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DISTRIBUTIONS OF PREFERENCES
AND
THE "LAW OF DEMAND"

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(CEPREMAP and MIT)

Number 356 November 1984

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A la memoire d'Andre Nataf

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Economic theory is plagued by quite a few "impossibility theorems". An obvious example is social choice theory with Arrow's famous impossibility result. No less important is the Debreu-Sonnenschein claim that summation over consumers does not place any other restrictions on competitive aggregate excess demand than Walras' law and homogeneity of degree 0. That sort of results - which is by no means confined to the two specific areas just mentioned - should be quite disquieting at times where in particular, it is increasingly recognized that macroeconomics must be appropriately rooted in microeconomic theory.

Many writers, in order to get specific results, often restrict the domain of the agents' allowable characteristics in their models, e.g. single peaked preferences in social choice, separable or homothetic preferences in equilibrium theory (competitive or not), and/or assume boldly that economic units are all the same, as it is often the case e.g., in recent "macroeconomics". While the study of these particular examples may be a useful and suggestive exercise, one keeps wondering how robust are the results obtained after having made such drastic restrictions.

* I wish to thank very much Philippe Aghion, Dale Jorgenson and Andreu Mas-Colell for very helpful conversations.

The principle of a possible solution to the problem has been known for some time, but has not yet been implemented much successfully. It is to put restrictions not so much on the support of the distribution of the agents' characteristics but on its <u>shape</u>. An early example of this approach in social choice was the result that aggregation is indeed possible through majority voting whenever the distribution of the voters' preferences has nice <u>symmetry</u> properties (Tullock [1967], Davis, de Groot and Hinich [1972], Grandmont [1978]). Other examples were the finding that suitably <u>dispersed</u> distributions over the space of consumers' characteristics (preferences, wealth) lead to a nice "smoothing" of competitive aggregate demand (see e.g., E. Dierker, H. Dierker and W. Trockel [1984] with references to earlier works).

A strong result along this line was obtained recently by W. Hildenbrand [1983] in demand analysis. He shows in particular that if the distribution of income, or expenditure, among individuals who have the same tastes, has a continuous <u>decreasing density</u>, then competitive aggregate demand has a negative definite Jacobian matrix (implying in particular that the weak axiom of revealed preference is satisfied in the aggregate, and that aggregate partial demands are decreasing functions of their own price), and this independently of the distribution of preferences in the society. The present work is in a sense complementary to the latter, since it aims at getting a similar outcome by placing restrictions on the shape of the distribution of preferences rather than on the income distribution.

The difficulty when trying to speak about the "shape" of the distribution of preferences is of course that one needs an algebraic structure on the space of preferences under consideration, e.g., as in Grandmont [1978] or Dierker *et al.* [1984]. This is achieved here by employing *homothetic tranformations* of preferences, that are particular instances of similar, more general transformations employed in related contexts by A. Mas-Colell and W. Neuefeind [1977, p.597] or by Dierker *et al.* [1984, pp.15-16]. More precisely, for every preference relation R defined on the nonnegative orthant of the commodity space and every income w, we generate a new pair (R_α, w) involving the same income but where the new preference R_α is derived from the original one through an homothecy of center O (the origin of the commodity space) with ratio e^α, α being an arbitrary real number. It is shown that under suitable regularity assumptions, there is a large class of distributions on the parameter α, including specific *gamma distributions*, that is independent of the particular pair (R,w) under consideration, such that, given (R,w), competitive aggregate demand has a negative definite Jacobian matrix. Since this property is preserved through addition, the result is of course still valid when the agents' characteristics are distributed over the pairs (R,w) in an arbitrary way.

The first section of this note deals briefly with homothetic transformations of preferences. The second part gives sufficient conditions on the distribution of preferences that lead to negative definiteness of the Jacobian matrix of competitive aggregate demand. We briefly comment in the concluding section on the prospects of using in this context more general "affine" transformations, as in Mas Colell and Neuefeind [1977] and Dierker *et al.* [1984]. We indicate there that, as pointed out to us by Dale Jorgenson, this concept is in fact identical to the notion of household equivalence scales introduced in demand analysis by A.P. Barten [1964], and subsequently used in econometric work (see A. Deaton and J.S. Muellbauer [1980], D.W. Jorgenson and D.T. Selsnick [1984], J.S. Muellbauer [1980].

1. HOMOTHETIC TRANSFORMATIONS OF PREFERENCES

Consider a consumption set equal to the nonnegative orthant of the commodity space $\mathbb{R}^{\ell}$ ($\ell > 2$), i.e., $X = \mathbb{R}^{\ell}_{+}$, and a preference relation R on X, i.e., a binary relation on X that is complete and transitive (for short, a preference) with the understanding that xRy means "x is preferred or equivalent to y". We assume R to be continuous (its graph is closed), strictly convex and locally nonsatiated.[1] It is well known (Debreu [1964]) that R is continuous if and only if it has a continous representation $u: X \to \mathbb{R}$.

