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Typical Peak Sidelobe Level of Binary Sequences

Noga Alon, Simon Litsyn, Senior Member, IEEE, and Alexander Shpunt

Abstract—For a binary sequence \(S_n = \{s_i : i = 1, 2, \ldots, n\} \in \{\pm 1\}^n, n > 1\), the peak sidelobe level (PSL) is defined as

\[
M(S_n) = \max_{k=1,2,\ldots,n-1} \left| \sum_{i=1}^{n-k} s_is_{i+k} \right|.
\]

It is shown that the distribution of \(M(S_n)\) is strongly concentrated, and asymptotically almost surely

\[
\gamma(S_n) = \frac{M(S_n)}{\sqrt{n \ln n}} \in [1 - o(1), \sqrt{2}].
\]

Explicit bounds for the number of sequences outside this range are provided. This improves on the best earlier known result due to Moon and Moser [19] and settles to the affirmative the conjecture of Dmitriev and Jedwab on the growth rate of the typical peak sidelobe. Finally, it is shown that modulo some natural conjecture, the typical \(\gamma(S_n)\) equals \(\sqrt{2}\).

Index Terms—Aperiodic autocorrelation, concentration, peak sidelobe level (PSL), random binary sequences autocorrelation, second moment method.

I. INTRODUCTION AND DEFINITIONS

Let \(S_n = \{s_i : i = 1, 2, \ldots, n\} \in \mathcal{A}_n, n > 1\), where \(\mathcal{A}_n = \mathbb{F}^n, \mathbb{F} = \{+1, -1\}\). Define

\[
M_k(S_n) = \sum_{i=1}^{n-k} s_is_{i+k}, \quad k = 1, 2, \ldots, n-1.
\]

The peak sidelobe level (PSL) \(M(S_n)\) of a sequence \(S_n\), is

\[
M(S_n) = \max_{k=1,2,\ldots,n-1} M_k(S_n), \quad n > 1.
\]

Let \(\mu_n\) stand for the optimal value of the PSL over the set \(\mathcal{A}_n\)

\[
\mu_n = \min_{S_n \in \mathcal{A}_n} M(S_n).
\]

Binary sequences with low PSL are important for synchronization, communications, and radar pulse design, see, e.g., [6], [7], [12], [21], [22]. In theoretical physics, study of the PSL landscape was introduced by Bernasconi via the so-called Bernasconi model [3], which is fascinating for the fact of being completely deterministic, but nevertheless having highly disordered ground states (sequences with the largest merit factor) and thus possessing similarities to the real glasses, with many features of a glass transition exhibited [3], [11].

Study of the problem started in the 1950s. Special attention has been given to the estimation of typical PSL. Since this is our central interest in this paper, let us mention several relevant results. Moon and Moser [19] proved that for almost all sequences

\[
\kappa(n) \leq M(S_n) \leq (2 + \epsilon)\sqrt{n \ln n}
\]

for any \(\kappa(n) = o(\sqrt{n})\). Mercer [18] showed that

\[
\mu_n \leq (\sqrt{2} + \epsilon)\sqrt{n \ln n}
\]

Apparently, the suggested approach also shows proving that this bound is indeed true for most of the sequences (see comments in [18, bottom of p. 670]). Dmitriev and Jedwab [4] conjectured and provided experimental evidence that the typical PSL behaves as \(\Theta(\sqrt{n \ln n})\). The same was presumed without proof by Ein-Dor, Kanter, and Kinzel [5].

In this paper, we prove that almost all binary sequences \(S_n\) of length \(n\), \(M(S_n) = \Theta(\sqrt{n \ln n})\). Moreover, it is shown that asymptotically almost surely

\[
\gamma(S_n) = \frac{M(S_n)}{\sqrt{n \ln n}} \in [1 - o(1), \sqrt{2}].
\]

The results of the paper have application to another problem related to estimation of the “level of randomness” of finite sequences from \(\mathbb{F}^n\). Mauduit and Sarkozy [17] introduced the correlation measure of order \(r\), which is defined for a sequence \(S_n\) as

\[
C_r(S_n) = \max_{0 \leq i_1 < \cdots < i_r \leq n-1} \left| \sum_{i=1}^{n-r+i} s_{i_1+i} s_{i_2+i} \cdots s_{i_r+i} \right|.
\]

In other words, this is just the maximum of the absolute value of the mutual correlation of \(r\) continuous runs of vector’s entries. In [1], Alon, Kohayakawa, Mauduit, Morreira, and Rödl showed that asymptotically almost surely

\[
\sigma_r(S_n) := \frac{C_r(S_n)}{\sqrt{n \ln n}} \in \left( \frac{2}{5}, \frac{7}{4} \right).
\]

Noticing that \(M(S_n) \leq C_2(S_n)\), we conclude that any lower bound on the typical \(M(S_n)\) is a lower bound for the typical \(C_2(S_n)\) as well. Therefore, our results (slightly) improve the lower bound on

\[
\lim_{n \to \infty} \inf_{S_n} \sigma_2(S_n) = \frac{2}{5} \text{ in (2) to} \frac{1}{\sqrt{2}} \approx 0.7.
\]
The same improvement can be easily achieved for any $r$ using the method in the paper.

The paper proceeds as follows. In the next short section we sketch a quick proof, based on the approach in [1], that for almost all sequences $S_n$, $(1 - o(1))\sqrt{n \ln n} \leq M(S_n) \leq (\sqrt{2} + o(1))\sqrt{n \ln n}$. We then proceed to give a detailed analysis that provides a somewhat better control of the error terms in the above estimates. In Section III, we recall a theorem due to Moon and Moser [19] for the number of sequences $S_n$ such that $M_k(S_n) = r$ for any $k = 1, 2, \ldots, n - 1$, and $r = \equiv -n, \ldots, -1, n$. We then provide estimates for binomial coefficients allowing approximation of the Moon–Moser formula by tails of the Gaussian distribution with vanishing error. Section IV is devoted to proving the upper bound in (1). To do so, we relate, via the Moon–Moser theorem, the number of sequences $S_n$ with $M(S_n) > \sqrt{2n(\ln n + \delta(n))}$ to certain binomial sums. Accurate estimates using bounds developed in Section III allow to establish the sought inequality. Section V derives a lower bound for the number of sequences $S_n$ with $M(S_n) > \sqrt{n(\ln n + \delta(n))}$, by looking only at autocorrelations with shifts $\geq n/2$. This allows to consider $M_k(S_n), k = n/2 + 1, \ldots, n$ as a collection of linear forms with coefficients $s_1, \ldots, s_n/2$ and variables $s_{n/2 + 1}, \ldots, s_n$. We then apply the Azuma inequality to show concentration of $M(S_n)$, and the results of Sections IV and V to accurately locate the mean of $M(S_n)$, and thus establish the lower bound in (1). In Section VI, we argue that modulo a plausible conjecture, and using the Azuma inequality, for most $S_n$

$$\gamma(S_n) = \sqrt{2}(1 + o(1)).$$

We attempted to make the paper as self-contained as possible. To achieve this, we have included several sketchy proofs of relevant results from other papers conveying ideas of importance for our presentation.

