Massachusetts Institute of Technology
Department of Economics
Working Paper Series

HEDGING SUDDEN STOPS & PRECAUTIONARY RECESSIONS: A QUANTITATIVE FRAMEWORK

Ricardo J. Caballero
Stavros Panageas

Working Paper 03-19
May 26, 2003

Room E52-251
50 Memorial Drive
Cambridge, MA 02142

Hedging Sudden Stops and Precautionary Recessions: A Quantitative Framework

Ricardo J. Caballero — MIT and NBER

Stavros Panageas — MIT

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Abstract

Even well managed emerging market economies are exposed to significant external risk, the bulk of which is financial. At a moment's notice, these economies may be required to reverse the capital inflows that have supported the preceding boom. Even if such a reversal does not take place, its anticipation often leads to costly precautionary measures and recessions. In this paper, we characterize the business cycle of an economy that on average needs to borrow but faces stochastic financial constraints. We focus on the optimal financial policy of such an economy under different imperfections and degrees of crowding out in its hedging opportunities. The model is simple enough to be analytically tractable but flexible and realistic enough to provide quantitative guidance.

JEL Codes: E2, E3, F3, F4, G0, C1.

Keywords: Capital flows, sudden stops, financial constraints, recessions, hedging, insurance, signals, contingent credit lines, asymmetric information.

*We are grateful to Victor Chernozhukov, Arvind Krishnamurthy and the participants at the macroeconomics and econometric lunches at MIT for their comments, and Andrei Levchenko for excellent research assistance. Caballero thanks the NSF for financial support.

1 Introduction

Most emerging economies need to borrow from abroad as they catch up with the developed world. Unfortunately, even well managed emerging economies are subject to the "sudden stop" of capital inflows. At a moment's notice, and with only limited consideration of initial external debt and conditions, these economies may be required to reverse the capital inflows that supported the preceding boom.

The deep contractions triggered by this sudden tightening of external financial constraints have great costs for these economies. Not surprisingly, local policymakers struggle to prevent such crises. During the cycle, the anti-crisis mechanism entails deep "precautionary recessions:" tight monetary and fiscal contractions at the first sign of symptoms of a potential external crunch. Over the medium run, the tools of choice are the build up of large stabilization funds and international reserves, and taxation of capital inflows. All of these *precautionary* mechanisms are extremely costly.

The case of Chile illustrates the point well. Chile's business cycle is highly correlated with the price of copper, its main export good: so much so that this price has become a signal to foreign and domestic investors, and to policymakers alike, of aggregate Chilean conditions. As a result of the many internal and external reactions to this signal, the decline in Chilean economic activity when the price of copper falls sharply is many times larger than the annuity value of the income effect of the price decline. This contrasts with the scenario in Australia, a developed economy not exposed to the possibility of a sudden stop, where similar terms-of-trade shocks are fully absorbed by the current account, with almost no impact on domestic activity and consumption.[1]

In this paper we do not attempt to explain why international financial markets treat Chile and Australia so differently, or even why the signal for the sudden stops should be so correlated with the price of copper. Instead, we take the "sudden stops" feature as a description of the environment and characterize the optimal hedging strategies under different assumptions about imperfections in hedging markets. We also characterize the precautionary business cycle that arises when hedging opportunities are very limited.[2]

The main technical contribution of the paper is a model that is stylized enough to allow extensive analytical characterization, but is also flexible and realistic enough to generate quantitative guidance. The model has two central features: First is the *sudden stop,* which we characterize as a probabilistic event that, once triggered, requires the country to reduce the pace of external borrowing significantly. Second is a *signal*

[1] See , e.g., Caballero (2001).

