2$  only) and the other case when two of the matrices have shifts ( $F_1$  and  $F_3$ ). Please note that we can't have both  $F_1$  and  $F_2$  to have a shift at the same time. We are trying to get three matrices in the product and that is possible only if no two consecutive matrices have the same structure. Otherwise, it would simply result in a degenerate case and we can only solve for two factors instead of three. (If  $F_1$  and  $F_2$  both have shifts, then they both have the same structure which means that the product will have the same structure. Hence  $F_{12}$  can be replaced by a single matrix with only one block in it and thus we would be able to solve only for ( $F_{12}$ ) and  $F_3$ . We will not get three matrices in the product and hence we cannot get three factors)

First let us consider the case when only  $F_2$  has a shift. For this situation, the structures of  $F_1$ ,  $F_2$  and  $F_3$  are shown. The structure of the product is also shown.

$$F_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} \end{bmatrix}$$

$$F_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{21} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{11} & b_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{21} & b_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{11} \end{bmatrix}$$

$$F_3 = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{11} & c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & c_{22} \end{bmatrix}$$

$F_{123}$

$$= \begin{bmatrix} a_{11}kc_{11} + a_{12}b_{11}c_{21} & a_{11}kc_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{21}kc_{11} + a_{22}b_{11}c_{21} & a_{21}kc_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \\ a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \dots \\ 0 & 0 & a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} \\ 0 & 0 & a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} & 0 & 0 \\ a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} & 0 & 0 \\ \dots a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \\ a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \end{bmatrix}$$

It can be seen from the structure of the product that once again, we would need to set some values to be able to solve for the remaining values. Although there is no hard and fast rule as to which values should be set, the best combination was found when the values of  $a_{11}$ ,  $b_{11}$ ,  $b_{21}$  and  $c_{11}$  were all set to unity. Using these values, the other variables can be solved for.

Before we start solving for the different entries of the matrices ‘a’, ‘b’ and ‘c’, let us label the matrices that are present in the product. We consider first only the blocks that are repeated in the product. Starting from left to right, we label the first 2-by-2 matrix as R ( $F_{123}(3:4,1:2)$ ), the second 2-by-2 matrix as S and the final 2-by-2 matrix as T. Please note that the ranks of the matrices R, S and T need to be 1, 2 and 1 in order to be able factorize the matrix. Denote the 2-by-2 matrix starting at the position (1, 1) as w.

$$R = \begin{bmatrix} a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} \end{bmatrix} \quad S = \begin{bmatrix} a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \end{bmatrix} \quad w = \begin{bmatrix} a_{11}kc_{11} + a_{12}b_{11}c_{21} & a_{11}kc_{12} + a_{12}b_{11}c_{22} \\ a_{21}kc_{11} + a_{22}b_{11}c_{21} & a_{21}kc_{12} + a_{22}b_{11}c_{22} \end{bmatrix}$$

Now, we are ready to solve for the entries of the matrices ‘a’, ‘b’ and ‘c’. In this case doing an element-wise comparison in each of R, S and T helps to get the remaining values. From this, we have:

$$c_{21} = R_{11}, \quad c_{22} = R_{12}, \quad a_{21} = R_{21}/R_{11} \text{ and } c_{12} = T_{12}/T_{11}$$

Please note that we haven’t solved for all the elements yet. It is prudent to stop at this juncture and to mention that some of the elements ( $b_{22}$  and  $a_{22}$ ) that we solve for will yield two different solutions when solved by two different methods. If  $F_{123}$  can indeed be broken up into different factors, then the two methods would yield an identical result. So what we can do is to actually get the values of some of the variables such that the two methods always yield the same solution. In essence, we make what is expected as what is required and solve from there. If we do this it leads to

$$a_{12} = [S_{11} - \{(T_{21}S_{11} - T_{11}S_{21})/(T_{21} - T_{11}a_{21})\}]/c_{21}$$

$$b_{12} = T_{11}/a_{12}$$

$$b_{22} = [S_{21} - \{c_{21}a_{12}(T_{21})/(T_{11})\}]/a_{21}$$

$$a_{22} = (T_{21}/T_{11})a_{12}$$

And

$$k = w_{11} - S_{11} + b_{22}$$

Now we have all the elements needed in order to get the matrices ‘a’, ‘b’ and ‘c’ which are present in the factors.

Moving on, we now consider solving for the case when both  $F_1$  and  $F_3$  have shifts and there is no shift in  $F_2$ . The structure of the product  $F_{123}$  is shown next:

$$F_{123} = \begin{bmatrix} kb_{11}m & kb_{12}c_{11} & kb_{12}c_{12} & 0 \\ a_{11}b_{21}m & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} \\ a_{21}b_{21}m & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} \\ 0 & a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} \\ 0 & a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} \\ 0 & 0 & 0 & a_{11}b_{21}c_{21} \\ 0 & 0 & 0 & a_{21}b_{21}c_{21} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{12}b_{12}c_{12} & 0 & 0 & 0 \\ a_{22}b_{12}c_{12} & 0 & 0 & 0 \\ a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} & 0 \\ a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} & 0 \\ a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} \\ a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} \\ 0 & a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} \end{bmatrix}$$

Once again, we label the different matrices and vectors that we would need. We denote as ‘g’ the first element of the product  $F_{123}$ , as ‘h’ the last element of  $F_{123}$ , ‘s’ the vector of non-zero elements in the first row excluding the first element, ‘w’ the vector of non-zero elements in the

first column in the first column excluding the first element, 'y' the matrix to the immediate right of the vector w, 'z' the matrix to the right of the matrix y, 'x' the matrix just below matrix y.

In this case we set  $b_{11}$ ,  $c_{11}$ ,  $b_{21}$  and  $m$  to be unity. The initial equations that we have are:

$$\begin{aligned} a_1 b_{21} r_2' &= x, & a_2 b_{12} r_1' &= z, & a_1 b_{22} r_1' + a_2 b_{11} r_2' &= y \\ k b_{12} r_1' &= s, & a_1 b_{21} m &= w, & k b_{11} m &= g, & a_{11} b_{22} c_{11} &= h \end{aligned}$$

Using the values we set leads us to

$$a_1 = w, \quad r_2' = \frac{a_1' x}{a_1' a_1}, \quad k = g, \quad a_2 = \frac{k z s'}{s s'}, \quad b_{22} = h,$$

$$r_1' = \frac{a_1' (y - a_2 r_2')}{b_{22} (a_1' a_1)}, \quad b_{12} = \frac{(s(1))}{k(r_1(1))}$$

Now we have all the entries of each of the matrices 'a', 'b' and 'c' in order to be able to get the factors  $F_1$ ,  $F_2$  and  $F_3$  along with the multipliers for the shifts - k and m.

### 3.3 Sample Problem: Toeplitz Case

In the two 2-by-2 factors case, we tested our code against the available factorization for the four Daubechies wavelet coefficients. Now, we take it to a higher level. We test it and look for factors for the six Daubechies wavelet coefficients. Unlike the previous case where we could compare it with the solution obtained by Strang, we do not have a formal set of factors to compare against. Our only way of ensuring that the factors are indeed correct would probably be to multiply the factors and to get the norm of the error between the original matrix and the product that we have just formed.

The six Daubechies coefficients [6] that we use are the following:

$$R = \begin{bmatrix} \sqrt{2} \left( 1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}} \right) & \sqrt{2} \left( 5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}} \right) \\ \sqrt{2} \left( 1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}} \right) & -\sqrt{2} \left( 5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}} \right) \end{bmatrix}$$

$$S = \begin{bmatrix} \sqrt{2} \left( 10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}} \right) & \sqrt{2} \left( 10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}} \right) \\ \sqrt{2} \left( 10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}} \right) & -\sqrt{2} \left( 10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}} \right) \end{bmatrix}$$

$$T = \begin{bmatrix} \sqrt{2} \left( 5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}} \right) & \sqrt{2} \left( 1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}} \right) \\ \sqrt{2} \left( 5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}} \right) & -\sqrt{2} \left( 1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}} \right) \end{bmatrix}$$

This then yields the following three factors –

$$F_1 = \begin{bmatrix} 1 & 0.0578 & 0 & 0 \\ 0.1059 & -0.5461 & 0 & 0 \\ 0 & 0 & 1 & \dots \\ 0 & 0 & \vdots & \ddots \end{bmatrix}, \quad F_2 = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 1 & -47.2820 & 0 \\ 0 & 1 & 14.1005 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1 & -0.4123 & 0 & 0 \\ 10.6455 & 25.8205 & 0 & 0 \\ 0 & 0 & 1 & \dots \\ 0 & 0 & \vdots & \ddots \end{bmatrix}$$

For completeness sake we also furnish the norm of the error between the original matrix with R, S and T as blocks and the product of F<sub>1</sub>, F<sub>2</sub> and F<sub>3</sub>.

$$\text{norm}(\text{error}) = 2.8377e - 014 = O(\text{round - off error})$$

Thus we can say that the factors that we obtained are correct to round off errors.