II. A QUICK SKETCH

In this short section, we sketch a quick proof that for almost all sequences $S_n$

$$(1 - o(1))\sqrt{n \ln n} \leq M(S_n) \leq (\sqrt{2} + o(1))\sqrt{n \ln n}.$$  

The upper bound is simple (see, e.g., [18]): for each fixed $k$, the sum $M_k(S_n)$ is easily seen to be a sum of $n - k$ independent random variables, each attaining the values $-1$ and 1 with equal probability. It thus follows by standard estimates (cf., for example, [2, Corollary A.1.2]) that for a random sequence $S_n$, the probability that $|M_k(S_n)| > a$ is at most $2e^{-a^2/2(n-k)} < 2e^{-a^2/2n}$. Thus, for $a = (\sqrt{2} + \delta)\sqrt{n \ln n}$, where $\delta > 0$ is arbitrarily small, this probability is much smaller than $1/n$ (for all sufficiently large $n$), and it thus follows that with high probability, all $n$ numbers $M_k(S_n)$ are smaller than $(\sqrt{2} + \delta)\sqrt{n \ln n}$, providing the required upper bound.

To prove the lower bound, we consider only values of $k$ satisfying $k > n/2$. It turns out that for $k, \ell$ which are both larger than $n/2$, the $2n - (k + \ell)$ products

are random and independent members of $\{-1, 1\}$. A detailed proof of this simple yet somewhat surprising fact appears in Section V. For each $k, n/2 < k \leq n/2 + n/\ln n$, let $X_k$ denote the indicator random variable whose value is 1 if the event $|M_k(S_n)| \geq (1 - \delta)\sqrt{n \ln n}$ (which we denote here by $E_k$) occurs, and 0 otherwise. Our objective is to show that asymptotically almost surely, the sum $X = \sum_{n/2 < k \leq n/2 + n/\ln n} X_k$ is positive. By standard estimates, for each fixed $k$, the probability that $E_k$ occurs is bigger than, say, $n^2 / \ln n$ (for every fixed $\delta > 0$ and all sufficiently large $n$). This means that the expectation $E(X)$ of $X$ is at least $\ln n$. The crucial point is that since the indicator random variables $X_k$ are pairwise independent, the variance $\text{Var}(X)$ of $X$ is the sum of variances of the variables $X_k$, and is thus smaller than the expectation of $X$. Therefore, by Chebyshev’s inequality, the probability that $X$ is zero is at most $\text{Var}(X)/(E(X))^2 < 1/E(X) < 1/\ln n$, implying that asymptotically almost surely $X$ is positive, as needed.

The detailed proof, with a more careful treatment of the error terms, is given in the next sections. We present The proof of the lower bound that we present in Section V is slightly different than the one indicated above, as it seems interesting to describe an alternative approach which derives the bound by combining the pairwise independence of the random variables described above with Azuma’s inequality.

III. AUXILIARY RESULTS

Let $g(n, k, r)$ denote the number of sequences $S_n$, such that $M_k(S_n) = r$. Throughout, we shall adopt the convention that the binomial coefficient $\binom{n}{m}$ equals 0 if $x$ is not an integer and 1 if $m = x - 1$.

**Theorem 3.1 (Moon–Moser [19]):** For $r = -n, \ldots, -1, n$, and $k = 1, \ldots, n - 1$,

$$g(n, k, r) = 2k \left( \binom{n}{k} \left( \frac{1}{2} + \frac{r}{2(n-k)} \right) \right).$$

**Proof:** See a sketch in the Appendix.

Note that one of the consequences of Theorem 3.1 is that $g(n, k, r) = g(n, k, r)$.

In the derivation of our bounds, we will need the following estimates for binomial coefficients.

**Lemma 3.2:** For $0 < \epsilon_1 < \sqrt{3}/2$, and all $n$, such that $n \cdot (1/2 - \epsilon_1)$ is an integer

$$2^{-n} \cdot \left( n \cdot \left( \frac{n}{2} - \epsilon_1 \right) \right) \leq \left( 1 + \epsilon_1 \right) \cdot \sqrt{\frac{2}{\pi n}} \cdot e^{-2n\epsilon_1^2} \quad (3)$$

$$\epsilon_1 = 3\epsilon_1^2. \quad (4)$$

Moreover, for $0 < \epsilon_1 < (2n)^{-1/4}$ and $n \geq 164$, such that $n \cdot (1/2 - \epsilon_1)$ is an integer

$$2^{-n} \cdot \left( n \cdot \left( \frac{n}{2} - \epsilon_1 \right) \right) \geq \left( 1 - \epsilon_2 \epsilon_1 \right) \cdot \sqrt{\frac{2}{\pi n}} \cdot e^{-2n\epsilon_1^2} \quad (5)$$

$$\epsilon_2 \epsilon_1 = \frac{3}{2} n\epsilon_1^2 + \frac{1}{2n}. \quad (6)$$

**Proof:** See the Appendix.
The next result addresses the question of how well sums of binomial coefficients can be approximated by the Gaussian complementary cumulative distribution function (CCDF). Henceforth, the Gaussian CCDF is defined by

\[ P_G(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{2}}^{\infty} e^{-t^2} dt. \]

**Lemma 3.3:** Let \( n \geq 164 \)

\[ S(n, d) = \sum_{k=\frac{n}{2}}^{n} \left( n \cdot \binom{n}{k} \right). \]

Then, for \( \sqrt{n} \ln \ln n < d < n/2 \), the following holds:

\[ 2^{-m} \cdot S(n, d) \leq \left( 1 + \zeta_3(n, d) \right) \cdot P_G\left( \frac{d}{\sqrt{n}} \right) \cdot \zeta_3(n, d) \geq \frac{7d}{n} \tag{7} \]

and for \( \sqrt{n} \ln \ln n < d < (2n)^{3/4} \)

\[ 2^{-m} \cdot S(n, d) \geq \left( 1 - \zeta_3(n, d) \right) \cdot P_G\left( \frac{d}{\sqrt{n}} \right) - e^{-\sqrt{n/32}} \tag{8} \]

\[ \zeta_3(n, d) = \frac{1}{2n} + \frac{5d^4}{n^3}. \tag{9} \]

**Proof:** See the Appendix.

The bounds on \( S(n, d) \) in Lemma 3.3 are given in terms of \( P_G(\frac{d}{\sqrt{n}}) \). In certain cases, we would like to provide more explicit bounds, which can be achieved with the following.

**Lemma 3.4:** For \( d > 0, n > 0 \)

\[ \frac{\sqrt{2n}}{\sqrt{n}d} \cdot e^{-\frac{d^2}{2n}} \cdot \left( 1 - \frac{1}{2n^2} \right) \leq P_G\left( \frac{d}{\sqrt{n}} \right) \leq \frac{\sqrt{2n}}{\sqrt{n}d} \cdot e^{-\frac{d^2}{2n^2}}. \]

**Proof:** Use, for \( x > 0 \)

\[ e^{-x^2} \left( 1 - \frac{1}{2n^2} \right) \leq \frac{2}{\sqrt{n}} \int_x^{\infty} e^{-t^2} dt \leq e^{-x^2} \frac{2}{\sqrt{n}x}. \]

It will turn out that a specific form of \( d \) is of interest. The following explicit bounds will be useful.