[2] We focus on the aggregate financial problem vis-a-vis the rest of the world, in an environment where domestic policy is managed optimally and decentralization is not a source of problems. Needless to say, these assumptions seldom hold in practice. Such failures compound the problems we deal with in this paper by exacerbating the country's exposure to sudden stops. See, e.g., Caballero and Krishnamurthy (2001, 2003) and Tirole (2002) for articles dealing with decentralization problems. The literature on government's excesses is very extensive. See, e.g., Burnside et al (2003) for a recent incarnation.

Sudden stops have some element of predictability to them. We start by studying a simple environment where there is a perfect signal (e.g., the high yield spread, or the price of an important commodity for the country) that triggers a sudden stop once it crosses a well defined threshold. We then study the more realistic case where the threshold is blurred and a sudden stop, while increasingly likely as the signal deteriorates, may occur at any time.

Within this model, we develop two substantive themes. First we characterize *precautionary recessions*: that is, in the absence of perfect hedging, the business cycle of the economy follows the signal even if no sudden stop actually takes place. This is because as the likelihood of a sudden stop rises, the country goes into a precautionary recession. Consumption is cut to reduce the extent of the adjustment required if a sudden stop does take place.

The second and main theme is aggregate hedging strategies. An adequate hedging strategy not only reduces the extent of the crisis in the case of a sudden stop, but also reduces the need for hoarding scarce assets and for incurring sharp precautionary recessions as the signal deteriorates. We study two polar-opposite types of generic hedging instruments or strategies, as well as their intermediate cases. At one end, the hedging contract relaxes the sudden stop constraint one-for-one. That is, each dollar of hedge can be used to relax the constraint. The other extreme is motivated by crowding out: if the resources obtained from the hedge facilitate the withdrawal of other lenders, then the main effect of the hedge is to reduce the country's debt in bad states of the world but not to provide fresh resources. By its nature, this type of hedging is an excellent substitute for precautionary recessions, but it cannot remove the sudden stop entirely.

The paper concludes with an illustration of our main results for the case of Chile. We estimate the probability of a sudden stop as a function of the price of copper and calibrate the parameters needed to obtain sharp consumption drops such as those experienced by Chile in the recent contraction of 1998/99. We then describe different hedging strategies and their impact on the volatility and levels of consumption. We discuss credit lines and their indexation to the signal as one way of reducing asymmetric information problems. For example, we argue that Chile could virtually eliminate sudden stops, precautionary recessions, and its large accumulation of precautionary assets, with a credit line that rises nonlinearly with the price of copper. The cost of this line, if fairly priced, should be around 1-2 percent of GDP. This is very little when compared with the costs of the precautionary measures currently undertaken, including large accumulation of reserves, limited short-term borrowing, and precautionary recessions.[3]

Currently, although futures markets exist for much of the income-flow effects of commodity price fluctuations, the size of the financial problem is much larger than that. The markets required for these strategies

[3] In his Nobel lecture, Robert Merton (1998) highlights the enormous savings than can be obtained by designing adequate derivatives and other contracting technologies to deal with risk management. He also argues that emerging-market economies stand to gain the most. Our findings in this paper fully support his views.

do not exist, at least in the magnitude required. In this sense, our framework also serves to highlight quantitatively the usefulness of these markets and allows us to begin gauging the potential size of the markets to be developed.[4]

In section 2 we describe the environment and characterize the optimization problem once a sudden stop has been triggered — this provides the boundary conditions for the "precautioning phase." In section 3 we study the phase that precedes a sudden stop when the country self-insures. The main goal of this section is to characterize precautionary recessions. Section 4 describes aggregate hedging strategies under different imperfections and the degrees of crowding-out in these markets. Section 5 illustrates our results through an application to the case of Chile. Along the way, we outline an econometric approach to gauging the likelihood of a sudden stop and its correlation with an underlying signal. Section 6 concludes and is followed by an extensive technical appendix.