For any real number α, we define a new preference R_α by

(1.1) $(e^{\alpha}x)R_{\alpha}(e^{\alpha}y)$ if and only if xRy

The indifferences surfaces of R_α are obtained from those of R through an homothecy of center 0 and ratio e^{α}, see Figure 1.a. If u(x) is a continuous representation of R, then $u(e^{-\alpha}x)$ is a representation of R_α, and it is continuous. Thus R_α is continuous, strictly convex,

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1 R is strictly convex if for every x_1, x_2, y in X with $x_i R y$, $x_1 \neq x_2$, then $\lambda x_1 + (1 - \lambda)x_2$ Py for all λ in (0,1)(xPy stands for "not yRx"). Local nonsatiation means that for any x and any neighborhood V of x, there exists y in V with yPx.

locally nonsatiated. Note that R is homothetic if and only if $R_\alpha = R$ for all α.

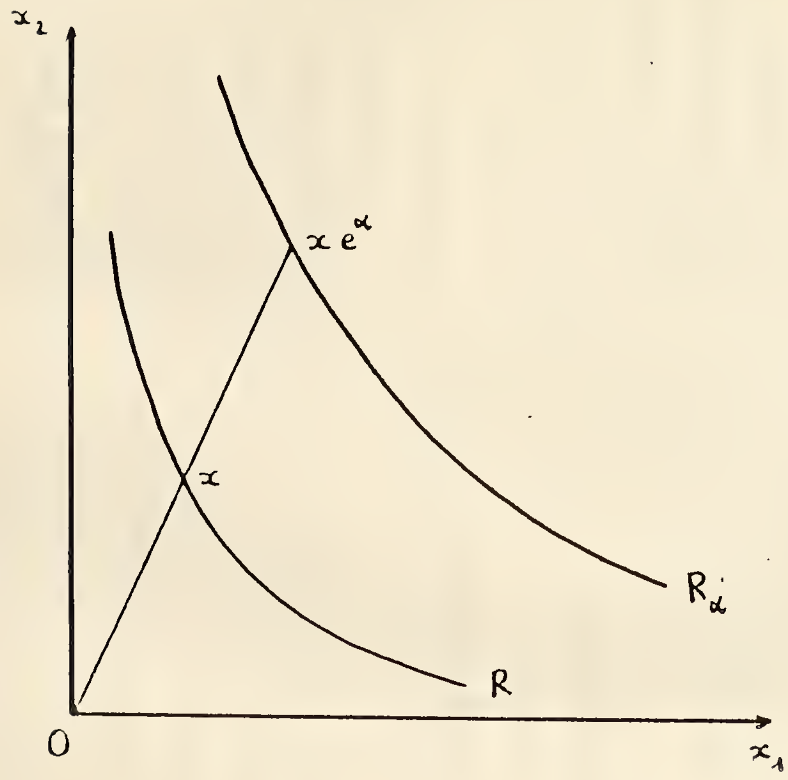
Let $P = \text{Int}\, \mathbb{R}^\ell_+$ be the set of positive vectors of $\mathbb{R}^\ell$. Given the preference R, the set of demands for goods at the vector of prices p in P and at the nonnegative income level w, is the set of commodity bundles that maximize the preference R, or equivalently any of its continuous representations u(x), under the constraints $x \varepsilon X$ and $p \cdot x \leq w$. This set is nonempty and reduces to a single element, noted $\xi(R,p,w)$. From local nonsatiation, we get Walras' law, $p \cdot \xi(R,p,w) \equiv w$ for all (p,w) in $P \times \mathbb{R}_+$, and one has $\xi(R,p,0) = 0$. Moreover ξ is homogenous of degree 0 in (p,w).

Consider now the transformed preference relation R_α. The demand function $\xi(R_\alpha,p,w)$ is obtained by maximizing $u(e^{-\alpha}x)$ under the constraints $x \varepsilon X$, and $p \cdot x \leq w$, or equivalently, $p \cdot (xe^{-\alpha}) \leq we^{-\alpha}$. Hence