# Chapter 4

## Banded Matrices – Three factors with 3-by-3 blocks

In this chapter, we will look at the results of the products of three matrices  $G_1$ ,  $G_2$  and  $G_3$ . All the three matrices are made up of 3-by-3 blocks.  $G_1$  has no shift,  $G_2$  has elements that are shifted one down and one to the right and  $G_3$  is further shifted one down and one to the right with respect to  $G_2$ . The typical structure of the three matrices is shown next

$$G_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$G_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & b_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{21} & b_{22} & b_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{31} & b_{32} & b_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} & b_{12} & b_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{21} & b_{22} & b_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{31} & b_{32} & b_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{21} & b_{22} \end{bmatrix}$$

$$G_3 = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} & c_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{31} & c_{32} & c_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} & c_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{21} & c_{22} & c_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{31} & c_{32} & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11} \end{bmatrix}$$

$$G_{123} = \begin{bmatrix} W_1 & U_1 & X_{11} & X_{12} & X_{13} & 0 & 0 & 0 & 0 \\ W_2 & U_2 & X_{21} & X_{22} & X_{23} & 0 & 0 & 0 & 0 \\ W_3 & U_3 & X_{31} & X_{32} & X_{33} & 0 & 0 & 0 & 0 \\ 0 & V_1 & Y_{11} & Y_{12} & Y_{13} & X_{11} & X_{12} & X_{13} & 0 \\ 0 & V_2 & Y_{21} & Y_{22} & Y_{23} & X_{21} & X_{22} & X_{23} & 0 \\ 0 & V_3 & Y_{31} & Y_{32} & Y_{33} & X_{31} & X_{32} & X_{33} & 0 \\ 0 & 0 & Z_{11} & Z_{12} & Z_{13} & Y_{11} & Y_{12} & Y_{13} & T_1 \\ 0 & 0 & Z_{21} & Z_{22} & Z_{23} & Y_{21} & Y_{22} & Y_{23} & T_2 \\ 0 & 0 & Z_{31} & Z_{32} & Z_{33} & Y_{31} & Y_{32} & Y_{33} & T_3 \end{bmatrix}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}$$

$$\begin{bmatrix} X_{11} = (a_{12}b_{12} + a_{13}b_{22})c_{11} + (a_{12}b_{13} + a_{13}b_{23})c_{21} & Y_{11} = (a_{11}b_{32}c_{11} + a_{11}b_{33}c_{21}) + (a_{12}b_{11} + a_{13}b_{21})c_{31} \\ X_{12} = (a_{12}b_{12} + a_{13}b_{22})c_{12} + (a_{12}b_{13} + a_{13}b_{23})c_{22} & Y_{12} = (a_{11}b_{32}c_{12} + a_{11}b_{33}c_{22}) + (a_{12}b_{11} + a_{13}b_{21})c_{32} \\ X_{13} = (a_{12}b_{12} + a_{13}b_{22})c_{13} + (a_{12}b_{13} + a_{13}b_{23})c_{23} & Y_{13} = (a_{11}b_{32}c_{13} + a_{11}b_{33}c_{23}) + (a_{12}b_{11} + a_{13}b_{21})c_{33} \\ X_{21} = (a_{22}b_{12} + a_{23}b_{22})c_{11} + (a_{22}b_{13} + a_{23}b_{23})c_{21} & Y_{21} = (a_{21}b_{32}c_{11} + a_{21}b_{33}c_{21}) + (a_{22}b_{11} + a_{23}b_{21})c_{31} \\ X_{22} = (a_{22}b_{12} + a_{23}b_{22})c_{12} + (a_{22}b_{13} + a_{23}b_{23})c_{22} & Y_{22} = (a_{21}b_{32}c_{12} + a_{21}b_{33}c_{22}) + (a_{22}b_{11} + a_{23}b_{21})c_{32} \\ X_{23} = (a_{22}b_{12} + a_{23}b_{22})c_{13} + (a_{22}b_{13} + a_{23}b_{23})c_{23} & Y_{23} = (a_{21}b_{32}c_{13} + a_{21}b_{33}c_{23}) + (a_{22}b_{11} + a_{23}b_{21})c_{33} \\ X_{31} = (a_{32}b_{12} + a_{33}b_{22})c_{11} + (a_{32}b_{13} + a_{33}b_{23})c_{21} & Y_{31} = (a_{31}b_{32}c_{11} + a_{31}b_{33}c_{21}) + (a_{32}b_{11} + a_{33}b_{21})c_{31} \\ X_{32} = (a_{32}b_{12} + a_{33}b_{22})c_{12} + (a_{32}b_{13} + a_{33}b_{23})c_{22} & Y_{32} = (a_{31}b_{32}c_{12} + a_{31}b_{33}c_{22}) + (a_{32}b_{11} + a_{33}b_{21})c_{32} \\ X_{33} = (a_{32}b_{12} + a_{33}b_{22})c_{13} + (a_{32}b_{13} + a_{33}b_{23})c_{23} & Y_{33} = (a_{31}b_{32}c_{13} + a_{31}b_{33}c_{23}) + (a_{32}b_{11} + a_{33}b_{21})c_{33} \end{bmatrix}$$



$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{31}c_{31} & a_{11}b_{31}c_{32} & a_{11}b_{31}c_{33} \\ a_{21}b_{31}c_{31} & a_{21}b_{31}c_{32} & a_{21}b_{31}c_{33} \\ a_{31}b_{31}c_{31} & a_{31}b_{31}c_{32} & a_{31}b_{31}c_{33} \end{bmatrix}$$

$$W = \begin{bmatrix} a_{11}km \\ a_{21}km \\ a_{31}km \end{bmatrix}, \quad U = \begin{bmatrix} (a_{12}b_{11} + a_{13}b_{21})n \\ (a_{22}b_{11} + a_{23}b_{21})n \\ (a_{32}b_{11} + a_{33}b_{21})n \end{bmatrix}, \quad V = \begin{bmatrix} a_{11}b_{31}n \\ a_{21}b_{31}n \\ a_{31}b_{31}n \end{bmatrix}, \quad T = \begin{bmatrix} (a_{12}b_{12} + a_{13}b_{22})c_{11} \\ (a_{22}b_{12} + a_{23}b_{22})c_{11} \\ (a_{32}b_{12} + a_{33}b_{22})c_{11} \end{bmatrix}$$

What we observe from the product is that as expected, there are three matrices in the product as a result of the different shifts in each of the individual matrices. As always we are interested in the ranks of the smaller matrices that are repeated in the product. With the current order of multiplication we see that the matrices are of ranks 1, 2 and 2 in that order. Please note that for this particular pattern for the factors –  $G_1$ ,  $G_2$  and  $G_3$  – yields this particular result. If however, we were to change the ordering, then we would get a bunch of different sequence for the ranks. Thus it would be impossible to furnish proofs in this particular situation. The 2-by-2 case was simple because there were only two possible ranks – one or two. Moreover, there were only two sequences for the Fs, while here we have many more.

There is an interesting point that we would like to point out when considering the three 3-by-3 blocks case. This concerns the order of multiplication of the factors  $G_1$ ,  $G_2$  and  $G_3$ . When the order of multiplication was changed, in some cases the order of the ranks also changed. What we found was that there seemed to be some sort of a cyclic relation for the products that shared the same ranks. We illustrate this with the following table:

	G1	G2	G3		R	S	T
no. of shifts	0	1	2	ranks	1	2	2
	0	2	1		2	2	1
	1	0	2		2	2	1
	1	2	0		1	2	2
	2	0	1		1	2	2
	2	1	0		2	2	1

**Table 1: Shifts and ranks for different  $G_1$ ,  $G_2$  and  $G_3$**

We see that in cases  $(0, 1, 2)$ ,  $(1, 2, 0)$ ,  $(2, 0, 1)$  share the same order for the ranks in the product  $G_{123} = (1, 2, 2)$ . In a similar manner, the cases  $(0, 2, 1)$ ,  $(2, 1, 0)$  and  $(1, 0, 2)$  share the same order for the ranks –  $(2, 2, 1)$ . In fact, it was also seen that the cases where the ranks were  $(2, 2, 1)$  can be obtained by transposing the cases with ranks  $(1, 2, 2)$ .