2 The Environment and the Sudden Stop Value Function

Intertemporal smoothing implies that, during the catch up process, emerging economies typically experience substantial needs for borrowing from abroad. For a variety of reasons that we do not model here, this dependence on borrowing is a source of fragility. The sudden tightening of financial constraints, or the mere anticipation of such an event, generates large drops in consumption. In this section, we formalize such an environment and characterize the optimization problem once a sudden stop has occurred. The next sections complete the description by characterizing the phase that precedes the crisis.

2.1 The Environment

2.1.1 Endowment and Preferences

We assume that the endowment grows at some constant rate, g, during $0 \leq t < \infty$. Thus, the income process $(y(t), t \geq 0)$, is described by:

$$\frac{dy(t)}{y(t)} = g\,dt, \qquad y(0) = y_0, \qquad g > 0.$$

Two aspects of this process are to be highlighted. First, it is deterministic. In the economies we wish to characterize, sudden stops are significantly more important than endowment shocks as triggers of deep

[4] See, e.g., Krugman (1988), Froot, Scharfstein, Stein (1989), Haldane (1999), Caballero (2001), for articles advocating commodity indexation of emerging markets debt. The contribution of this paper relative to that literature is to offer a quantitative framework and to link the hedging need not to commodities per se, but simply as a signal of much costlier financial constraints.

contractions.[5] From this perspective, the main importance of endowment shocks is their "collateral" effect, and hence their potential to trigger a sudden stop. We simplify the model along this dimension and group any collateral effects contained in endowment shocks with the signal process (to be described below). Second, since $g > 0$, there is an incentive for the country to borrow early on.

Reflecting its initial net debtor position, the country starts with financial wealth, $X(0) = X_0 < 0$. However, the country's total wealth must be positive at all times:

$$X_t > -\frac{y_t}{r-g} \qquad t \geq 0,$$

where r denotes the riskless interest rate and it exceeds the rate of growth of the endowment, $r > g$.

Let c_t and c_t^* represent date t consumption and "excess" consumption, respectively, with:

$$c_t^* \equiv c_t - \kappa y_t, \qquad 0 \leq \kappa < 1.$$

The representative consumer maximizes:

$$\mathrm{E}_t \left[\int_t^\infty u(c_s^*) e^{-\delta(s-t)} \, ds \right] \tag{1}$$

with

$$u(c_t^*) = \frac{c^{*1-\gamma}}{1-\gamma}, \qquad \delta > 0, \gamma > 0.$$

The parameters δ and γ are the discount rate and risk aversion coefficient, respectively. For simplicity, we assume $r = \delta$ throughout. The functional form of the utility function captures an external habit formation, with a habit level that is increasing at the rate of the country's growth rate. This is a natural assumption in economies that exhibit strong growth, since future generations are likely to have a higher standard of living than the current ones.[6]

A frictionless benchmark

It is instructive to pause and study the solution of problem (1) subject to a standard intertemporal budget constraint (and absent a sudden stop constraint):

$$dX_t = (rX_t - c_t^* + y_t^*) \, dt$$

$$X_0 \text{ given}$$

[5] Of course, sudden stops reduce growth as well, but we capture these effects directly through the decline in consumption. In this sense, y can be thought of as potential, rather than actual, output.

[6] Accordingly, this type of utility function allows us to impose (in a simple reduced form) a barrier on the amount of indebtedness of the country at any point in time of:

$$X_t > -\frac{(1-\kappa)y_t}{r-g} \qquad t \geq 0.$$

and on saving, since:

$$y_t - c_t \leq (1-\kappa)y_t \qquad t \geq 0.$$

with $y^* \equiv (1-\kappa)y$.

By standard methods it can be shown that the solution to this deterministic problem is given by:

$$c_t^* = r\left(X_0 + \frac{y_0^*}{r-g}\right) \quad \text{for all } t > 0,$$

and the "total resources" of the country remain constant throughout:

$$X_t + \frac{y_t^*}{r-g} = X_0 + \frac{y_0^*}{r-g} \tag{2}$$

so that

$$\lim_{t\to\infty} \frac{X_t}{y_t^*} = -\frac{1}{r-g}$$

and, accordingly:

$$\lim_{t\to\infty} \frac{X_t}{y_t} = -\frac{1-\kappa}{r-g}.$$

Moreover, for any level of the debt-to-income ratio below its limit value, the ratio $\frac{X_t}{y_t^*}$ decreases monotonically to $-\frac{1}{r-g}$.