$$(1.2) \qquad \xi(R_\alpha,p,w) = e^\alpha \xi(R,p,we^{-\alpha})$$

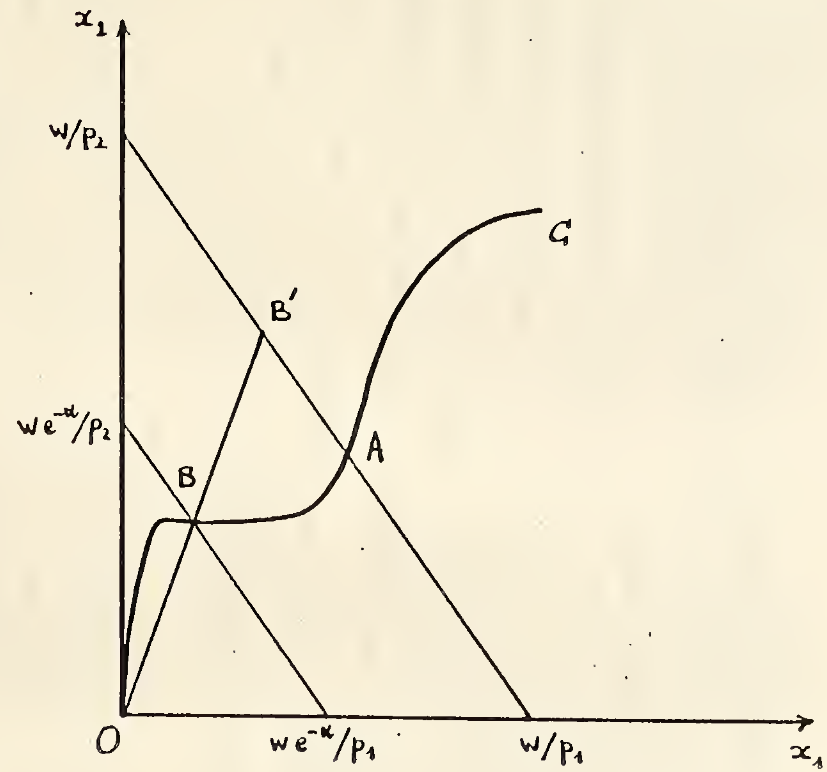
The effect of the homothetic transformation on the demand function is shown in Figure 1.b, in the case where $e^\alpha = 2$. The curve OBAC is the Engel curve of the preference R corresponding to the price system p. The point A represents the demand for R at (p,w). To obtain the demand $\xi(R_\alpha,p,w)$, one considers first the point of the Engel curve of the original preference R at the income $we^{-\alpha}$, i.e., the point B, then one "rescales" it to get back to the original budget set, which yields B'.

It is routine to verify that $\xi(R,p,w)$ is a continuous function of (p,w) on $P \times \mathbb{R}_+$. We shall in fact assume that R is such that



x_2
x
$x e^{\alpha}$
R_{α}
R
O
x_1

Fig. 1.a



x_2
w/p_2
$we^{-\alpha}/p_2$
B'
B
A
G
O
$we^{-\alpha}/p_1$
w/p_1
x_1

Fig. 1.b

(H.1) $\xi(R, p, w)$ <u>is a continuously differentiable function of</u> (p, w) <u>on</u> $P \times \mathbb{R}_+$.

One would obtain indeed continuous differentiability for all p in P and all positive incomes w if R were <u>smooth</u> in the sense of Debreu [1972] or Mas Colell [1974]. What is added by (H.1) is that the partial derivative $\frac{\partial \xi_h}{\partial w}(R, p, w)$ tends to a finite limit and that $\frac{\partial \xi_h}{\partial p_k}(R, p, w)$ tends to zero, when income decreases to 0. It is immediately apparent from (1.2) that R_α satisfies (H.1) if and only if R does, for all α. In fact, $\xi(R_\alpha, p, w)$ is then continuously differentiable in (α, p, w) on $\mathbb{R} \times P \times \mathbb{R}_+$. Differentiation of (1.2) yields

$$\frac{\partial \xi_h}{\partial p_k}(R_\alpha, p, w) = e^\alpha \frac{\partial \xi_h}{\partial p_k}(R, p, we^{-\alpha}) \tag{1.3}$$

$$\frac{\partial \xi_h}{\partial w}(R_\alpha, p, w) = \frac{\partial \xi_h}{\partial w}(R, p, we^{-\alpha}) \tag{1.4}$$

Differentiation of (1.2) with respect to α yields then the following relation, which will turn out to be important when aggregating income effects

$$w \frac{\partial \xi_h}{\partial w}(R_\alpha, p, w) = \xi_h(R_\alpha, p, w) - \frac{\partial}{\partial \alpha}[\xi_h(R_\alpha, p, w)] \tag{1.5}$$

The elements of the <u>Slutsky matrix</u> $SM(R, p, w)$ of a preference R satisfying (H.1) are the substitution terms of the corresponding Slutsky equation

$$S_{hk}(R, p, w) = \frac{\partial \xi_h}{\partial p_k}(R, p, w) + A_{hk}(R, p, w)$$

where the income term is

$$A_{hk}(R,p,w) = \xi_k(R,p,w)\,\frac{\partial\xi_h}{\partial w}(R,p,w).$$

It is well known that the Slutsky matrix is negative semidefinite. On the other hand $S(R,p,w)$ cannot have rank ℓ since for every h,

$$\sum_k p_k S_{hk}(R,p,w) = 0.$$

This implies in particular

$$\sum_{h,k} v_h v_k S_{hk}(R,p,w) = 0$$

for every vector v of $\mathbb{R}^\ell$ that is collinear with p, i.e., such that $v = rp$ for some real number r. We shall focus attention on the case where R satisfies the regularity condition