We make here a short remark about the case involving two factors in the 2-by-2 blocks case – there we did not have any such problems with the order based on the shift. This is because the ranks of the matrices were  $(1, 1)$  and as a result no matter which order we obtain the product, the ranks will always be  $(1, 1)$ . Even in the three factors 2-by-2 block situation, this was the case. The ranks in the product were  $(1, 2, 1)$  which was symmetric and independent of the order we chose for  $F_1$ ,  $F_2$  and  $F_3$  (admittedly, there were only two possible choices – both  $F_1$  and  $F_3$  have a shift or  $F_2$  has a shift).

We now look at trying to solve for the entries of the factors –  $G_1$ ,  $G_2$  and  $G_3$  – given the product  $G_{123}$ . We can see that just like in the 2-by-2 blocks case, we would need to look at the end rows and columns. This would give us an idea of what the order of multiplication should be. However, since each of the matrices ‘a’, ‘b’ and ‘c’ contain 9 unknowns, it is difficult to solve for all of them. We could set a few entries to be specific values, but which ones is in itself a big question. For the 2-by-2 case, there were far fewer unknowns and as a result, the unknowns that needed to

be set could be arrived at by brute force in the worst case scenario. That however, would not be useful in the 3-by-3 block case. What we can do is to examine what are the different matrices and where they come from. An exact solution (as in the 2-by-2 block case) could not be obtained in the duration of the current thesis. We present some of the formulations in the product below:

$$X = \underline{a_2 r'_1} b_{12} + \underline{a_3 r'_1} b_{22} + \underline{a_2 r'_2} b_{13} + \underline{a_3 r'_2} b_{23}, \quad Y = \underline{a_1 r'_1} b_{32} + \underline{a_1 r'_2} b_{33} + \underline{a_2 r'_3} b_{11} + \underline{a_3 r'_3} b_{21}$$

$$Z = \underline{a_1 r'_3} b_{31}, \quad W = \underline{a_1} km, \quad U = (\underline{a_2} b_{11} + \underline{a_3} b_{21}) n$$

$$V = \underline{a_1} b_{31} n, \quad T = (\underline{a_2} b_{12} + \underline{a_3} b_{22}) c_{11}$$



# Chapter 5

## Circulant Matrices

### 5.1 Two factors with 2-by-2 blocks – Toeplitz and Non-Toeplitz

In this chapter, we consider the effects of adding elements to the top right and bottom left of the factor matrices. In order to be able to understand the process better, we only consider factors with 2-by-2 blocks. However, we consider both the Toeplitz and the non-Toeplitz cases.

For the most part, the product that we obtain is very similar to what we obtained previously. The difference, as expected, is seen only at the ends. The middle rows remain unaffected. In order to make the discussion general, we consider a 2-by-2 matrix at the ends of the factors. Please note that we add the 2-by-2 matrix only to the factor  $F_2$  that already has a shift. We do not add anything to the matrix  $F_1$  that is free of the shift. The structures of  $F_1$  and  $F_2$  are shown next.

$$F_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{21} & a_{22} \end{bmatrix}$$

$$F_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} \\ 0 & b_{11} & b_{12} & 0 & 0 & 0 & c_{21} & c_{22} \\ 0 & b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{21} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{11} & b_{12} & 0 \\ d_{11} & d_{12} & 0 & 0 & 0 & b_{21} & b_{22} & 0 \\ d_{21} & d_{22} & 0 & 0 & 0 & 0 & 0 & b_{11} \end{bmatrix}$$

The product F12 has the following structure:

$$F_{12} = \begin{bmatrix} A_{11} & X_{11} & X_{12} & 0 & 0 & 0 & W_{11} & W_{12} \\ A_{21} & X_{21} & X_{22} & 0 & 0 & 0 & W_{21} & W_{22} \\ 0 & Y_{11} & Y_{12} & X_{11} & X_{12} & 0 & 0 & 0 \\ 0 & Y_{21} & Y_{22} & X_{21} & X_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{11} & Y_{12} & X_{11} & X_{12} & 0 \\ 0 & 0 & 0 & Y_{21} & Y_{22} & X_{21} & X_{22} & 0 \\ Z_{11} & Z_{12} & 0 & 0 & 0 & Y_{11} & Y_{12} & A_{12} \\ Z_{21} & Z_{22} & 0 & 0 & 0 & Y_{21} & Y_{22} & A_{22} \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad d = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$\underline{A}_1 = \begin{bmatrix} a_{11}k \\ a_{21}k \end{bmatrix}, \quad \underline{A}_2 = \begin{bmatrix} a_{12}b_{11} \\ a_{22}b_{11} \end{bmatrix}, \quad X = \begin{bmatrix} a_{12}b_{11} & a_{11}b_{12} \\ a_{22}b_{11} & a_{21}b_{12} \end{bmatrix}, \quad Y = \begin{bmatrix} a_{11}b_{21} & a_{11}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} \end{bmatrix}$$

$$W = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} a_{11}d_{11} + a_{12}d_{21} & a_{11}d_{12} + a_{12}d_{22} \\ a_{21}d_{11} + a_{22}d_{21} & a_{21}d_{12} + a_{22}d_{22} \end{bmatrix}$$

In a generic scenario, we see that the addition of the end elements and matrices in  $F_2$  has resulted in full rank matrices at the ends.

## 5.2 Solution Process: Toeplitz Case

In order to solve for the factors  $F_1$  and  $F_2$  in this case, we use the information from the middle rows of the matrix  $F_{12}$  first and solve it as for the two factor 2-by-2 case. The only addition that we make here is to solve for the blocks  $c$  and  $d$  also. The different steps in this process are shown next:

We set  $k$  and  $b_{11}$  to 1 to get:

$$\underline{A}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad \underline{A}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \quad A = [\underline{A}_1 \quad \underline{A}_2] = a$$

Then

$$\begin{aligned} A_2 r_1' &= Y, & A_1 r_2' &= X, & Ac &= W, & Ad &= Z \\ \Rightarrow r_1' &= \frac{A_2 Y}{A_2' A_2}, & r_2' &= \frac{A_1 X}{A_1' A_1}, & c &= A^{-1} W, & d &= A^{-1} Z \\ a &= [\underline{A}_1 \quad \underline{A}_2], & b &= \begin{bmatrix} r_1' \\ r_2' \end{bmatrix}, & c &= A^{-1} W, & d &= A^{-1} Z \end{aligned}$$

Which then implies that all the blocks –  $a$ ,  $b$ ,  $c$  and  $d$  – are known and from which the matrices  $F_1$  and  $F_2$  can be approximately constructed (approximately because we don't know the original factors)

One point to note in the above solution process is that we do not specify that the matrices 'c' and 'd' have to be of a particular rank. In fact  $c$  and  $d$  can contain a single non-zero entry and this solution method will still work.

At this juncture, we also make a few remarks under some of the cases. Let us assume for the moment that the matrices 'c' and 'd' contain only a single non-zero entry. We assume that these are the extreme top right ( $c_{12}$ ) and the extreme bottom left ( $d_{21}$ ) entries. We then get:

$$\begin{aligned} W &= \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}, & Z &= \begin{bmatrix} a_{11}d_{11} + a_{12}d_{21} & a_{11}d_{12} + a_{12}d_{22} \\ a_{21}d_{11} + a_{22}d_{21} & a_{21}d_{12} + a_{22}d_{22} \end{bmatrix} \\ \Rightarrow W &= \begin{bmatrix} 0 & a_{11}c_{12} \\ 0 & a_{21}c_{12} \end{bmatrix}, & Z &= \begin{bmatrix} a_{12}d_{21} & 0 \\ a_{22}d_{21} & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow F_{12} = \begin{bmatrix} A_{11} & X_{11} & X_{12} & 0 & 0 & 0 & 0 & a_{11}c_{12} \\ A_{21} & X_{21} & X_{22} & 0 & 0 & 0 & 0 & a_{21}c_{12} \\ 0 & Y_{11} & Y_{12} & X_{11} & X_{12} & 0 & 0 & 0 \\ 0 & Y_{21} & Y_{22} & X_{21} & X_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{11} & Y_{12} & X_{11} & X_{12} & 0 \\ 0 & 0 & 0 & Y_{21} & Y_{22} & X_{21} & X_{22} & 0 \\ a_{12}d_{21} & 0 & 0 & 0 & 0 & Y_{11} & Y_{12} & A_{12} \\ a_{22}d_{21} & 0 & 0 & 0 & 0 & Y_{21} & Y_{22} & A_{22} \end{bmatrix}$$