This case serves as a frictionless benchmark in what follows.

2.1.2 Signal

There is a publicly observable signal, s_t, correlated with the sudden stop and, for simplicity, uncorrelated with world endowments. This signal follows a diffusion process:

$$ds_t = \mu dt + \sigma dB_t.$$

In our basic model, s_t is a perfect signal and a crisis is triggered the first time the signal reaches a threshold, $\underline{s}$, from above. We associate with this event the stochastic time, τ:

$$\tau = \inf\{t \in (0,\infty) : s_t \leq \underline{s}\}.$$

In our second model, the signal is imperfect. In this case, a sudden stop can happen at any point in time. It is only its likelihood that is (smoothly) influenced by the signal s_t. In this case, the specification of the stochastic time, τ, depends on the realization of a stochastic jump process, the intensity of which is given by

$$\lambda_t = \exp(\alpha_0 - \alpha_1 s_t).$$

2.1.3 Sudden Stops

We place no limits on the country's borrowing ability up to the stochastic time τ. At this time, the country faces a "sudden stop." We do not model the informational or contractual factors behind this constraint, or the complex bargaining and restructuring process that follows once the sudden stop is triggered. Since our goal is to produce a quantitative assessment of the hedging aspects of the problem, we look for a realistic and fairly robust (across models) constraint. For this, we simply model the sudden stop as a temporary and severe constraint on the rate of external borrowing. In particular, since the "natural" aggregator of a country's total wealth is the net present value of its total resources, we place the constraint on:

$$X_\tau + \frac{y_\tau^*}{r-g}.$$

We assume that at time τ, financial markets require the country to increase its total resources by $\zeta \geq e^{gT}$ within T periods. Formally:

$$\left(X_{\tau+T} + \frac{y_{\tau+T}^*}{r-g}\right) \geq \zeta \left(X_\tau + \frac{y_\tau^*}{r-g}\right). \tag{3}$$

It is obvious that this constraint will be always binding because as we showed in a previous subsection, at the unconstrained benchmark $\left(X_t + \frac{y_t^*}{r-g}\right)$ remains constant at all times. It is then straightforward to show that this constraint can be expressed as a constraint on the maximum allowable amount of debt/gdp at time $\tau + T$, as a function of the debt/gdp ratio at time τ. To see this, redefine

$$\zeta = e^{gT} + \phi(1 - e^{-(r-g)T}) \tag{4}$$

and observe that the constraint becomes:

$$X_{\tau+T} \geq X_\tau \left[e^{gT} + \phi(1 - e^{-(r-g)T})\right] + \frac{y_\tau^* \phi(1 - e^{-(r-g)T})}{r-g} \tag{5}$$

or

$$\frac{X_{\tau+T}}{y_{\tau+T}^*} \geq \frac{X_\tau}{y_\tau^*} \frac{\left[e^{gT} + \phi(1 - e^{-(r-g)T})\right]}{e^{gT}} + \frac{\phi(1 - e^{-(r-g)T})}{e^{gT}(r-g)}$$