(H.2) <u>For all</u> p <u>in</u> P <u>and all</u> $w > 0$, $\sum_{h,k} v_h v_k S_{hk}(R,p,w) < 0$ <u>for every</u> v <u>that is not collinear with</u> p.

The restriction to a positive income in the foregoing condition is of course necessary since $S_{hk}(R,p,0) = 0$. The preference will fulfill (H.2) if the rank of its Slutsky matrix is $\ell-1$ for all p in P and all $w > 0$, and thus in particular if it is <u>smooth</u> and <u>regular</u> in the sense of Debreu [1972] or Mas-Colell [1974].

If R and thus R_α satisfy (H.1), their Slutsky matrices are related by

(1.6) $$S_{hk}(R_\alpha,p,w) = e^\alpha\, S_{hk}(R,p,we^{-\alpha})$$

Hence R_α verifies (H.2) if and only if R does. In the sequel, $\mathcal{R}^*$ will denote the set of preferences on $X = \mathbb{R}_+^\ell$ that are continuous, strictly convex, locally nonsatiated and satisfy (H.1), (H.2). The above arguments show that R_α belongs to $\mathcal{R}^*$ whenever R does, for all α.

2. THE LAW OF DEMAND

There is no reason why an individual demand function should have a negative definite Jacobian matrix since there are income effects. We show in this section that there are restrictions on the shape of the distribution of the consumers' preferences (not on its support) such that aggregate demand does have this property, independently of the distribution of income.

We must first make the problem more precise. For the present purpose, we say that the characteristics of a consumer are his preference relation R in $\mathcal{R}^*$ and his income $w > 0$, although this viewpoint is somewhat restrictive since it means that income is independent of prices. We define a probability distribution over the agents' characteristics in the following way.[2] There is first a probability γ on characteristics, with finite support, say $\{(R^1,w^1),\ldots,(R^m,w^m)\}$. For reasons that will become clear below, we may call (R^i,w^i) a generator. Then $\gamma_i > 0$ is the weight assigned by γ to the ith generator, with $\sum_i \gamma_i = 1$. The key assumption is that for each generator (R^i,w^i) there is a whole distribution of individuals who have the same income w^i but preferences that are homothetic transforms R^i_α of R^i, in the sense of the preceding section. In order to specify this distribution, we need only to specify the probability distribution

[2] This way is directly inspired from Dierker et al [1984].

of the corresponding parameter α, conditionally upon i, say μ_i. The conditional probabilities μ_i together with the probability γ over generators yield an overall probability distribution over the space of characteristics.[3]

Aggregate demand per capita or <u>mean demand</u> is then given by $\sum_i \gamma_i \bar{\xi}_i(p)$, with

$$\bar{\xi}_i(p) = \int \xi(R^i_\alpha, p, w^i)\ \mu_i(d\alpha)$$

The problem to be studied can then be phrased very simply. We look for conditions on the <u>shape of the probability distributions</u> μ_i which ensure that the mean demand is continuously differentiable and has a negative definite Jacobian matrix, and this independently of the particular distribution γ over generators. One gets then a problem which we may hope to be tractable since this formulation enables us to work with probabilities over the real line instead of being stuck with probabilities over the space of preferences, which has <u>a priori</u> no algebraic structure.

The first remark is that it suffices to answer this question for each $\bar{\xi}_i$, since continuous differentiability and negative definiteness are preserved through addition. One may accordingly assume without <u>loss of generality</u> that γ gives full weight to a single characteristic

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[3] More precisely, let $\mathcal{R}^*$ be endowed with the topology of closed convergence [Hildenbrand, 1974]. Let ν_i be the probability over $\mathcal{R}^* \times \mathbb{R}_+$ that is the image of μ_i by the continuous map which associates to each α the characteristic (R^i_α, w^i). The proportion of individuals who have characteristics in the (Borel) subset A of $\mathcal{R}^* \times \mathbb{R}_+$ is then

$$\nu(A) = \sum_i \gamma_i\ \nu_i(A).$$

The assumption that γ has a finite support is made for expositional convenience. The argument may be adapted to the case where γ has a compact support contained in $\mathcal{R}^* \times \text{Int}\ \mathbb{R}_+$.