We can now re-arrange this particular form of  $F_{12}$  to the following form (and denote it as  $f_{12}$ ):

$$\Rightarrow f_{12} = \begin{bmatrix} a_{11}c_{12} & A_{11} & X_{11} & X_{12} & 0 & 0 & 0 & 0 \\ a_{21}c_{12} & A_{21} & X_{21} & X_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{11} & Y_{12} & X_{11} & X_{12} & 0 & 0 \\ 0 & 0 & Y_{21} & Y_{22} & X_{21} & X_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_{11} & Y_{12} & X_{11} & X_{12} \\ 0 & 0 & 0 & 0 & Y_{21} & Y_{22} & X_{21} & X_{22} \\ A_{12} & a_{12}d_{21} & 0 & 0 & 0 & 0 & Y_{11} & Y_{12} \\ A_{22} & a_{22}d_{21} & 0 & 0 & 0 & 0 & Y_{21} & Y_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = \begin{bmatrix} a_{11}k \\ a_{21}k \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} a_{12}b_{11} \\ a_{22}b_{11} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

This then implies that the:

$$\text{rank} \left( \begin{bmatrix} a_{11}c_{12} & a_{11} \\ a_{21}c_{12} & a_{21} \end{bmatrix} \right) = 1, \quad \text{rank} \left( \begin{bmatrix} a_{12} & a_{12}d_{21} \\ a_{22} & a_{22}d_{21} \end{bmatrix} \right) = 1$$

In essence, this is like analyzing an infinite dimensional banded matrix to see if it too has a banded inverse. The major difference however, lies in the final two rows where the position of the second matrix is not the same as where it would have been in the case of an infinite dimensional matrix.

This particular re-arranged form can be solved for using the code that has been written to solve for the cyclic factors. In essence, all this requires is a permutation matrix to be multiplied at some point to change the position of the columns to get the standard formulation. The way to check if a permutation matrix is needed or not is very simple and straight forward – we check the



first and last rows of the matrix  $f_{12}$ . If the four elements are consecutively placed (in the first row; and similarly there are two elements in the first four slots in the last row), then we know that we should shift the first column over to the last. Now we solve as usual on this  $F_{12}$ . We essentially then have this:

$$f_{12} = F_{12}P$$

$$\text{But: } F_{12} = F_1F_2$$

$$\Rightarrow f_{12} = F_1F_2P = f_1f_2, \quad \text{where } f_1 = F_1 \text{ and } f_2 = F_2P$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the non-Toeplitz case, there is not much of a difference between the current solution and the solution that we obtained in the purely non-Toeplitz case. What this means is that, we solve for the constituents of the factors as per normal, by setting the first entries of all the matrices to 1. Finally, we use the very first matrix in  $F_1$  (the one with no shift) to get the matrix ‘c’ and then use the last matrix in  $F_1$  to get the factor ‘d’. In this manner, we solve for all the factors as well as all extra entries present in the factors.

### 5.3 Three factors with 2-by-2 blocks – Toeplitz

Now we deal with the three factors 2-by-2 blocks case. For this, we only deal with trying to solve for the factors when  $F_2$  is the only matrix that has a shift and consequently, is the one with entries in the off-diagonal positions. The structure of the product is similar to that what we had obtained

previously. The difference is the presence of the two blocks in the top right and bottom left corners of the matrix. The structures of  $F_2$  and of the product are shown next:

$$F_2 = \begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 & d_{11} & d_{12} \\ 0 & b_{11} & b_{12} & 0 & 0 & 0 & d_{21} & d_{22} \\ 0 & b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{21} & b_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{11} & b_{12} & 0 \\ e_{11} & e_{12} & 0 & 0 & 0 & b_{21} & b_{22} & 0 \\ e_{21} & e_{22} & 0 & 0 & 0 & 0 & 0 & b_{11} \end{bmatrix}$$

$F_{123}$

$$= \begin{bmatrix} a_{11}kc_{11} + a_{12}b_{11}c_{21} & a_{11}kc_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{21}kc_{11} + a_{22}b_{11}c_{21} & a_{21}kc_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \\ a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \\ 0 & 0 & a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} \\ 0 & 0 & a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} \\ x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & y_{11} & y_{12} \\ 0 & 0 & y_{21} & y_{22} \\ a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} & 0 & 0 \\ a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} & 0 & 0 \\ \dots a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} & a_{12}b_{12}c_{11} & a_{12}b_{12}c_{12} \\ a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} & a_{22}b_{12}c_{11} & a_{22}b_{12}c_{12} \\ a_{11}b_{21}c_{21} & a_{11}b_{21}c_{22} & a_{11}b_{22}c_{11} + a_{12}b_{11}c_{21} & a_{11}b_{22}c_{12} + a_{12}b_{11}c_{22} \\ a_{21}b_{21}c_{21} & a_{21}b_{21}c_{22} & a_{21}b_{22}c_{11} + a_{22}b_{11}c_{21} & a_{21}b_{22}c_{12} + a_{22}b_{11}c_{22} \end{bmatrix}$$

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = aec$$

$$y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = adc$$

To solve, we follow the exact same procedure stated previously to solve for the matrices ‘a’, ‘b’ & ‘c’ and the factor ‘k’. We can then see that

$$x = aec \Rightarrow e = a^{-1}xc^{-1}, \quad y = adc \Rightarrow d = a^{-1}yc^{-1}$$

From which we see that everything about the matrices  $F_1$ ,  $F_2$  and  $F_3$  is known. Hence we have the complete solution for this case.



# Chapter 6

## Conclusions and Future Work

This chapter serves as a final recap of all that we have looked at. We started by looking at the two factor 2-by-2 blocks case. We proved that the ranks of the matrices in a banded matrix with two blocks needed to be unity. Only under this situation was the banded matrix actually invertible and also banded. We then looked at solving for the factors in this case. Owing to the large number of unknowns vs. the number of equations to solve for them, we realized that we had to assign values to a few of the variables. Once this was done, we could solve for the others easily. We solved for the factors in situations where the factors were either Toeplitz or non-Toeplitz. The non-Toeplitz case added a lot more unknowns into the mix than the number of equations available. However, we were able to circumvent this problem by assuming a kind of normalization for each of the blocks – we set the first element of every block to be unity.

We then went on to the case where we had three factors with 2-by-2 blocks. Once again we proved that the ranks needed to be  $(1, 2, 1)$  for the inverse of the banded matrix to also be banded and factorizable. Like the two factor case, here also we solved for the different factors after assuming a few values.

We then had a brief overview of the three factor 3-by-3 blocks case. We saw that simply going from 2-by-2 to 3-by-3 causes a lot more hassles and it is a lot more complicated to solve. We made observations regarding the different entries in the banded matrix but did not solve for them. We believe that trying to solve for the three factors would be a nice and logical extension to the current work done.

Finally, we looked at the case of circulant matrices, where we had non-zero entries in the minor diagonals of the factors. We saw that for the most part we needed to add only an extra step to compute the off-diagonal entries. We did this for the two factor Toeplitz and non-Toeplitz case as well as for the three factor Toeplitz case. In this manner, we covered most of the ground for the Toeplitz and non-Toeplitz cases.

We wrote solvers for each of the cases that we have discussed and the algorithm used in the solvers is seen in the solution process section of Chapters 2 and 3. In some cases, we ensured that given a random banded matrix, the solver itself would determine if it is a two factor Toeplitz or non-Toeplitz case or if it needs to be permuted or transposed before being solved. The actual solving though was done in a separate solvers – one for the two factor Toeplitz 2-by-2 blocks case, one separate for the two factor 2-by-2 non-Toeplitz blocks case and so on. We also think it would be great to see how much of a generalized solver we can write in order to be able to solve for factors that have  $n$ -by- $n$  sized blocks.