**Lemma 3.5:** For \( \Delta = \frac{2n(\ln n + \delta(n))}{n} < \frac{n}{4} \)

\[ P_G\left( \frac{\Delta}{\sqrt{n-k}} \right) \leq \frac{1}{2n} \cdot \frac{e^{-\delta(n)}}{\sqrt{n}} \cdot e^{-\frac{\ln(n) + \delta(n)}{n} k} \]

\[ P_G\left( \frac{\Delta}{\sqrt{n-k}} \right) \geq \frac{1}{2n} \cdot \frac{e^{-\delta(n)}}{\sqrt{n}} \cdot e^{-\frac{\ln(n) + \delta(n)}{n} k} \times (1 - kn^{-1} - (2\ln n + \delta(n))^{-1}). \]

**Proof:** See Appendix.

IV. AN UPPER BOUND ON \( M(S_n) \) FOR ALMOST ALL \( S_n \)

For \( d > 0 \), the number of sequences \( S_n \) such that \( M_k(S_n) \geq d \) is given by

\[ G(n, k, d) = \sum_{r=d}^{n-k} \binom{n-r}{k} \]

\[ = 2^k \sum_{r=d}^{n-k} \frac{n-k}{(n-k) \cdot \left( \frac{1}{2} - \frac{r}{n-k} \right)} \]

\[ = 2^k \cdot S(n-k, d). \]

The total number of binary sequences of length \( n \) is \( 2^n \), therefore

\[ \Pr(M_k(S_n) \geq d) = 2^{-n} \cdot G(n, k, d) = 2^{k-n} \cdot S(n-k, d), \]

Similarly, the number of sequences \( S_n \) such that \( M_k(S_n) \leq -d \) is given by

\[ \tilde{G}(n, k, d) = \sum_{r=-d}^{n-k} \binom{n-r}{k} \]

\[ = 2^k \sum_{r=-d}^{n-k} \frac{n-k}{(n-k) \cdot \left( \frac{1}{2} + \frac{r}{n-k} \right)} \]

\[ = 2^k \cdot S(n-k, d). \]

Therefore

\[ \Pr(M_k(S_n) \leq -d) = 2^{-n} \cdot \tilde{G}(n, k, d) = 2^{k-n} \cdot G(n, k, d) = \Pr(M_k(S_n) \geq d). \]

We have the following lemma.

**Lemma 4.1:** Let \( n - k \geq 164 \). For \( \sqrt{(n-k) \ln(n-k)} < d < (n-k)/2 \)

the following holds:

\[ \Pr(M_k(S_n) \geq d) \leq P_G\left( \frac{d}{\sqrt{n-k}} \right) \cdot \left( 1 + \zeta_3(n-k, d) \right) \]

\[ \zeta_3(n-k, d) = \frac{7d}{n-k} \]

and for \( \sqrt{(n-k) \ln(n-k)} < d < (n-k)^{3/4} \)

the following holds:

\[ \Pr(M_k(S_n) \geq d) \geq P_G\left( \frac{d}{\sqrt{n-k}} \right) \left( 1 - \zeta_3(n-k, d) \right) - e^{-\sqrt{n/32}} \]

\[ \zeta_3(n-k, d) = \frac{5d^4}{(n-k)^{3/2}}. \]

**Proof:** Use Lemma 3.3 with \( n \rightarrow k \) in place of \( n \).

Combining Lemmas 4.1 and 3.5, and using \( e^x > 1 + x \) for \( x > 0 \), we have

\[ 2^{-n} \sum_{k=1}^{n-k} \binom{n-k}{(n-k)} \leq \frac{1}{2n} \cdot \frac{e^{-\delta(n)}}{\sqrt{n} \ln n + \delta(n)} \sum_{k=1}^{n-k} e^{-\frac{\ln(n) + \delta(n)}{n} k} \cdot \left( 1 + \frac{7d}{n-k} \right) \]

\[ \leq \frac{1}{2n} \cdot \frac{e^{-\delta(n)}}{\sqrt{n} \ln n + \delta(n)} \times \frac{n}{\ln n + \delta(n)} \cdot (1 + o(1)). \]

Consequently, we have the following corollary.
We see that the lower and upper bounds for $2^{-n} \sum_{k=1}^{n} G(n,k,\Delta)$ differ only by a multiplicative constant.

V. A LOWER BOUND ON $M(S_n)$ FOR ALMOST ALL $S_n$

Notice that $M_{\frac{n}{2}}, M_{\frac{n}{2}+1}, \ldots M_{n-1}$ are linear in $s_1, s_2, \ldots, s_{\frac{n}{2}}$ and in $s_{\frac{n}{2}+1}, \ldots, s_n$, and therefore can be written collectively as a linear system

$$
\begin{pmatrix}
M_{\frac{n}{2}} \\
M_{\frac{n}{2}+1} \\
\vdots \\
M_{n-1}
\end{pmatrix} =
\begin{pmatrix}
s_1 & s_2 & \cdots & s_{\frac{n}{2}-1} & s_{\frac{n}{2}} \\
0 & s_1 & s_2 & \cdots & s_{\frac{n}{2}-2} & s_{\frac{n}{2}-1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & s_1 & s_{n-1} \\
0 & 0 & 0 & \cdots & 0 & s_n
\end{pmatrix}.
$$

This linearity allows us to prove independence of $M_{\frac{n}{2}+i-1}$ and $M_{\frac{n}{2}+j-1}$ for $1 \leq i < j \leq \frac{n}{2}$, in Lemma 5.1. Using the independence and the inclusion–exclusion principle, we provide a lower bound for the upper tail on the probability of the number of sequences $S_n$ with $M(S_n) \geq \sqrt{n \ln n + \delta(n)}$, Theorem 5.3.

Next, we use Azuma’s bound to show that since $M(S_n)$ satisfies a Lipschitz condition, the distribution of $M(S_n)$ is concentrated, though we cannot indicate where its expectation lies. However, noticing that the expectation cannot be too small, since otherwise its upper tail—an upper bound on the probability that $M(S_n) \geq \sqrt{n \ln n + \delta(n)}$—will contradict the earlier derived lower bound on the probability of the same event, we conclude that the expectation cannot be less than approximately $\sqrt{n \ln n}$.

**Lemma 5.1:** For any $1 \leq i < j \leq \frac{n}{2}$ and $d > 0$

$$
\Pr\left(M_{\frac{n}{2}+i-1} > d \cap M_{\frac{n}{2}+j-1} > d \right) = \Pr\left(M_{\frac{n}{2}+i-1} > d\right) \cdot \Pr\left(M_{\frac{n}{2}+j-1} > d\right).
$$

**Remark 5.2:** As suggested by a referee, the key idea of the statement is that $2n - (k + \ell)$ products are independent random variables. A consequence is that $M_{n/2+i-1}$ and $M_{n/2+j-1}$ are independent, which in turn implies the above lemma.

**Proof:** For $1 \leq i < j \leq \frac{n}{2}$, we consider two forms

$$
M_{\frac{n}{2}+i-1} = s_1 s_{\frac{n}{2}+i} + s_2 s_{\frac{n}{2}+i+1} + \cdots + s_{\frac{n}{2}} s_{i+1} s_n
$$

and

$$
M_{\frac{n}{2}+j-1} = s_1 s_{\frac{n}{2}+j} + s_2 s_{\frac{n}{2}+j+1} + \cdots + s_{\frac{n}{2}} s_{j+1} s_n.
$$

Notice that the number of product terms in the second form, $\frac{n}{2} - j + 1$, is less than the number of product terms in the first one, $\frac{n}{2} - i + 1$. Let us form a vector of length $n - 2j + 2$, having the first half consisting of the first $\frac{n}{2} - j + 1$ product terms from $M_{\frac{n}{2}+i-1}$ and the second half containing the product terms from $M_{\frac{n}{2}+j-1}$, namely

$$
\mathbf{x} = (x^{(1)}, x^{(2)}) = (s_1 s_{\frac{n}{2}+i} s_2 s_{\frac{n}{2}+i+1} \cdots s_{\frac{n}{2}-j+1} s_{n-j+i}, s_1 s_{\frac{n}{2}+j} s_2 s_{\frac{n}{2}+j+1} \cdots s_{\frac{n}{2}-j+1} s_{n-j+i}).
$$

For example, taking $\delta(n) = -1.5 \ln \ln n + \beta \ln \ln n$, we obtain the following.