(R,w). The second (obvious) remark is that one needs indeed restrictions on the μ_i to get the desired property. For if μ_i assigns probability one to a single number $\bar{\alpha}$, then $\bar{\xi}_i(p)$ is equal to the individual demand $\xi(R^i_{\bar{\alpha}},p,w^i)$ and as noted earlier, there is no reason to get negative definiteness since there are income effects. Intuitively, one should run into the same sort of trouble if μ_i is distributed over a small number of points or is too concentrated. One may however hope to get the desired property when μ_i is distributed over a large number of points and is dispersed appropriately. An example is provided by the following result.

PROPOSITION. Consider a preference R in $\mathcal{R}^*$ and an income $w > 0$. Let μ be a probability distribution over the real line. Assume that the support of μ has a finite lower bound, i.e., $-\infty < a = \text{Inf supp } \mu$. Then if e^α is μ-integrable, the mean demand

$$\bar{\xi}(p) = \int \xi(R_\alpha,p,w)\, \mu(d\alpha)$$

is continuously differentiable and

$$\frac{\partial\bar{\xi}_h}{\partial p_k}(p) = \int \frac{\partial \xi_h}{\partial p_k}(R_\alpha,p,w)\, \mu(d\alpha)$$

Assume that in addition μ has a continuous density $\rho(\alpha)$ with respect to the Lebesgue measure, that satisfies

(1) The restriction of ρ to the interval $[a,+\infty)$ is continuously differentiable and eventually nonincreasing, i.e., there exists $b < +\infty$ such that $\rho'(\alpha) \leq 0$ for $\alpha \geq b$,

(2) for every α in $[a,+\infty)$, one has $\rho'(\alpha) + 2\rho(\alpha) \geq 0$.

Then $\bar{\xi}(p)$ has a negative definite Jacobian matrix, i.e.,

$$\sum_{h,k} v_h v_k \frac{\partial \bar{\xi}_h}{\partial p_k}(p) < 0 \text{ for every vector } v \neq 0 \text{ of } \mathbb{R}^{\ell}.$$

Remark. When μ has a density $\rho(\alpha)$, $\bar{\xi}_h(p)$ and $\frac{\partial \bar{\xi}_h}{\partial p_k}(p)$ are respectively equal to the (improper) Riemann integrals

$$\int_a^{+\infty} \xi_h(R_\alpha, p, w)\rho(\alpha)\, d\alpha \quad \text{and} \quad \int_a^{+\infty} \frac{\partial \xi_h}{\partial p_k}(R_\alpha, p, w)\rho(\alpha)\, d\alpha .$$

In fact these Riemann integrals are then absolutely convergent. Similarly, the condition that e^α is μ-integrable reads

$$\int_a^{+\infty} e^\alpha \rho(\alpha)\, d\alpha < +\infty .$$

Proof. The principle of the proof is simple and is similar to Hildenbrand [1983, p.1018]. The only complications arise from the fact that integration takes place over an unbounded domain, so that we have to verify integrability at each step.

Remark that $\xi(R_\alpha, p, w)$ is continuously differentiable in (α, p) on $\mathbb{R} \times P$, and that $0 \leq \xi_h(R_\alpha, p, w) \leq w/p_h$. Thus as a function of α, $\xi(R_\alpha, p, w)$ is μ-integrable.

The first step is to show that if $-\infty < a = \text{Inf supp } \mu$ and if e^α is μ-integrable, then $\bar{\xi}$ is continuously differentiable on P and

$$(2.1)\qquad \frac{\partial\bar{\xi}_h}{\partial p_k}(p) = \int \frac{\partial\xi_h}{\partial p_k}(R_\alpha, p, w)\ \mu(d\alpha)$$

Let $a_{hk}(p,c)$ be the maximum of $\frac{\partial\xi_h}{\partial p_k}(R,p,z)$ when z varies in the interval $[0,c]$. Since R satisfies (H.1), $a_{hk}(p,c)$ exists and is continuous in (p,c). From (1.3), we have

$$\left| \frac{\partial\xi_h}{\partial p_k}(R_\alpha, p, w) \right| < e^\alpha\ a_{hk}(p, we^{-a})$$

when $\alpha > a$. Thus continuous differentiability of $\bar{\xi}$ and (2.1) follows from the dominated convergence theorem.