# Appendix A

## Conjecture for an $n$ -by- $n$ matrix formed from the various permutations of $n$ elements

In this section we conjecture that the minimum rank possible for an  $n$ -by- $n$  matrix formed from the permutation of  $n$  elements is given by the smallest possible  $q$  which satisfies the relation:

$$n \leq q!$$

Let us now look at an example to illustrate the point. Consider a set of 7 distinct elements –  $a_1$  through  $a_7$ . We now form a 7-by-7 matrix using only the different permutations of the 7 elements

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_7 & a_6 \\ a_1 & a_2 & a_3 & a_4 & a_6 & a_5 & a_7 \\ a_1 & a_2 & a_3 & a_4 & a_6 & a_7 & a_5 \\ a_1 & a_2 & a_3 & a_4 & a_7 & a_5 & a_6 \\ a_1 & a_2 & a_3 & a_4 & a_7 & a_6 & a_5 \\ a_1 & a_2 & a_3 & a_5 & a_4 & a_6 & a_7 \end{bmatrix}$$

According to this conjecture, we say that the minimum rank of  $M$  is 4

$$q = 3 \Rightarrow 7 > 3!, \quad q = 4 \Rightarrow 7 < 4!$$

Now we try to show that this is indeed the case. Perform row operations to get:

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & 0 & 0 & a_7 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_5 - a_6 & 0 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_7 - a_6 & a_5 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & a_5 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & 0 & a_5 - a_7 \\ 0 & 0 & 0 & a_5 - a_4 & a_4 - a_5 & 0 & 0 \end{bmatrix}$$

Next, we perform column operations to get:

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 + a_5 + a_6 + a_7 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & 0 & 0 & a_7 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_5 - a_6 & 0 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_7 - a_6 & a_5 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & a_5 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & 0 & a_5 - a_7 \\ 0 & 0 & 0 & 0 & a_4 - a_5 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & a_4 + a_5 + a_6 + a_7 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & 0 & 0 & a_7 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_5 - a_6 & 0 \\ 0 & 0 & 0 & 0 & a_6 - a_5 & a_7 - a_6 & a_5 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & a_5 - a_6 & a_6 - a_7 \\ 0 & 0 & 0 & 0 & a_7 - a_5 & 0 & a_5 - a_7 \\ 0 & 0 & 0 & 0 & a_4 - a_5 & 0 & 0 \end{bmatrix}$$

Now we prove that the last four columns are linearly independent. Denote

$$c_1 = \begin{bmatrix} a_4 + a_5 + a_6 + a_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; c_2 = \begin{bmatrix} a_5 \\ 0 \\ a_6 - a_5 \\ a_6 - a_5 \\ a_7 - a_5 \\ a_7 - a_5 \\ a_4 - a_5 \end{bmatrix}; c_3 = \begin{bmatrix} a_6 \\ a_7 - a_6 \\ a_5 - a_6 \\ a_7 - a_6 \\ a_5 - a_6 \\ 0 \\ 0 \end{bmatrix}; c_4 = \begin{bmatrix} a_7 \\ a_6 - a_7 \\ 0 \\ a_5 - a_7 \\ a_6 - a_7 \\ a_5 - a_7 \\ 0 \end{bmatrix}$$

Need:  $\underline{c_1}, \underline{c_2}, \underline{c_3}$  and  $\underline{c_4}$  are linearly independent

$$\Leftrightarrow \alpha_1 \underline{c_1} + \alpha_2 \underline{c_2} + \alpha_3 \underline{c_3} + \alpha_4 \underline{c_4} = \underline{0}, \quad \text{s.t. } \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$



We need the above vector equation to be satisfied. In particular it should be satisfied no matter which row we consider. We then get:

$$\alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \alpha_3 \cdot (a_7 - a_6) + \alpha_4 \cdot (a_6 - a_7) = 0, \quad \text{considering 2}^{nd} \text{ row}$$

$$\alpha_1 \cdot 0 + \alpha_2 \cdot (a_6 - a_5) + \alpha_3 \cdot (a_5 - a_6) + \alpha_4 \cdot 0 = 0, \quad \text{considering 3}^{rd} \text{ row}$$

$$\alpha_1 \cdot 0 + \alpha_2 \cdot (a_7 - a_5) + \alpha_3 \cdot 0 + \alpha_4 \cdot (a_5 - a_7) = 0, \quad \text{considering 6}^{th} \text{ row}$$

$$\alpha_1 \cdot 0 + \alpha_2 \cdot (a_4 - a_5) + \alpha_3 \cdot 0 + \alpha_4 \cdot 0 = 0, \quad \text{considering 7}^{th} \text{ row}$$

Which then leads to:

$$(a_7 - a_6)(\alpha_3 - \alpha_4) = 0, \quad (a_6 - a_5)(\alpha_2 - \alpha_3) = 0$$

$$(a_5 - a_7)(\alpha_4 - \alpha_2) = 0, \quad \alpha_2(a_4 - a_5) = 0$$

As per the construction, the  $a_i$  are all distinct. This then leads to:

$$\alpha_3 = \alpha_4, \quad \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_2, \quad \alpha_2 = 0$$

$$\Rightarrow \alpha_3 = \alpha_4 = \alpha_2 = \alpha = 0$$

Finally this yields:

$$\alpha_1 \cdot (a_4 + a_5 + a_6 + a_7) + \alpha_2 \cdot a_5 + \alpha_3 \cdot a_6 + \alpha_4 \cdot a_7 = 0$$

$$\alpha_1 \cdot (a_4 + a_5 + a_6 + a_7) + \alpha \cdot (a_5 + a_6 + a_7) = 0$$

$$\alpha_1 \cdot (a_4 + a_5 + a_6 + a_7) = 0$$

Now, if we also impose the condition  $(a_4 + a_5 + a_6 + a_7) \neq 0$ , we get

$$\alpha_1 = 0$$

Thus

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha = 0$$

And the four columns  $c_1$  through  $c_4$  are linearly independent.

We see that the smallest integer  $q$  which satisfies the conjecture:

$$n \leq q!$$

$$7 \leq q!$$

*The smallest  $q$  would be:  $q = 4$*

We see that the rank of the matrix  $M$  that we constructed is also the same as  $q=4$ .

The reason we call this the lowest possible rank is because in this particular case we change the positions of only very few elements. The number of elements that need to be permuted is much lesser than the size of the matrix. Hence we end up with a matrix that can be reduced to one with a large number of zeros.

We now try to give a possible proof to support the conjecture. There are two cases we should consider - when  $n = q!$  and when  $n < q!$

i)  $n = q!$

$$M = \begin{bmatrix} a_1 & a_2 & \dots & \dots & \dots & \dots & a_{q-1} & a_q & a_{q+1} & \dots & \dots & \dots & a_n \\ a_{p+m} & a_{q-k} & \dots & \dots & \dots & \dots & a_{q-p} & a_i & a_{q+1} & \dots & \dots & \dots & a_n \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_6 & a_{k-p} & \ddots & \ddots & \ddots & \ddots & a_m & a_j & a_{q+1} & \dots & \dots & \dots & a_n \\ a_k & a_{i+j} & \dots & \dots & \dots & \dots & a_6 & a_p & a_{q+1} & \dots & \dots & \dots & a_n \end{bmatrix}$$

$$\text{Construct: } A_{n \times q} = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_{q-1} & a_q \\ a_{p+m} & a_{q-k} & \cdots & \cdots & a_{q-p} & a_i \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_6 & a_{k-p} & \cdots & \cdots & a_m & a_j \\ a_k & a_{i+j} & \cdots & \cdots & a_6 & a_p \end{bmatrix}$$

We need to prove now that  $A$  is of full column rank. Denote coefficients  $\alpha_i$ - one for each one of the columns of the matrix  $A$ . Then we need to prove:

$$\alpha_1 \underline{A_1} + \alpha_2 \underline{A_2} + \cdots + \alpha_{q-1} \underline{A_{q-1}} + \alpha_q \underline{A_q} = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{q-1} = \alpha_q = 0$$

Perform row operations to get:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_{q-1} & a_q \\ a_{p+m} - a_1 & a_{q-k} - a_2 & \cdots & \cdots & a_{q-p} - a_{q-1} & a_i - a_q \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_6 - a_1 & a_{k-p} - a_2 & \cdots & \cdots & a_m - a_{q-1} & a_j - a_q \\ a_k - a_1 & a_{i+j} - a_2 & \cdots & \cdots & a_6 - a_{q-1} & a_p - a_q \end{bmatrix}$$

It can be seen that each of the  $q$  numbers appear at each of the  $q$  positions exactly  $(q-1)!$  times.