**Corollary 4.3:**

$$
\Pr(M(S_n) \geq \sqrt{2n \ln n - 1.5 \ln \ln n + \beta \ln \ln n}) \leq O\left(\frac{1}{\ln^3 n}\right).
$$

For the sake of comparison, let us derive a lower bound for $2^{-n} \sum_{k=1}^{n} G(n,k,\Delta)$. For notational convenience, in what follows $f \ll g$ stands for $f = o(g)$.

**Lemma 4.4:** For

$$
\Delta = \sqrt{2n \ln n + \delta(n)}, \quad \sqrt{n} \ll \Delta \ll n^{3/4}
$$

$$
2^{-n} \sum_{k=1}^{n} G(n,k,\Delta) \geq \frac{1}{2} e^{-\frac{1}{2} \sqrt{\frac{n^2}{\ln n + \delta(n)}}} (1 - o(1)).
$$

**Proof:** Use Lemma 3.3 and note that for

$$
\sqrt{n} \ll \Delta \ll n^{3/4} \quad \text{and} \quad k \leq 2(n/\Delta)^2
$$

$$
s_{\frac{n}{2}}^{(n-k)} \cdot \frac{e^{-\sqrt{\frac{n}{\ln n}}}}{P_G\left(\frac{\Delta}{\sqrt{n/\ln n}}\right)} = o(1).
$$

Hence

$$
2^{-n} \sum_{k=1}^{n} G(n,k,\Delta) \geq \frac{1 - o(1)}{2n} \cdot \frac{e^{-\frac{1}{2} \sqrt{\frac{n^2}{\ln n + \delta(n)}}}}{\sqrt{\frac{n^2}{\ln n + \delta(n)}}} \sum_{k=1}^{n} \frac{n-k}{n} \cdot \frac{e^{-\frac{1}{2} \sqrt{\frac{n^2}{\ln n + \delta(n)}}}}{\sqrt{\frac{n^2}{\ln n + \delta(n)}}} \cdot (1 - o(1)). \quad \square
$$
Let us show that when
\[
\mathbf{y} = (s_1, s_2, \ldots, s_{\frac{n}{2} - j + 1})
\]
assumes all possible values from \(F_{2^j}^{\frac{n}{2} - j + 1}\), and
\[
\mathbf{z} = (s_{\frac{n}{2} + i}, s_{\frac{n}{2} + i + 1}, \ldots, s_n)
\]
asumes all possible values from \(F_{2^j}^{-2^j + 2}\), then \(\mathbf{z}\) assumes all possible values from \(F_{2^j}^{\frac{n}{2} - i + 1}\). Notice that \(\mathbf{w} = (\mathbf{w}^{(1)}, \mathbf{w}^{(2)}) \in (F_{2^j}^{\frac{n}{2} - j + 1})^2\), assumes all possible values from \(F_{2^j}^{\frac{n}{2} - j + 2}\) exactly once if and only if the vector \((\mathbf{w}^{(1)}, \mathbf{w}^{(2)} \ast \mathbf{w}^{(2)})\), where \(\ast\) stands for the coordinatewise multiplication of vectors, also assumes all possible values of \(F_{2^j}^{\frac{n}{2} - j + 2}\) exactly once. Indeed, for a fixed \(\mathbf{w}^{(1)}\), \(\mathbf{w}^{(2)}\) assumes all possible values exactly once. The same is clearly true for \((\mathbf{w}^{(1)} \ast \mathbf{w}^{(2)})\) for a fixed \(\mathbf{w}^{(1)}\) and \(\mathbf{w}^{(2)}\) running over all possibilities in \(F_{2^j}^{\frac{n}{2} - j + 1}\). In the opposite direction, the same is correct since the transform is involutive.

Using \(\mathbf{z}^{(1)}\) and \(\mathbf{z}^{(2)}\) in place of \(\mathbf{w}^{(1)}\) and \(\mathbf{w}^{(2)}\) in the preceding expression, and noticing that the variables \(s_{\frac{n}{2} - j + 2}, \ldots, s_{\frac{n}{2} - i + 1}\) do not appear in either \(\mathbf{z}^{(1)}\) or \(\mathbf{z}^{(2)}\) we conclude that
\[
(\mathbf{x}^{(1)}, s_{\frac{n}{2} - j + 2} s_{\frac{n}{2} - i + 1}, \ldots, s_{\frac{n}{2} - i + 1} \mathbf{x}^{(2)})
\]
assumes each of its \(2^{n - i - j + 2}\) possible values exactly \(2j - i\) times. Consequently, \(M_{\frac{n}{2} + i - 1}\) and \(M_{\frac{n}{2} + j - 1}\) are independent for all \(1 \leq i < j \leq \frac{n}{2}\).

**Theorem 5.3:** Let
\[
\Delta = \sqrt{n \ln n + \delta(n)}, \quad \sqrt{n \ln \ln n} < \Delta \leq n^{3/4}.
\]

Then
\[
\Pr(M(S_n) \geq \Delta) \geq \frac{2e^{-\delta(n)} \ln n \cdot \sqrt{\pi(n + \delta(n))}}{\ln n \ln n} (1 - o(1)).
\]

**Proof:** For any subset of sequences \(S_n\) from \(A_n\), and any subset \(K\) of indices \(k\) belonging to \(\{1, 2, \ldots, n - 1\}\), we have
\[
\Pr\left(\max_{k=1,2,\ldots,n-1} |M_k(S_n)| \geq \Delta\right) \geq \Pr\left(\max_{k\in K} |M_k(S_n)| \geq \Delta\right).
\]

For \(m = o(n)\) and \(K = \{\frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + m\}\), we have
\[
\Pr\left(\max_{k=1,2,\ldots,n-1} |M_k(S_n)| \geq \Delta\right) \geq \Pr\left(\max_{k=\frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + m} |M_k(S_n)| \geq \Delta\right).
\]

Moreover, by the inclusion–exclusion we have
\[
\Pr\left(\max_{k=\frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + m} |M_k(S_n)| \geq \Delta\right) \geq \sum_{k=\frac{n}{2}}^{\frac{n}{2} + m} \Pr(\max |M_k(S_n)| \geq \Delta) - \sum_{i,j=\frac{n}{2}, i \neq j} \Pr(\max |M_i(S_n)| \geq \Delta \cap |M_j(S_n)| \geq \Delta).
\]

The proof of the theorem is to be continued after we demonstrate the following auxiliary result.

**Lemma 5.4:** For any \(i < m, m = o(n)\), and \(\delta(n) > -\ln n + \ln \ln n\)
\[
\Pr\left(|M_{\frac{n}{2} + i - 1}(S_n)| \geq \sqrt{n \ln n + \delta(n)}\right)
\]
\[
\geq \frac{2e^{-\delta(n)}}{n} \cdot \sqrt{n \ln n + \delta(n)} (1 - o(1)).
\]

**Proof:** By construction
\[
M_{\frac{n}{2} + i - 1}(S_n) = s_1 s_{\frac{n}{2} + 1} s_{\frac{n}{2} + i + 1} + \cdots + s_{\frac{n}{2} - i + 1} s_n
\]
therefore, by Theorem 3.1
\[
\Pr\left(\frac{n}{2} + i - 1(S_n) = d\right) = 2^{\left(\frac{n}{2} - i + 1\right)} \cdot \left(\frac{n}{2} + i - 1\right) - \frac{\frac{n}{2} - i + 1}{2}.
\]

Applying Lemmas 3.3, 3.4 with \(n \rightarrow \frac{n}{2} - i + 1\) and
\[
d = \Delta = \sqrt{n \ln n + \delta(n)}
\]
we have
\[
\Pr\left(M_{\frac{n}{2} + i - 1}(S_n) \geq \sqrt{n \ln n + \delta(n)}\right)
\]
\[
= 2^{\left(\frac{n}{2} - i + 1\right)} \cdot \left(\frac{n}{2} + i - 1\right) - \frac{\frac{n}{2} - i + 1}{2} \cdot \frac{1 - \delta(n)}{2\ln n + \delta(n)} (1 - o(1)).
\]