Using the Slutsky equation, we get

$$\frac{\partial\bar{\xi}_h}{\partial p_k}(p) = \int \left[S_{hk}(R_\alpha, p, w) - A_{hk}(R_\alpha, p, w)\right] \mu(d\alpha)$$

which we may rewrite as $\frac{\partial\bar{\xi}_h}{\partial p_k}(p) = \bar{S}_{hk}(p) - \bar{A}_{hk}(p)$, with

$$\bar{S}_{hk}(p) = \int S_{hk}(R_\alpha, p, w)\ \mu(d\alpha)$$

$$\bar{A}_{hk}(p) = \int A_{hk}(R_\alpha, p, w)\ \mu(d\alpha)$$

if and only if $A_{hk}(R_\alpha, p, w)$ is a μ-integrable function of α. It is clearly continuous and $0 < \xi_k(R_\alpha, p, w) < w/p_k$. On the other hand, from (1.4), $\left|\frac{\partial\xi_h}{\partial w}(R_\alpha, p, w)\right|$ is bounded when α varies in the interval $[a, +\infty)$ since R satisfies (H.1). Clearly, A_{hk} is μ-integrable.

Since every R_α satisfies (H.2), one has

$$\sum_{h,k} v_h v_k \xi_{hk}(p) \leq 0$$

for all v in $\mathbb{R}^\ell$, with equality if and only if v is collinear with p. To show that $\bar{\xi}$ has a negative definite Jacobian matrix, it suffices accordingly to prove that

$$\sum_{h,k} v_h v_k \bar{A}_{hk}(p) \geq 0$$

with strict inequality when $v \neq 0$ is collinear with p.

Given p and w, we may consider $\xi_h(R_\alpha,p,w)$ as a function of α, say $f_h(\alpha)$. With this notation, (1.5) reads

$$w \frac{\partial \xi_h}{\partial w}(R_\alpha,p,w) = f_h(\alpha) - f_h'(\alpha)$$

and thus

$$w\bar{A}_{hk}(p) = \int f_k(\alpha)\,[f_h(\alpha) - f_h'(\alpha)]\,\mu(d\alpha)$$

Consider any vector v of $\mathbb{R}^\ell$ and let $g(\alpha) = \sum_h v_h f_h(\alpha)$. We get

$$w \sum_{h,k} v_h v_k \bar{A}_{hk}(p) = \int g(\alpha)[g(\alpha) - g'(\alpha)]\,\mu(d\alpha)$$

Now if $v = rp$, $r \neq 0$, one has $g(\alpha) = rw$, hence $(g^2(\alpha))' = 0$, and the above expression is equal to $(rw)^2 > 0$. It remains to show that it is nonnegative for all v, under the assumptions of the Proposition.

If we take into account that μ has a density ρ with respect to the

Lebesgue measure λ on $\mathbb{R}$, we may rewrite the preceding equality as

$$(2.2) \qquad 2w \sum_{h,k} v_h v_k \bar{A}_{hk}(p) = \int [2g^2(\alpha)\rho(\alpha) - (g^2(\alpha))'\rho(\alpha)] \lambda(d\alpha)$$

Remark that since $0 < f_h(\alpha) < w/p_h$, the function $g^2\rho$ is λ-integrable. Thus, from (2.2), $(g^2(\alpha))'\rho(\alpha)$ is also λ-integrable. The next step is to note that the fact that the restriction of ρ to the interval $[a,+\infty)$ is continuously differentiable, and that ρ is eventually nonincreasing, implies first that $\lim_{\alpha\to+\infty} \rho(\alpha) = 0$ (otherwise one could not have $\int \rho(\alpha) \lambda(d\alpha) = 1$), and second that $\rho'(\alpha)$ is λ-integrable (the improper Riemann integral $\int_a^{+\infty} \rho'(\alpha)\, d\alpha = 0$ is then absolutely convergent and is thus equal to the Lebesgue integral $\int \rho'(\alpha) \lambda(d\alpha)$). Hence, since g^2 is bounded, $g^2(\alpha)\rho'(\alpha)$ is also λ-integrable. We get accordingly,

$$\begin{aligned}
\int (g^2(\alpha))' \rho(\alpha) \lambda(d\alpha) &= \int [(g^2(\alpha)\rho(\alpha))' - g^2(\alpha)\rho'(\alpha)] \lambda(d\alpha) \\
&= [g^2(\alpha)\rho(\alpha)]_a^{+\infty} - \int g^2(\alpha)\rho'(\alpha) \lambda(d\alpha) \\
&= -\int g^2(\alpha)\rho'(\alpha) \lambda(d\alpha)
\end{aligned}$$

Combining this result with (2.2) yields

$$2w \sum_{h,k} v_h v_k \bar{A}_{hk}(p) = \int g^2(\alpha)[\rho'(\alpha) + 2\rho(\alpha)] \lambda(d\alpha)$$

which is clearly nonnegative whenever $\rho'(\alpha) + 2\rho(\alpha) > 0$ for $\alpha > a$.

The proof is complete[4]. Q.E.D.