As a result, for each  $a_i, (i = 1 \text{ to } q)$  we can construct  $q-1$  equations in the remaining  $q-1$  numbers. Please note that every row sums to the same value (because we have the same set of numbers in every row, only the ordering has changed). This trivial fact is highly important and is used recursively. We then consider this reduced system of equations which we call  $A_{red}$

$$A_{red} \underline{\alpha} = \begin{bmatrix} a_{p+m} - a_1 & a_{q-k} - a_2 & \cdots & \cdots & a_{q-p} - a_{q-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_6 - a_1 & a_{k-p} - a_2 & \cdots & \cdots & a_m - a_{q-1} \\ a_k - a_1 & a_{i+j} - a_2 & \cdots & \cdots & a_6 - a_{q-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \end{bmatrix} = \underline{0}$$

$$\begin{bmatrix} a_{p+m} - a_1 & a_{q-k} - a_2 & \cdots & \cdots & a_{q-p} - a_{q-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_6 - a_1 & a_{k-p} - a_2 & \cdots & \cdots & a_m - a_{q-1} \\ a_k - a_1 & a_{i+j} - a_2 & \cdots & \cdots & a_6 - a_{q-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

In a similar manner we can prove that all the  $\alpha$  are equal. This then leads to (from the first row of A):

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{q-1} a_{q-1} + \alpha_q a_q = 0 \Rightarrow \alpha a_1 + \alpha a_2 + \cdots + \alpha a_{q-1} + \alpha a_q = 0$$

$$\Rightarrow \alpha(a_1 + a_2 + \cdots + a_{q-1} + a_q) = 0$$

$$\Rightarrow \alpha(a_1 + a_2 + \cdots + a_{q-1} + a_q) = 0$$

$$\Rightarrow \alpha = 0 \text{ or } (a_1 + a_2 + \cdots + a_{q-1} + a_q) = 0$$

For the moment, let us suppose that  $(a_1 + a_2 + \cdots + a_{q-1} + a_q) \neq 0$ . Then we have  $\alpha = 0$  and so the  $q$  columns of A are linearly independent. Also, we have changed only the  $q$  elements in the big matrix M. So the rank of the matrix M is essentially the same as the rank of A. And this gives us that the rank of the matrix M is

$$\text{rank}(M) = q, \quad \text{where: } q! = n$$

ii)  $n < q!$

In this situation, an exact proof could not be obtained. We leave the current problem in this state with the hope that the second part can also be proved.

# Appendix B

## Miscellaneous Properties

The thesis dealt with the problem of obtaining the factors for the 2-by-2 block matrices cases. We also looked at the results that we obtained from the 3-by-3 block matrices cases. We made an observation that the ranks of the matrices in the product of  $G_{123}$  seemed to follow a particular pattern. We furnished a table that showed the same. We now briefly extend that result that we obtained to the case where we have 4-by-4 block matrices in the product. The table for this is presented below

No. of shifts				Rank of matrix			
H1	H2	H3	H4	rank(R)	rank(S)	rank(T)	rank(U)
0	1	2	3	1	2	3	3
1	2	3	0	1	2	3	3
2	3	0	1	1	2	3	3
3	0	1	2	1	2	3	3
0	1	3	2	1	3	3	1
1	3	2	0	2	4	3	1
3	2	0	1	1	3	3	1
2	0	1	3	1	3	4	2
0	2	1	3	2	4	3	1
2	1	3	0	1	3	3	1
1	3	0	2	1	3	4	2
3	0	2	1	1	3	3	1
0	2	3	1	1	3	4	2
2	3	1	0	1	3	3	1
3	1	0	2	2	4	3	1
1	0	2	3	1	3	3	1
0	3	1	2	1	3	3	1
3	1	2	0	1	3	4	2
1	2	0	3	1	3	3	1
2	0	3	1	2	4	3	1
0	3	2	1	3	3	2	1
3	2	1	0	3	3	2	1
2	1	0	3	3	3	2	1
1	0	3	2	3	3	2	1

**Table 2: Shifts and ranks for different  $H_1, H_2, H_3$  and  $H_4$**

We see that for the 4-by-4 blocks case, in some cases cyclically switching the order of multiplication of the matrices results in the same order of ranks for the products. However, this is not always the case as can be seen from the second set that is considered. This table might be useful for those who want to have a quick idea of what the ranks look like without having to perform the experiments in full.

From the work done over the period of the thesis, a couple of interesting patterns were obtained.

We list a couple of them here:

- i) Suppose we have  $n$  factors, each of which is made up of  $n$ -by- $n$  block matrices. Let us denote them to be  $f_1$  through  $f_n$ . Assume further that the each matrix from  $f_2$  is shifted one down and one to the right from its predecessor's position. This would mean that  $f_1$  would have no shift,  $f_2$  would have a shift one down and one to the right in position with respect to  $f_1$  and so on. Now, let us consider the product of the matrices  $f_1$  through  $f_k$ , We then have

$$\text{ranks of matrices in the product } f_{1\dots k} = 1, 2, 3 \dots k, \quad 2 \leq k \leq n - 1$$

$$\text{ranks of matrices in the product } f_{1\dots k} = 1, 2, 3 \dots k - 1, k - 1, \quad k = n$$

Let us apply this to the three factor 3-by-3 case. We then have

$$n = 3 \Rightarrow \text{ranks of matrices in the product } f_{1\dots k} = 1, 2, 3 \dots k, \quad 1 \leq k \leq 2$$

$$k = 2 \Rightarrow \text{ranks of matrices in the product } f_{12} = 1, 2$$

$$k = 3 \Rightarrow \text{ranks of matrices in the product } f_{123} = 1, 2, 2$$

- ii) Another pattern that we obtained was related to the product factors made up of 2-by-2 block matrices. We saw that in the two factor case the ranks were simply (1, 1). This became (1, 2, 1) when three factors were considered. We then multiply the product  $F_{123}$  with another matrix  $F_4$  (also made up of 2-by-2 blocks and with the same structure as  $F_2$ ). The rank pattern we got in this case was (1, 2, 2, 1). When this was continued with  $F_5$ , it lead to (1, 2, 2, 2, 1). Hence it became apparent that with every extra factor that we add in, we get another full rank matrix in the product. There is a logical explanation for this. Every additional multiplication essentially means that we combine entries that are themselves linear combination of the previous cases. So

initially we have two rank-1 matrices adding up to give a rank-2 (full rank) matrix. Since we are looking at 2-by-2 blocks, more such additions will only give a full rank matrix and that is why we keep getting rank-2 blocks in the product.



# Appendix C

## Extension of results obtained in [1]

We now look at some of the properties that we can derive for the matrices P and Q as mentioned in the paper by Strang[1]. We examine the construction of the matrices R, S and T among others.

Basically we look at the matrices P and Q. We need them to be of rank-1 each and we also need to satisfy a few conditions. The properties we need from them are the following

$$PQ = QP = 0, \quad P + Q = I, \quad P^2 = P, \quad Q^2 = Q$$

$$P = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad Q = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

Since P and Q should be of rank-1, we can re-write them as:

$$P = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \quad Q = \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix}$$

$$P^2 = P \Rightarrow \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix}$$

$$\Rightarrow A_1^2 \begin{bmatrix} 1 + kp & p + kp^2 \\ k + k^2p & kp + k^2p^2 \end{bmatrix} = A_1 \begin{bmatrix} 1 & p \\ k & kp \end{bmatrix}$$

$$\Rightarrow A_1 \begin{bmatrix} (1 + kp)A_1 & p(1 + kp)A_1 \\ k(1 + kp)A_1 & kp(1 + kp)A_1 \end{bmatrix} = A_1 \begin{bmatrix} 1 & p \\ k & kp \end{bmatrix}$$

$$\Rightarrow (1 + kp)A_1 = 1, \quad \Rightarrow (1 + kp) = \frac{1}{A_1}, \quad \Rightarrow kp = \frac{1}{A_1} - 1$$

$$\Rightarrow k = \left(\frac{1}{A_1} - 1\right)/p$$

In a similar manner using  $Q^2 = Q$ , we get:

$$\Rightarrow l = \left(\frac{1}{B_1} - 1\right)/q$$

Next, we make use of  $PQ = QP = 0$  to get some more relations:

$$PQ = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix} = A_1 B_1 \begin{bmatrix} 1 + pl & q(1 + pl) \\ k(1 + pl) & kq(1 + pl) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$QP = \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix} \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} = A_1 B_1 \begin{bmatrix} 1 + kq & p(1 + kq) \\ l(1 + kq) & lp(1 + kq) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 1 + pl = 0, \quad 1 + qk = 0$$

$$\Rightarrow l = -\frac{1}{p}, \quad q = -\frac{1}{k}$$

Finally, we make use of the last requirement:

$$P + Q = I$$

$$\begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} + \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_1 + B_1 & pA_1 + qB_1 \\ kA_1 + lB_1 & kpA_1 + lqB_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A_1 + B_1 = 1, \quad pA_1 + qB_1 = 0, \quad kA_1 + lB_1 = 0, \quad kpA_1 + lqB_1 = 1$$

Now, we make use of the relations connecting q with k, l with p and k with p to get:

$$pA_1 + qB_1 = 0 \Rightarrow pA_1 - \frac{1}{k}B_1 = 0 \Rightarrow pkA_1 - B_1 = 0$$

$$\left(\frac{1}{A_1} - 1\right)A_1 - B_1 = 0 \Rightarrow 1 - A_1 - B_1 = 0 \Rightarrow A_1 + B_1 = 1$$

In a similar manner, the other two equations yield the same result. Hence we need:

$$A_1 + B_1 = 1$$

To recap, the following need to hold:

$$k = \left(\frac{1}{A_1} - 1\right)/p \quad l = \left(\frac{1}{B_1} - 1\right)/q \quad l = -\frac{1}{p} \quad q = -\frac{1}{k} \left. \vphantom{\begin{matrix} k \\ l \\ l \\ q \end{matrix}} \right\} \Rightarrow (A_1 + B_1 = 1)$$