For \(i = 1, 2, \ldots, m, m = o(n)\), we then have
\[
\Pr\left(M_{\frac{n}{2} + i - 1}(S_n) \geq \sqrt{n \ln n + \delta(n)}\right)
\]
\[
= \frac{1}{n} \cdot \frac{e^{-\delta(n)} \ln n \ln n}{\sqrt{n \ln n + \delta(n)}} \cdot (1 - o(1)).
\]

Let us now return to the proof of the theorem. Note that by symmetry
\[
\Pr(\max |M_k(S_n)| \geq \Delta \cap |M_j(S_n)| \geq \Delta) = 4 \cdot \Pr(\max |M_k(S_n)| \geq \Delta) - \Pr(\max |M_k(S_n)| \geq \Delta) \cdot \Pr(\max |M_j(S_n)| \geq \Delta).
\]

Moreover, by the inclusion–exclusion we have
\[
\Pr\left(\max_{k=\frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + m} |M_k(S_n)| \geq \Delta\right) \geq \frac{2m \cdot e^{-\delta(n)} \cdot (1 - o(1))}{\ln n \ln n + \delta(n)} \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{e^{-2\delta(n)} \ln n}{\pi(n \ln n + \delta(n))}.
\]

For \(m = n/\ln n\)
\[
\Pr\left(\max_{k=1,2,\ldots,n-1} |M_k(S_n)| \geq \Delta\right) \geq \frac{2e^{-\delta(n)} \ln n}{\ln n \cdot \sqrt{n \ln n + \delta(n)}} \cdot \frac{2e^{-2\delta(n)} (1 + o(1))}{(\ln n)^2 \cdot \pi(n \ln n + \delta(n))}.
\]

This accomplishes the proof of Theorem 5.3.

Using the lower bound from Theorem 5.3 and the Azuma inequality, we will provide a lower bound for the mean of \(M(S_n)\).
Lemma 5.5 (Azuma, cf. e.g., [15], [16]): Let $z_1, z_2, \ldots, z_n$ be independent random variables, with $z_j$ taking values in a set $\Lambda_j$. Assume that a function $f : \Lambda_1 \times \Lambda_2 \times \cdots \Lambda_n \rightarrow R$ satisfies, for some constants $b_j, j = 1, 2, 3, \ldots, n$, the following Lipschitz condition: if two vectors $\mathbf{z}, \mathbf{z}'$ differ only in the $j$th coordinate, then $|f(\mathbf{z}) - f(\mathbf{z}')| \leq b_j$.

Then, the random variable $X = f(z_1, z_2, \ldots, z_n), \lambda \equiv E(X)$ satisfies, for any $t \geq 0$

$$\Pr(X \geq \lambda + t) \leq \exp\left(-\frac{2t^2}{\sum_{j=1}^{N} b_j^2}\right)$$

$$\Pr(X \leq \lambda - t) \leq \exp\left(-\frac{2t^2}{\sum_{j=1}^{N} b_j^2}\right).$$

For

$$X = \max_{k=1, \ldots, n-1} |M_k|$$

we note that $b_j = 2, j = 1, 2, \ldots, n - 1$, and therefore, for $t = \sqrt{n \ln n - \lambda}$

$$\Pr(\max_{k=1, \ldots, n-1} |M_k| \geq \lambda + t) = \Pr(\max_{k=1, \ldots, n-1} |M_k| \geq \sqrt{n \ln n})$$

$$\leq \exp\left(-\frac{(\sqrt{n \ln n} - \lambda)^2}{2n}\right).$$

On the other hand

$$\Pr(\max_{k=1, \ldots, n-1} |M_k| \geq \sqrt{n \ln n}) \geq \frac{2}{\sqrt{\pi(n \ln n)^3}}(1-o(1)).$$

For consistency, we require

$$\exp\left(-\frac{(\sqrt{n \ln n} - \lambda)^2}{2n}\right) \geq \frac{2}{\sqrt{\pi(n \ln n)^3}}(1-o(1))$$

and therefore

$$(\sqrt{n \ln n} - \lambda)^2 \leq 3n \ln n \cdot (1-o(1))$$

and consequently

$$\lambda \geq \sqrt{n \ln n} - \sqrt{3n \ln n \cdot (1-o(1))}$$

$$\geq \sqrt{n \ln n} \left(1 - \sqrt{\frac{3 \ln n}{n \ln n}}\right) (1-o(1)).$$

Written differently

$$\lambda \geq n \ln n - 2\sqrt{3 \ln n \ln n + 3 \ln n}(1-o(1)).$$

We have thus shown that

$$E\left\{\max_{k=1, \ldots, n-1} |M_k|\right\} \geq \sqrt{n \ln n} \cdot (1-o(1)).$$

From here, straightforward application of the Azuma inequality gives us the following.

**Corollary 5.6:** For almost all $S_n$, $M(S_n) = \Theta(\sqrt{n \ln n})$

$$\Pr\left(\max_{k=1, \ldots, n} |M_k| \leq \sqrt{n \ln n} - \sqrt{2/\ln n} \ln n\right)$$

$$\leq \exp\left(-\frac{2/\ln n \ln n}{2n}\right) = (\ln n)^{-\beta}.$$

**VI. THE $M(S_n)$ CONCENTRATION, MODULO A CONJECTURE**

For notational convenience, let us introduce the following definitions:

$$A_k(d) \equiv \{|M_k| \geq d\}, \quad k = 1, \ldots, n - 1$$

$$A_{k_1}k_2(d) \equiv A_{k_1}(d) \cap A_{k_2}(d), \quad k_1 \neq k_2$$

$$E_{k_1}k_2(d_1, d_2) \equiv \{|M_{k_1}| = d_1 \cap \{|M_{k_2}| = d_2\}, \quad k_1 \neq k_2.$$}

The number of sequences in a set $S$ is denoted by $\#S$.

The following idea was suggested to the authors by Alex Korolev.

**Conjecture 6.1 (Korolev [10]):** For $d_1, d_2 = 0(\sqrt{n \ln n})$

$$\#E_{12}(d_1, d_2) = \#E_{k_1}k_2(d_1, d_2), \quad k_1, k_2 = 1, 2, \ldots, n - 1, k_1 \neq k_2.$$ $\diamond$

Though we were not able to prove that the previous is correct, we can show that modulo Conjecture 6.1, the following holds.