We may note that, as in Hildenbrand [1983], under the assumptions of the Proposition, the mean demand function $\bar{\xi}(p)$ is <u>strictly monotone</u>, i.e., for every p,q in P with $p \neq q$,

$$(q - p) \cdot [\bar{\xi}(q) - \bar{\xi}(p)] < 0 .$$

This in turn implies that $\bar{\xi}$ is <u>one to one</u> ($p \neq q$ implies $\bar{\xi}(p) \neq \bar{\xi}(q)$) and that it satisfies the <u>weak axiom of revealed preference</u>, i.e., for every p,q in P with $p \neq q$, $q \cdot \bar{\xi}(p) \leq w$ implies $p \cdot \bar{\xi}(q) > w$.

We wish now to assess more precisely the functional forms of the density function implied by the foregoing Proposition. In doing so, we may assume without any loss of generality that a = 0, since one can always make the change of variable $\alpha' = \alpha - a$, and that ρ is defined only on $\mathbb{R}_+$, with $\rho(0) = 0$.

We note first that the fact that ρ is continuously differentiable on $\mathbb{R}_+$ means that we are dealing with a <u>continuum</u> of individuals. This conforms to our earlier remarks that we needed a large number of consumers to have any hope to get negative definiteness in the aggregate. All other conditions on ρ are of a "technical" nature -- they are there to guarantee integrability — except the last one, namely $\rho'(\alpha) + 2\rho(\alpha) \geq 0$ for all $\alpha \geq 0$, which is the key assumption to ensure negative definiteness of the aggregate Jacobian matrix. The

[4] The proof shows that one can replace the assumption that ρ is eventually nonincreasing by the two assumptions (i) $\lim_{\alpha \to +\infty} \rho(\alpha) = 0$ and (ii) the improper Riemann integral $\int_a^{+\infty} \rho'(\alpha)\, d\alpha$ is absolutely convergent. The loss of simplicity did not seem to be offset by the gain in generality.

meaning of this condition is most easily seen by considering

$$\frac{d}{d\alpha}\left[e^{2\alpha}\,\rho(\alpha)\right] = e^{2\alpha}\left[\rho'(\alpha) + 2\rho(\alpha)\right] > 0.$$

Therefore, ρ must be the product of $e^{-2\alpha}$ and of a nondecreasing function. This implies immediately that the support of ρ must be unbounded, or in other words, that we need a distribution with a "tail". For if one had $\rho(\alpha) = 0$ when $\alpha > b$ for some b, then $e^{2\alpha}\rho(\alpha)$, and thus $\rho(\alpha)$, would vanish everywhere, which is impossible. On the other hand, the tail must not be too thick to meet the integrability requirements.

It is now relatively simple to generate all densities ρ we are looking for. Choose any nonnegative, nondecreasing, continuously differentiable real valued function F defined on $\mathbb{R}_+$, with $F(0)=0$, which of course is not identically zero, and such that

(i) $\int_0^{+\infty} e^{-\alpha}\,F(\alpha)\,d\alpha < +\infty$,

(ii) $e^{-2\alpha}\,F(\alpha)$ is nonincreasing on $[b,+\infty)$ for some b.

Remark that (i) implies, since $e^{-\alpha} < 1$ for $\alpha > 0$,

$$c = \int_0^{+\infty} e^{-2\alpha}\,F(\alpha)\,d\alpha < +\infty.$$

Then it suffices to set

$$\rho(\alpha) = e^{-2\alpha}\,F(\alpha)/c$$

for $\alpha > 0$, and $\rho(\alpha) = 0$ for $\alpha < 0$, to get the corresponding density. Indeed (i) is then equivalent to the integrability condition $\int_0^{+\infty} e^{\alpha}\rho(\alpha)\,d\alpha < +\infty$, while (ii) means that ρ is eventually nonincreasing.

It is clear that the set of the functions F that meet these requirements is <u>convex</u> and that it is very large. For instance, any <u>bounded</u> F would satisfy (i) and (ii). Alternatively, one may choose $F(\alpha) = e^{s\alpha}\alpha^r$ with $0 < s < 1$ and $r > 1$. The corresponding ρ is then the density of a <u>gamma distribution</u>

$$\rho(\alpha) = \frac{v^{r+1}}{\Gamma(r+1)} \quad e^{-v\alpha} \ \alpha^r$$

with parameters $r > 1$ and $1 < v < 2$.

3. CONCLUSION

The result presented in this note does not go very far. The main limitations come from the fact that income has been treated as independent of prices, which precludes apparently any meaningful application to, say, general equilibrium theory. There may be possible applications of the approach, however, to models of competition with spatially separated markets or differentiated products. Another topic of interest would be to investigate whether or not the theory presented here, or some simple versions of it, has testable implications for demand analysis.

From a purely theoretical viewpoint, the most interesting feature of the analysis seems to be that it exploits a particularly simple algebraic structure of the space of preferences. In this respect, it is perhaps worthwhile to draw attention to a notion contained in the papers by A. Mas-Colell and W. Neuefeind, and by E. Dierker, H. Dierker and W. Trockel, already mentioned. These authors introduce indeed a group of "affine" transformations acting on the space of preferences, which seems to be quite promising for aggregation purposes.