Now let us use these relations in the simple case of a matrix formed from the product  $F_{12}$  (where  $F_{12}$  is obtained from the matrices  $F_1$  and  $F_2$ , with  $F_2$  having a shift in its structure). As usual we let the block matrices in  $F_1$  be 'a' and those in  $F_2$  be 'b':

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then we have:

$$P = \begin{bmatrix} a_{11}b_{21} & a_{11}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} \end{bmatrix} = a_{11}b_{21} \begin{bmatrix} 1 & \frac{b_{22}}{b_{21}} \\ \frac{a_{21}}{a_{11}} & \frac{a_{21}b_{22}}{a_{11}b_{21}} \end{bmatrix} \Rightarrow A_1 = a_{11}b_{21} \quad p = \frac{b_{22}}{b_{21}} \quad k = \frac{a_{21}}{a_{11}}$$

$$Q = \begin{bmatrix} a_{12}b_{11} & a_{12}b_{12} \\ a_{22}b_{11} & a_{22}b_{12} \end{bmatrix} = a_{12}b_{11} \begin{bmatrix} 1 & \frac{b_{12}}{b_{11}} \\ \frac{a_{22}}{a_{12}} & \frac{a_{22}b_{12}}{a_{12}b_{11}} \end{bmatrix} \Rightarrow B_1 = a_{12}b_{11} \quad q = \frac{b_{12}}{b_{11}} \quad l = \frac{a_{22}}{a_{12}}$$

We use the previously mentioned equations to get:

$$\begin{aligned}
k &= \left(\frac{1}{A_1} - 1\right)/p & l &= \left(\frac{1}{B_1} - 1\right)/q \\
\Rightarrow \boxed{a_{11}b_{21} + a_{21}b_{22} = 1} & & \Rightarrow \boxed{a_{12}b_{11} + a_{22}b_{12} = 1} \\
l &= -1/p & q &= -1/k \\
\Rightarrow \boxed{a_{12}b_{21} + a_{22}b_{22} = 0} & & \Rightarrow \boxed{a_{11}b_{11} + a_{21}b_{12} = 0}
\end{aligned}$$

Solving this set of equations leads to:

$$\begin{aligned}
a_{11} &= -\frac{b_{12}}{\det(b)} & a_{12} &= \frac{b_{22}}{\det(b)} \\
a_{21} &= \frac{b_{11}}{\det(b)} & a_{22} &= -\frac{b_{21}}{\det(b)} \\
a &= (1/\det(b)) \begin{bmatrix} -b_{12} & b_{22} \\ b_{11} & -b_{21} \end{bmatrix} & b &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
\end{aligned}$$

We now look at the construction of 2-by-2 matrices R, S and T in the product and look at the means of constructing them. We know that the matrices R and T need to be of rank-1 while the matrix S is of full rank.

$$\begin{aligned}
r &= \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} & t &= \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} & u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & v &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
R = ru' &= \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} r_1u_1 & r_1u_2 \\ r_2u_1 & r_2u_2 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \\
T = tv' &= \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} t_1v_1 & t_1v_2 \\ t_2v_1 & t_2v_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}
\end{aligned}$$

We construct two different S matrices – S<sub>1</sub> and S<sub>2</sub> – to determine which should be the one that needs to be used to get a banded inverse.

$$S_1 = rv' + tu' = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} [v_1 \ v_2] + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} [u_1 \ u_2]$$

$$S_1 = \begin{bmatrix} r_1 v_1 & r_1 v_2 \\ r_2 v_1 & r_2 v_2 \end{bmatrix} + \begin{bmatrix} t_1 u_1 & t_1 u_2 \\ t_2 u_1 & t_2 u_2 \end{bmatrix} = \begin{bmatrix} r_1 v_1 + t_1 u_1 & r_1 v_2 + t_1 u_2 \\ r_2 v_1 + t_2 u_1 & r_2 v_2 + t_2 u_2 \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}$$

$$S_2 = rv' + \alpha tu' = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} [v_1 \ v_2] + \alpha \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} [u_1 \ u_2]$$

$$S_2 = \begin{bmatrix} r_1 v_1 & r_1 v_2 \\ r_2 v_1 & r_2 v_2 \end{bmatrix} + \alpha \begin{bmatrix} t_1 u_1 & t_1 u_2 \\ t_2 u_1 & t_2 u_2 \end{bmatrix} = \begin{bmatrix} r_1 v_1 + \alpha t_1 u_1 & r_1 v_2 + \alpha t_1 u_2 \\ r_2 v_1 + \alpha t_2 u_1 & r_2 v_2 + \alpha t_2 u_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix}$$

Now construct  $M_1$  and  $M_2$  as follows:

$$M_1 = R + S_1 z + T z^2 \quad M_2 = R + S_2 z + T z^2$$

$$M_1 = \begin{bmatrix} r_1 u_1 + (r_1 v_1 + t_1 u_1)z + (t_1 v_1)z^2 & r_1 u_2 + (r_1 v_2 + t_1 u_2)z + (t_1 v_2)z^2 \\ r_2 u_1 + (r_2 v_1 + t_2 u_1)z + (t_2 v_1)z^2 & r_2 u_2 + (r_2 v_2 + t_2 u_2)z + (t_2 v_2)z^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} r_1 u_1 + (r_1 v_1 + \alpha t_1 u_1)z + (t_1 v_1)z^2 & r_1 u_2 + (r_1 v_2 + \alpha t_1 u_2)z + (t_1 v_2)z^2 \\ r_2 u_1 + (r_2 v_1 + \alpha t_2 u_1)z + (t_2 v_1)z^2 & r_2 u_2 + (r_2 v_2 + \alpha t_2 u_2)z + (t_2 v_2)z^2 \end{bmatrix}$$

Let us consider  $M_1$  first:

$$M_1 = R + S_1 z + T z^2$$

$$M_1 = \begin{bmatrix} r_1 u_1 + (r_1 v_1 + t_1 u_1)z + (t_1 v_1)z^2 & r_1 u_2 + (r_1 v_2 + t_1 u_2)z + (t_1 v_2)z^2 \\ r_2 u_1 + (r_2 v_1 + t_2 u_1)z + (t_2 v_1)z^2 & r_2 u_2 + (r_2 v_2 + t_2 u_2)z + (t_2 v_2)z^2 \end{bmatrix}$$

Using our determinant notation we have:

$$\text{constant} = \begin{vmatrix} R_1 & R_2 \\ R_3 & R_4 \end{vmatrix} = 0, \quad \text{coefficient of } z = \begin{vmatrix} R_1 & R_2 \\ \beta_3 & \beta_4 \end{vmatrix} + \begin{vmatrix} \beta_1 & \beta_2 \\ R_3 & R_4 \end{vmatrix}$$

$$\text{coefficient of } z^2 = \begin{vmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{vmatrix} + \begin{vmatrix} R_1 & R_2 \\ T_3 & T_4 \end{vmatrix} + \begin{vmatrix} T_1 & T_2 \\ R_3 & R_4 \end{vmatrix}$$

$$\text{coefficient of } z^3 = \begin{vmatrix} T_1 & T_2 \\ \beta_3 & \beta_4 \end{vmatrix} + \begin{vmatrix} \beta_1 & \beta_2 \\ T_3 & T_4 \end{vmatrix}, \quad \text{coefficient of } z^4 = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix} = 0$$

Consider now, the remaining terms – coefficient of  $z$ , coefficient of  $z^3$  and coefficient of  $z^2$ :

$$\text{We get: coefficient of } z = 0, \quad \text{coefficient of } z^3 = 0$$

$$\text{And: coefficient of } z^2 = 0$$

$$\Rightarrow \det(M_1) = 0$$

This implies that the matrix with  $R$ ,  $S_1$  and  $T$  as blocks will not be invertible.