**Lemma 6.2:** Let

$$d = \sqrt{2n(\ln n + \delta(n))}.$$ 

For $k_1, k_2 = 1, 2, \ldots, n - 1, k_1 \neq k_2$, we have (modulo Conjecture 6.1)

$$\Pr(A_{k_1}k_2(d)) \leq \frac{2(\ln n + \delta(n))}{n^2}e^{-2\delta(n)}(1+o(1)).$$

**Proof:** As put forward by Moon and Moser in [19]

$$\#E_{12}(d_1, d_2) = 2\left\{\left(\frac{(n-1)-d_1}{2}\right)^{-1} \left(\frac{(n-1)-d_2}{2}\right)^{-1}\right\}$$

$$+ \left\{\left(\frac{(n-1)+d_1}{2}\right)^{-1} \left(\frac{(n-1)+d_2}{2}\right)^{-1}\right\}.$$ 

For

$$d_i = \sqrt{2n(\ln n + \delta_i(n))} = \Theta(\sqrt{n \ln n}), \quad i = 1, 2.$$ 

we have the inequality at the top of the following page. For

$$\Delta = \sqrt{2n(\ln n + \delta(n))}$$

$$\#A_{k_1}k_2(\Delta) = \sum_{d_1=\Delta}^{n-1} \sum_{d_2=\Delta}^{n-1} \#E_{k_1}k_2(d_1, d_2)$$

$$\leq \sum_{d_1=\Delta}^{n-1} \sum_{d_2=\Delta}^{n-1} \#E_{12}(d_1, d_2) + \Delta G$$

$$\leq \Delta^2 \cdot \#E_{12}(\Delta, \Delta) + \Xi_G$$

$$\leq 2n \cdot \left(\frac{2(\ln n + \delta(n))}{n^2}\right) e^{-2\delta(n)}(1+o(1)).
\[ \#E_{12}(d_1, d_2) = 2 \left( \frac{(n-3)+d_1}{2} \left( \frac{1}{2} - \frac{d_1+d_2-1}{2(n-3)+d_1} \right) \right) \left( \frac{(n-3)-d_1}{2} \left( \frac{1}{2} + \frac{d_1+d_2-1}{2(n-3)-d_1} \right) \right) \]
\[ + 2 \left( \frac{(n-3)+d_1}{2} \left( \frac{1}{2} - \frac{d_1+d_2+1}{2(n-3)+d_1} \right) \right) \left( \frac{(n-3)-d_1}{2} \left( \frac{1}{2} + \frac{d_1+d_2+1}{2(n-3)-d_1} \right) \right) \]
\[ \leq 2^n \frac{1+o(1)}{\pi \sqrt{(n-3)^2 - d_1^2}} \exp \left\{ \frac{-0.5(n-3)((d_1-1)^2 + d_2^2) - (d_1-1)d_1d_2}{(n-3)^2 - d_1^2} \right\} \]
\[ + 2^n \frac{1+o(1)}{\pi \sqrt{(n-3)^2 - d_1^2}} \exp \left\{ \frac{-0.5(n-3)((d_1+1)^2 + d_2^2) - (d_1+1)d_1d_2}{(n-3)^2 - d_1^2} \right\} \]
\[ \leq 2^n \frac{2e^{-b_1(n)+b_2(n)}}{n^3} (1+o(1)). \]

where \(\Xi_G \equiv \Xi(n, G(n, k_1, 2\Delta) + G(n, k_2, 2\Delta)) \ll \#A_{k_1,k_2}(\Delta).\)

Lemma 6.2 along with Lemma 4.4 enable us to derive a tight lower bound for

\[ \Pr \left( \max_{k=1,2,\ldots,n-1} |M_k| \geq \sqrt{2n(\ln n + \delta(n))} \right). \]

**Corollary 6.3:** For \(3\ln n \leq \delta(n) \leq \frac{2}{n} \ln n\)

\[ \Pr \left( \max_{k=1,2,\ldots,n-1} |M_k| \geq \sqrt{2n(\ln n + \delta(n))} \right) \geq \frac{e^{-\delta(n)}}{e^{\sqrt{n}(\ln n + \delta(n))^2}} (1-\alpha(1)). \]

**Proof:** Let \(d = \sqrt{2n(\ln n + \delta(n))}\)

\[ \Pr(\max_{k=1,2,\ldots,n-1} |M_k| > d) \]
\[ = 2 \cdot \Pr(\max_{k=1,2,\ldots,n-1} M_k > d) \]
\[ \geq 2 \sum_{k=1}^{n-1} \Pr(M_k > d) - 4 \sum_{k=1}^{n-1} \sum_{k_2=k+1}^{n-1} \Pr(M_{k_1} > d, M_{k_2} > d) \]
\[ \geq \left( \frac{e^{-\delta(n)}}{e^{\sqrt{n}(\ln n + \delta(n))^2}} - 8(\ln n + \delta(n)) e^{-2\delta(n)} \right) (1-\alpha(1)). \]

For \(\delta(n) > \frac{3}{2} \ln n (1+o(1))\), the union bound dominates and we obtain the claim. \(\square\)

Now we may repeat the steps in the end of Section V, but this time using Corollary 6.3, giving

\[ \Pr(\max_{k=1,2,\ldots,n-1} |M_k| \geq \sqrt{2n(\ln n + 3\ln \ln n)}) \]
\[ \geq \frac{2}{e^{\sqrt{n}(\ln n)^2}} (1-\alpha(1)). \]

Together with the Azuma inequality, it provides us with a tighter lower bound for the mean of \(M(S_n)\). Indeed, the consistency requirement yields

\[ \exp \left\{ -\frac{(t-\lambda)^2}{2n} \right\} \geq \frac{1}{e^{\sqrt{n}(\ln n)^2}} (1-\alpha(1)). \]

Therefore

\[ (t-\lambda)^2 \leq \frac{9}{2} n \ln n (1-\alpha(1)) \]

and consequently

\[ \lambda \geq \sqrt{2n(\ln n + 3\ln \ln n)} - 3 \left( \frac{n \ln n}{2} \right) (1-\alpha(1)) \]
\[ \geq \sqrt{2n \ln n} \left( 1 - 3 \left( \frac{\sqrt{n \ln n}}{2} \right) \right) (1-\alpha(1)). \]

Written differently, it is

\[ \lambda \geq \sqrt{2n(\ln n - 3\sqrt{n \ln n \ln n + 2.25 \ln n})} (1-\alpha(1)). \]

We have thus shown that

\[ E \left\{ \max_{k=1,2,\ldots,n-1} |M_k| \right\} \geq \sqrt{2n \ln n} \cdot (1-\alpha(1)). \]

From here, the straightforward application of the Azuma inequality gives the following result.

**Corollary 6.4:** For almost all \(S_n, M(S_n) \approx \sqrt{2n \ln n} \).

\[ \diamond \]

**APPENDIX I**

A. Proof of Theorem 3.1

The original proof appears in [19], here it is sketched for completeness. If

\[ M_k(S_n) = \sum_{i} s_i s_{i+k} = r \]

there must be an excess of \(r/2\) values of \(i\) with \(s_i s_{i+k} = +1\). Hence, the set \{1, 2, \ldots, \(n-k\)\} can be partitioned into two subsets \(A\) and \(B\), with \((n-k+r)/2\) and \((n-k-r)/2\), respectively, such that

\[ s_{i+k} = s_i, \quad \text{if } i \in A \]
\[ s_{i+k} = -s_i, \quad \text{if } i \in B. \]

(17)

(18)

There are \(\binom{n-k}{r/2} \binom{n-k-r}{r/2}\) choices for the subsets \(A\) and \(B\) and there are \(2^k\) choices for the first \(k\) elements of
$S_n$. Once these choices are made, the remaining elements $s_{m} = s_{(m-k)+k}$ are determined recursively by (17) or (18).

B. Proof of Lemma 3.2

Let us first show the upper bound. For $0 < \epsilon_1 < 1/2$ and any $n > 0$, we have
\[
\frac{n!}{(n \cdot (\frac{1}{2} - \epsilon_1))!(n \cdot (\frac{1}{2} + \epsilon_1))!} \leq \frac{1}{\sqrt{2\pi n}} \cdot \frac{1}{(\frac{1}{2} + \epsilon_1)^{(\frac{1}{2} + \epsilon_1)n}(\frac{1}{2} - \epsilon_1)^{(\frac{1}{2} - \epsilon_1)n}}
\]

where we have used (cf. e.g., [14])
\[
\sqrt{2\pi n} \cdot e^{-n + \frac{1}{12n} - \frac{1}{36n^3}} < n! < \sqrt{2\pi n} \cdot n^{n+1/2} e^{-n + \frac{1}{12n}}.
\]

Therefore, for $0 < \epsilon_1 < \frac{1}{2}$ and any $n > 0$
\[
\left( n \cdot \left( \frac{n}{2} - \epsilon_1 \right) \right) \leq \frac{1}{\sqrt{2\pi n}} \cdot e^{nH_e(\frac{1}{2} - \epsilon_1)}
\]

(20)
where $H_e(x) \equiv -x \ln x - (1 - x) \ln(1 - x)$ stands for the natural entropy function.