A version of this group is the following. Let R be a preference relation on $X = \mathbb{R}^{\ell}_{+}$. If x is a commodity bundle, and t any vector of $\mathbb{R}^{\ell}$, one defines a new commodity bundle by

$$(3.1) \qquad x*t = (x_1 e^{t_1}, \ldots , x_{\ell} e^{t_{\ell}})$$

The new commodity bundle $x*t$ is obtained through a sequence of affine transformations. Then one may define a transformed preference R_t by

$$(3.2) \qquad (x*t)\ R_t\ (y*t) \quad \text{if and only if} \quad xRy$$

The indifference surfaces of R_t are obtained here too from those of R through a sequence of affine transformations. The particular case of an homothetic transformation that has been considered in the present work arises when t belongs to the diagonal of $\mathbb{R}^\ell$, i.e., $t = (\alpha,\ldots,\alpha)$. If u(x) is a (continuous) representation of R, then

$$u(x^*(-t)) = u(x_1 e^{-t_1}, \ldots, x_\ell e^{-t_\ell})$$

is a continuous representation of R_t.

Therefore (3.2) defines consistently a map (<u>transformation</u>) σ_t from, say, the space $\mathcal{R}$ of continuous preferences on X into itself by $\sigma_t(R) = R_t$. These transformations form a group. Given R, we have $R_0 = R$, $(R_t)_s = R_{t+s}$, and thus

$$\sigma_0 = id \ , \ \sigma_t \circ \sigma_s = \sigma_{t+s}$$

where id stands for the identity map of $\mathcal{R}$ into itself.[5,6] Equivalently, one may view (3.2) as defining a <u>dynamical system</u> (<u>flow</u>) acting on the space of preferences, in which the vector t plays the role of "time". Assuming that R is continuous, strictly convex, one

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[5] Dierker <u>et al.</u> [1984] work directly with the vectors $(e^{t_1},\ldots,e^{t_\ell})$ which form a <u>multiplicative</u> group. The present formulation seems more natural. In particular, the Haar measure that they consider is apparently the image of the Lebesgue measure of $\mathbb{R}^\ell$ by the map $(t_1,\ldots,t_\ell) \to (e^{t_1},\ldots,e^{t_\ell})$.

[6] A lot of subsets of the space of preferences are left invariant by these affine transformations (continuous, (strictly) convex, homothetic, etc.) — in particular the space $\mathcal{R}^*$ considered in this note. It is amusing to note that a preference that has a Cobb-Douglas utility representation is invariant, i.e., satisfies $R_t = R$ for all t. It is not known if other preferences have this property.

sees easily that the effect on demand of these transformations is particularly simple, since it is given by $\xi(R_t,p,w) = \xi(R, p*t,w)*t$, or equivalently for every h

$$\xi_h(R_t,p,w) = e^{t_h}\,\xi_h(R,p_1e^{t_1},\ldots,p_\ell e^{t_\ell},w)$$

Affine transformations, of course, can be defined for preferences on arbitrary consumption sets X. It suffices indeed to define R_t through (3.2) on the transformed consumption set $X_t = X*t$.

Affine transformations generate a simple neat algebraic structure on the space of preferences.. We have not used in this note the more general multidimensional structure of this Section for it does not add anything for the treatment of the problem at hand, i.e., the analysis of the probability distributions on α , or on t, that give rise to negative definiteness of the aggregate Jacobian matrix, independently of the distribution over generators (R,w). This is so because any vector t of $\mathbb{R}^\ell$ has a unique representation of the form $t = \alpha\,\bar{1} + \beta$, where $\bar{1} = (1,\ldots,1)$ and β verifies $\sum_h \beta_h = 0$. We would not have gained by considering probability distributions over $\mathbb{R}^\ell$ on the whole vector t, for in the end, through an application of the Fubini theorem, what matters is the shape of the conditional distributions of the variable α : they should meet the requirements of the Proposition. But it seems likely that multidemensional affine transformations should play a useful role when dealing with aggregation issues. This feeling is reinforced by the remark made to us by Dale Jorgenson, that the concept of an affine transformation is in fact identical to the notion of a <u>household equivalent scale</u> introduced by A. Barten [1964] in applied demand analysis to account for differences in preferences, and subsequently used in econometric work (see A. Deaton and J.S. Muellbauer [1980], J.S. Muellbauer [1980], D.W. Jorgenson and D.T. Slesnick [1984]). The hypothesis is there that all individuals have the same preference up to a rescaling of the units of measurement of commodities. This is exactly the same as saying that preferences are affine transformations of each other, in the sense of the present section.

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