Let us now consider  $M_2$ :

$$M_2 = R + S_2 z + T z^2$$

$$M_2 = \begin{bmatrix} r_1 u_1 + (r_1 v_1 + a t_1 u_1) z + (t_1 v_1) z^2 & r_1 u_2 + (r_1 v_2 + a t_1 u_2) z + (t_1 v_2) z^2 \\ r_2 u_1 + (r_2 v_1 + a t_2 u_1) z + (t_2 v_1) z^2 & r_2 u_2 + (r_2 v_2 + a t_2 u_2) z + (t_2 v_2) z^2 \end{bmatrix}$$

$$\text{constant} = \begin{vmatrix} R_1 & R_2 \\ R_3 & R_4 \end{vmatrix} = 0, \quad \text{coefficient of } z = \begin{vmatrix} R_1 & R_2 \\ \gamma_3 & \gamma_4 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \gamma_2 \\ R_3 & R_4 \end{vmatrix}$$

$$\text{coefficient of } z^2 = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{vmatrix} + \begin{vmatrix} R_1 & R_2 \\ T_3 & T_4 \end{vmatrix} + \begin{vmatrix} T_1 & T_2 \\ R_3 & R_4 \end{vmatrix}$$

$$\text{coefficient of } z^3 = \begin{vmatrix} T_1 & T_2 \\ \gamma_3 & \gamma_4 \end{vmatrix} + \begin{vmatrix} \gamma_1 & \gamma_2 \\ T_3 & T_4 \end{vmatrix}, \quad \text{coefficient of } z^4 = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix} = 0$$

Consider now, the remaining terms – coefficient of  $z$ , coefficient of  $z^3$  and coefficient of  $z^2$ :

$$\text{We get: coefficient of } z = 0, \quad \text{coefficient of } z^3 = 0$$

$$\text{And: coefficient of } z^2 = (1 - \alpha)(r_2 t_1 - r_1 t_2)(u_2 v_1 - u_1 v_2)$$

$$\Rightarrow \det(M_2) = ((1 - \alpha)(r_2 t_1 - r_1 t_2)(u_2 v_1 - u_1 v_2)) z^2$$

Let us look at  $\det(M_2)$  a bit more closely:

$$\det(M_2) = ((1 - \alpha)(r_2 t_1 - r_1 t_2)(u_2 v_1 - u_1 v_2))z^2$$

This will be zero when one of the terms is zero:

$$(1 - \alpha)(r_2 t_1 - r_1 t_2)(u_2 v_1 - u_1 v_2) = 0$$

$$\Rightarrow (1 - \alpha) = 0 \text{ or } (r_2 t_1 - r_1 t_2) = 0 \text{ or } (u_2 v_1 - u_1 v_2) = 0$$

i)  $(r_2 t_1 - r_1 t_2) = 0$

$$\Rightarrow r_2 t_1 = r_1 t_2, \quad \Rightarrow \frac{r_2}{r_1} = \frac{t_2}{t_1}, \quad \Rightarrow \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \mu \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \quad \Rightarrow r = \mu t$$

$$\Rightarrow S_2 = r v' + a t u' = \mu t v' + a t u' = t(\mu v' + a u')$$

This then implies that  $S_2$  is of rank-1. However, we need it to be of rank-2 if we need the big matrix to be factorizable with a banded inverse.

ii)  $(u_2 v_1 - u_1 v_2) = 0$

$$\Rightarrow u_2 v_1 = u_1 v_2, \quad \Rightarrow \frac{u_2}{u_1} = \frac{v_2}{v_1}, \quad \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \rho \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \Rightarrow u = \rho v$$

$$\Rightarrow S_2 = r v' + a t u' = r \rho v + a t u' = (r \rho + a t) u'$$

This once again implies that  $S_2$  is of rank-1. However, we need it to be of rank-2 if we need the big matrix to be factorizable with a banded inverse.

iii)  $(1 - \alpha) = 0$

$$\Rightarrow \alpha = 1, \quad \Rightarrow S_2 = r v' + t u' = S_1$$

We get that the matrix  $S_2$  in this case is the same as  $S_1$ . But we also know that  $S_1$  does not lead to a monomial determinant for  $M_1$ . The determinant will instead be zero implying that the matrix is not invertible.

Thus we see that as long as  $\alpha \neq 1$ , it is possible to have a monomial determinant for  $M_2$ .

We also need the following to hold true:

$$PT = QR = 0$$

$$PT = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \begin{bmatrix} t_1v_1 & t_1v_2 \\ t_2v_1 & t_2v_2 \end{bmatrix} = 0 \quad QR = \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix} \begin{bmatrix} r_1u_1 & r_1u_2 \\ r_2u_1 & r_2u_2 \end{bmatrix} = 0$$

$$PT = \begin{bmatrix} A_1t_1v_1 + pA_1t_2v_1 & A_1t_1v_2 + pA_1t_2v_2 \\ kA_1t_1v_1 + kpA_1t_2v_1 & kA_1t_1v_2 + kpA_1t_2v_2 \end{bmatrix} = 0$$

$$QR = \begin{bmatrix} B_1r_1u_1 + qB_1r_2u_1 & B_1r_1u_2 + qB_1r_2u_2 \\ lB_1r_1u_1 + lqB_1r_2u_1 & lB_1r_1u_2 + lqB_1r_2u_2 \end{bmatrix} = 0$$

$$PT = \begin{bmatrix} t_1v_1 + pt_2v_1 & t_1v_2 + pt_2v_2 \\ kt_1v_1 + kpt_2v_1 & kt_1v_2 + kpt_2v_2 \end{bmatrix} = \begin{bmatrix} v_1(t_1 + pt_2) & v_2(t_1 + pt_2) \\ kv_1(t_1 + pt_2) & kv_2(t_1 + pt_2) \end{bmatrix} = 0$$

$$\Rightarrow (t_1 + pt_2) = 0, \quad \Rightarrow \frac{t_1}{t_2} = -p, \quad \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -p \\ 1 \end{bmatrix}$$

$$QR = \begin{bmatrix} r_1u_1 + qr_2u_1 & r_1u_2 + qr_2u_2 \\ lr_1u_1 + lqr_2u_1 & lr_1u_2 + lqr_2u_2 \end{bmatrix} = \begin{bmatrix} u_1(r_1 + qr_2) & u_2(r_1 + qr_2) \\ lu_1(r_1 + qr_2) & lu_2(r_1 + qr_2) \end{bmatrix} = 0$$

$$\Rightarrow (r_1 + qr_2) = 0, \quad \Rightarrow \frac{r_1}{r_2} = -q = \frac{1}{k}, \quad \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$Pr = r, \quad Qt = t$$

$$Pr = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = r \quad Qt = \begin{bmatrix} B_1 & qB_1 \\ lB_1 & lqB_1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t$$



$$Pr = \begin{bmatrix} A_1 r_1 + pA_1 r_2 \\ kA_1 r_1 + kpA_1 r_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = r \quad Qt = \begin{bmatrix} B_1 t_1 + qB_1 t_2 \\ lB_1 t_1 + lqB_1 t_2 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t$$

It can be shown that r and t are linked to the eigenvectors of P and Q

$$(P - \lambda I)x = 0$$

$$(P - \lambda I) = \begin{bmatrix} A_1 - \lambda & pA_1 \\ kA_1 & kpA_1 - \lambda \end{bmatrix} = 0 \Rightarrow \det(P - \lambda I) = 0 = (A_1 - \lambda)(kpA_1 - \lambda) - (pA_1)(kA_1)$$

$$\Rightarrow (A_1 - \lambda)(kpA_1 - \lambda) = kpA_1^2$$

$$\Rightarrow (kpA_1^2 - \lambda A_1(1 + kp) + \lambda^2) = kpA_1^2$$

$$\Rightarrow \lambda(\lambda - A_1(1 + kp)) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = A_1(1 + kp) = 1$$

$$Px = \begin{bmatrix} A_1 & pA_1 \\ kA_1 & kpA_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x$$

$$A_1 x_1 + pA_1 x_2 = x_1 \text{ and } kA_1 x_1 + kpA_1 x_2 = x_2$$

$$x_1 = 1 \Rightarrow x_2 = \frac{1}{p} \left( \frac{1}{A_1} - 1 \right) = k \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = r$$

Thus r is an eigenvector of P. In a similar manner, it can be shown that t is an eigenvector of Q.



# Bibliography

- [1]. Strang G., *Fast transforms: Banded matrices with banded inverses*, Proc. Natl. Acad. Sci. 107 (2010) 12413-12416
- [2]. Burden, R. L., Faires, J. D., *Numerical Analysis, 8<sup>th</sup> ed.*, Thomson Brooks/Cole, (2005) 388-389
- [3]. Daubechies, I., *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math., 41(1988) 909-996
- [4]. Olshevsky, V., Strang, G., and Zhlobich, P., *Green's matrices*, Lin. Alg. Appl. 432 (2010) 218-241
- [5]. Barrett, W. W., and Feinsilver, P. J., *Inverses of Banded Matrices*, Lin. Alg. Appl. 41 (1981) 111-130
- [6]. Daubechies, I., and Sweldens, W., *Factoring Wavelet Transforms into Lifting Steps*, The Journal of Fourier Analysis and Applications, Volume 4, Issue 3, (1998) 247-269