Using also
\[
H_e\left( \frac{1}{2} - \epsilon_1 \right) \leq \ln 2 - 2\epsilon_1^2, \quad \text{for } 0 < \epsilon_1 < \frac{1}{2}
\]
and
\[
\frac{1}{1 - x} \leq 1 + \frac{3}{4} x, \quad \text{for } 0 \leq x \leq \frac{3}{8}
\]
we have ($\epsilon_1^2 \leq 3/32$)
\[
\left( n \cdot \left( \frac{n}{2} - \epsilon_1 \right) \right) \leq \frac{1}{\sqrt{2\pi n}} \cdot e^{n\ln 2 - 2n\epsilon_1^2}
\]

(22)

Now to the lower bound. Using (19), we have for $0 < \epsilon_1 < \frac{1}{2}$ and any $n > 0$
\[
\frac{n!}{(n \cdot (\frac{1}{2} - \epsilon_1))!(n \cdot (\frac{1}{2} + \epsilon_1))!} \geq \frac{1}{\sqrt{2\pi n}} \cdot e^{-12(\frac{1}{2} + \epsilon_1)n - 12(\frac{1}{2} - \epsilon_1)n}.
\]

Since $e^{-x} \geq 1 - x$ for $x > 0$, we have
\[
\left( n \cdot \left( \frac{n}{2} - \epsilon_1 \right) \right) \geq 2^n \cdot \frac{\sqrt{2 \pi n}}{2} \cdot e^{-2n\epsilon_1^2} \cdot \left( 1 - \frac{3}{2} n \epsilon_1^2 - \frac{1}{2n} \right).
\]

(24)

C. Proof of Lemma 3.3

Let us first prove (7). Let
\[
k_0 = \left[ \frac{3}{32n} \right].
\]

Then
\[
2^{-n} \cdot S(n,d) = \sum_{k=\frac{n}{2}}^{n} \left( n \cdot \left( \frac{n}{2} - \frac{k}{n} \right) \right) + \sum_{k=k_0+1}^{n} \left( n \cdot \left( \frac{n}{2} - \frac{k}{n} \right) \right) = S_1(n,d) + S_2(n,d).
\]

Taking into account that the terms of $S_2(n,d)$ are monotonically decreasing, let us bound $S_2(n,d)$ from above by the product of the first (biggest) term and the number of terms in the sum, using Lemma 3.2
\[
S_2(n,d) < \frac{41}{64} \sqrt{\frac{2}{\pi}} \left( 1 - \frac{3}{8} \right) \cdot \sqrt{n} \cdot e^{-\frac{3n^2}{16}}.
\]

(25)
As for $S_1(n,d)$, we apply the upper bound of Lemma 3.2, to get
\[
S_1(n,d) \leq \sqrt{\frac{2}{\pi n}} \cdot \sum_{k=\frac{n}{2}}^{\infty} \left( 1 + 3k^2 \right) \cdot e^{-2k^2/n}.
\]

Bounding the sum with an integral, noting that for $d > \sqrt{n \ln \ln n}$ the integrands are monotonically decreasing functions of $k$, and recalling
\[
PG\left( \frac{d}{\sqrt{n}} \right) = \frac{1}{\sqrt{\pi}} \int_{\frac{d}{\sqrt{n}}}^{\infty} e^{-z^2} dz,
\]
we have
\[
\sqrt{\frac{2}{\pi n}} \cdot \sum_{k=\frac{n}{2}}^{\infty} e^{-2k^2/n} < \sqrt{\frac{2}{\pi n}} \cdot e^{-\frac{d^2}{2n}} + PG\left( \frac{d}{\sqrt{n}} \right)
\]
\[
\sqrt{\frac{2}{\pi n}} \cdot \frac{3}{2n} \cdot \sum_{k=\frac{n}{2}}^{\infty} \frac{2k^2}{n} e^{-2k^2/n} < \frac{3d(2d + 1)}{\sqrt{32\pi n^5}} \cdot e^{-\frac{d^2}{2n}} + \frac{3}{4n} \cdot PG\left( \frac{d}{\sqrt{n}} \right).
\]

By assumption, we have
\[
3d(2d + 1)/\sqrt{32} < \sqrt{2d^2}, \quad \text{and } (d/n)^2 + 1 < \sqrt{\pi/2}
\]
and
\[
\sqrt{\frac{2}{\pi n}} + \frac{3d(2d + 1)}{\sqrt{32\pi n^5}} < \frac{1}{n}.
\]
Summing up and using (25)

\[ 2^{-n} \cdot S(n,d) \leq P_G \left( \frac{d}{\sqrt{n}} \right) \cdot \left( 1 + \frac{3}{4n} \right) + e^{-\frac{d^2}{8n}} \]

\[ + \sqrt{\frac{n}{4\pi}} \cdot e^{-3n/36} \]  

(26)

Noting that

\[ \sqrt{\frac{n}{2\pi d^2}} \cdot e^{-\frac{d^2}{4n}} \cdot \left( 1 + \frac{n}{d^2} - 1 \right) \leq P_G \left( \frac{d}{\sqrt{n}} \right) \leq \sqrt{\frac{n}{2\pi d^2}} \cdot e^{-\frac{d^2}{2n}} \]  

under the imposed conditions

\[ \frac{e^{-\frac{d^2}{8n}}}{\sqrt{n}} \leq \sqrt{\frac{2\pi d}{n}} \cdot \left( 1 + \frac{n}{d^2} - 1 \right) \cdot P_G \left( \frac{d}{\sqrt{n}} \right) \]

\[ < \frac{6.5d}{n} \cdot P_G \left( \frac{d}{\sqrt{n}} \right) \]  

(28)

We finally have

\[ 2^{-n} \cdot S(n,d) \leq P_G \left( \frac{d}{\sqrt{n}} \right) \cdot \left( 1 + \frac{6.5d}{n} + e^{-\frac{d^2}{2n}} + e^{-\frac{d^2}{8n}} \right) \]  

(29)

\[ < P_G \left( \frac{d}{\sqrt{n}} \right) \cdot \left( 1 + \frac{6.6d}{n} \right) \]  

(30)

where we have bounded

\[ 3/4 < d/20, \quad d^2/(2n) < n/8, \quad \text{and} \quad e^{-\frac{d^2}{2n}} < d/(25n). \]

Now, let us prove (8). Starting from the lower bound in Lemma 3.2

\[ 2^{-n} \cdot S(n,d) \geq \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \left( 1 - \frac{3k^4}{2n^3} - \frac{1}{2n} \right) \cdot \sqrt{\frac{2\pi d}{n}} \cdot e^{-2k^2/n} \]

\[ \geq \left( 1 - \frac{1}{2n} \right) \cdot P_G \left( \frac{d}{\sqrt{n}} \right) - P_G \left( \left( \frac{n}{8} \right)^{\frac{1}{3}} \right) \]

\[ - \sqrt{\frac{2\pi}{n}} \cdot \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \frac{3k^4}{2n^3} \cdot e^{-2k^2/n}. \]

To complete the proof, let us provide an upper bound for

\[ S_3(n,d) = \sqrt{\frac{2\pi}{n}} \cdot \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \frac{3k^4}{2n^3} \cdot e^{-2k^2/n}. \]

Note that the maximum of \(k^4 e^{-2k^2/n}\) is reached for \(k^2 = n\).

If \(d > 2\sqrt{n}\), the summands in \(S_3(n,d)\) are monotonically decreasing and the sum can be bounded from above by an integral as follows:

\[ \sqrt{\frac{2\pi}{n}} \cdot \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \frac{3k^4}{2n^3} \cdot e^{-2k^2/n} < \frac{3}{8n} \int_{\frac{2\pi^2}{2n^2}}^{\infty} x^4 e^{-x^2/8} dx. \]

On the other hand, if \(\sqrt{\ln \ln n} < d/\sqrt{n} \leq 2\) (which can only happen when \(\ln \ln n < 2\), i.e., for \(n < 1619\)), the summands increase for \(d/2 < k \leq \sqrt{n}\) and decrease thereafter. The biggest summand is \(\leq 4d^2 e^{-2d^2}/8\).

Clearly, \(S_3(n,d)\) can be bounded from above in both cases as

\[ \sqrt{\frac{2\pi}{n}} \cdot \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \frac{3k^4}{2n^3} \cdot e^{-2k^2/n} \]

\[ \leq \frac{3}{8n} \sqrt{\frac{2\pi}{n}} \cdot \int_{\frac{2\pi^2}{2n^2}}^{\infty} x^4 e^{-x^2/d^2} dx \]

\[ = \frac{3}{32n} \left[ \frac{d-1}{\sqrt{2\pi n}} \left( \frac{(d-1)^2}{n} \right) + 3 + \frac{32}{(d-1)^2} \right] e^{-\frac{(d-1)^2}{2n}} \]

\[ + 3 \cdot \frac{d-1}{n} \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right). \]

Analogously to (28), we have

\[ \frac{e^{-\frac{(d-1)^2}{2n}}}{\sqrt{n}} < \frac{8.26(d-1)}{n} \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right) \]

\[ < \frac{8.26d}{n} \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right). \]

(31)

Lumping the contributions

\[ \frac{(d-1)^2}{n} + 3 + \frac{32}{(d-1)^2} < 4.55 \frac{(d-1)^2}{n} < 4.55 \frac{d^2}{n} \]

we get

\[ \sqrt{\frac{2\pi}{n}} \cdot \sum_{k=\frac{d}{3}}^{(n^3/2)^{\frac{1}{3}}} \frac{3k^4}{2n^3} \cdot e^{-2k^2/n} \]

\[ < \frac{3}{8n} \cdot \frac{1}{\sqrt{\pi}} \cdot \left( \frac{37.6d^4}{4} + \frac{3\sqrt{\pi}}{4} \right) \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right) \]

\[ < \frac{45d^4}{16\sqrt{\pi} n^3} \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right) \]

\[ < \frac{5d^4}{n^3} \cdot P_G \left( \frac{d-1}{\sqrt{n}} \right). \]

where in the last inequality we used

\[ P_G \left( \frac{d-1}{\sqrt{n}} \right) \leq e^{(2d-1)/2n} \cdot \frac{d-1}{d} \cdot \frac{d^2}{d^2-1} \cdot P_G \left( \frac{d}{\sqrt{n}} \right) \]

\[ < 3.04 \cdot P_G \left( \frac{d}{\sqrt{n}} \right). \]

Finally, noting that

\[ P_G \left( \left( \frac{n}{8} \right)^{\frac{1}{3}} \right) \leq \sqrt{\frac{8}{\pi}} \cdot n^{-1/4} \cdot e^{-\sqrt{n}/32} < e^{-\sqrt{n}/32}, \]

we have

\[ 2^{-n} \cdot S(n,d) \geq \left( 1 - \frac{1}{2n} - \frac{5d^4}{n^3} \right) \cdot P_G \left( \frac{d}{\sqrt{n}} \right) - e^{-\sqrt{n}/32}. \]

(33)
D. Proof of Lemma 3.5

For $x > 0$, we have

$$\frac{e^{-x^2}}{\sqrt{\pi} x} \left(1 - \frac{1}{2x^2}\right) \leq \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \leq \frac{e^{-x^2}}{\sqrt{\pi} x}.$$  

Noting that

$$P_G \left( \frac{d}{\sqrt{n-k}} \right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{d} e^{-t^2} dt$$

we have

$$x = \sqrt{\frac{n(\ln n + \delta(n))}{n-k}} = \sqrt{\frac{(\ln n + \delta(n)) + k(\ln n + \delta(n))}{n-k}}$$

$$P_G \left( \frac{\Delta}{\sqrt{n-k}} \right) \leq \frac{1}{2n} \frac{e^{-\Theta(n)}}{\sqrt{\pi(\ln n + \delta(n))}} \cdot \frac{1}{\sqrt{\pi(\ln n + \delta(n))}} \cdot \frac{1}{\sqrt{\pi(\ln n + \delta(n))}} \cdot \left( n \delta(n) \right)^{-\frac{k}{n} \frac{1}{\ln n + \delta(n)}} \left( 1 - \frac{1}{2(\ln n + \delta(n))} \right).$$

Further, using

$$1 - x \leq \sqrt{1-x} \leq 1, \quad 1 + x \leq \frac{1}{1-x}, \quad \text{for } 0 \leq x < 1,$$

$$P_G \left( \frac{\Delta}{\sqrt{n-k}} \right) \leq \frac{1}{2n} \frac{e^{-\Theta(n)}}{\sqrt{\pi(\ln n + \delta(n))}} \left( n \delta(n) \right)^{-\frac{k}{n} \frac{1}{\ln n + \delta(n)}} \left( 1 - \frac{1}{2(\ln n + \delta(n))} \right).$$

\[\Box\]

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REFERENCES


Noga Alon received the Ph.D. degree in mathematics from the Hebrew University of Jerusalem, Israel, in 1983.

He is a Baumanr Professor of Mathematics and Computer Science at Tel-Aviv University, Israel. He has had visiting positions in various research institutes including MIT, Cambridge, MA; The Institute of Advanced Study, Princeton, NJ; IBM Almaden Center, San Jose, CA; Bell Labs, Murray Hill, NJ; Bellcore, Morristown, NJ; and Microsoft Research, Redmond, WA. He published more than four hundred research papers, mostly in Combinatorics and in Theoretical Computer Science, and one book.

Dr. Alon has been a member of the Israel National Academy of Sciences since 1997 and of the Academia Europaea since 2008. He received the Erdős prize in 1989, the Feher prize in 1991, the Polya Prize in 2000, the Bruno Memorial Award in 2001, the Landau Prize in 2005, the Goedel Prize in 2005, and the Israel Prize in 2008.


Since 1991, he has been with the Department of Electrical Engineering—Systems, Tel-Aviv University, Israel, where he is a Professor. His research interests include coding, information theory, communications, and application of discrete mathematics. He authored Covering Codes (Elsevier, 1997) and Peak Power Control in Multicarrier Communications (Cambridge University Press, 2007).

Dr. Litsyn received the Guastallo Fellowship in 1992. During 2000–2003, he served as an Associate Editor for Coding Theory for the IEEE TRANSACTIONS ON INFORMATION THEORY. He is an editorial board member of Advances in Mathematics and Communications (AMC) and Applicable Algebra in Engineering Communication and Computing (AAECC).

Alexander Shpunt received the B.Sc. degree in physics/computer science in 1999, followed by the M.Sc. degree in electrical engineering in 2005, both with high honors, from Tel-Aviv University, Tel-Aviv, Israel. In 2005, he moved to the Department of Physics at the Massachusetts Institute of Technology, Cambridge, where he currently is a fourth year Graduate Fellow.