Higher levels of ϕ imply higher levels of ζ and thus make the constraint tighter. It is also trivial to verify that for the special case $\phi = 0$ this constraint literally reduces to the requirement that debt/gdp cannot grow any further between τ and $\tau + T$:[7]

$$\frac{X_{\tau+T}}{y_{\tau+T}^*} \geq \frac{X_\tau}{y_\tau^*}$$

Finally, it is interesting to note that one can think of (5) as a constraint on the minimum balance-of-trade surpluses the country has to accumulate over the next T periods. To see this, start with the intertemporal budget constraint:

$$X_\tau + \int_\tau^{\tau+T} e^{-r(t-\tau)} y_t^* \, dt = \int_\tau^{\tau+T} e^{-r(t-\tau)} c_t^* \, dt + e^{-rT} X_{\tau+T}$$

or:

$$X_\tau - e^{-rT} X_{\tau+T} = \int_\tau^{\tau+T} e^{-r(t-\tau)} c_t^* \, dt - \frac{y_\tau^*(1 - e^{-(r-g)T})}{r-g}$$

[7] Even for the $\phi = 0$ case the constraint we consider is binding at the time of the sudden stop. This is due to the fact that the debt/gdp ratio is always growing for an unconstrained country as we showed in section 2.1.1.

and replace (5) into it, to obtain:

$$\int_{\tau}^{\tau+T} e^{-r(t-\tau)}\left(c_t^* - y_t^*\right) dt \leq X_\tau(1 - (e^{-(r-g)T} + \phi e^{-rT}(1 - e^{-(r-g)T}))) - \phi e^{-rT} \int_{\tau}^{\tau+T} e^{-r(t-\tau)} y_t^* \, dt$$

or, finally:

$$\int_{\tau}^{\tau+T} e^{-r(t-\tau)}\left(y_t^* - c_t^*\right) dt \geq X_\tau((e^{-(r-g)T} + \phi e^{-rT}(1 - e^{-(r-g)T})) - 1) + \phi e^{-rT} \int_{\tau}^{\tau+T} e^{-r(t-\tau)} y_t^* \, dt.$$

This gives us the constraint in terms of the balance-of-trade surpluses required from a country that is faced with a sudden stop. These required surpluses rise with the level of debt (recall that $X_\tau < 0$), ϕ, and the endowment of the country.

2.2 The Optimization Problem

Let us assume for now that the only financial instrument is riskless debt, so there are no hedging instruments indexed to s or τ. In this case, the country maximizes the expected utility of a representative consumer:

$$V(X_t, y_t^*) = \max_{c_t^*} \mathrm{E}\left[\int_0^\infty \frac{{c_t^*}^{1-\gamma}}{1-\gamma} e^{-rt}\, dt\right]$$

$$s.t.$$

$$dX_t = (rX_t - c_t^* + y_t^*)\, dt$$

$$X_{\tau+T} \geq \overline{X}_{\tau+T} \equiv X_\tau\left[e^{gT} + \phi(1 - e^{-(r-g)T})\right] + \phi\frac{y_\tau^*(1 - e^{-(r-g)T})}{r-g}, \quad 0 \leq \phi < e^{rT}$$

$$\frac{dy_t^*}{y_t^*} = g\, dt$$

$$ds_t = \mu dt + \sigma dB_t$$

$$\lim_{t\to\infty} X(t)e^{-rt} = 0, \qquad a.s.$$

and either

Perfect signal: $\tau = \inf\{t \in (0,\infty) : s_t < \underline{s}\}$ or

Imperfect Signal: $\tau = \inf\{t \in (0,\infty) : \int_0^\tau \lambda(s_t)dt < z\}, \quad \lambda_t = e^{\alpha_0 - \alpha_1 s_t}, \quad z \sim \exp(1)$

where z is independent of the standard Brownian F_t-Filtration.

2.3 The Sudden Stop Value Function and Amplification

We solve the optimization problem in three steps. Starting backwards, we first solve for the post-crisis period, then for the sudden-stop period, and finally for the period preceding the sudden stop. In this section we present the first two, and trivial, steps. The goal of these is to find the value function at the time of the sudden stop, $V^{SS}(X_\tau, y_\tau)$, which then can be used to find the solution of the optimization and hedging problems before the crisis takes place.

2.3.1 Post Sudden Stop: $t > \tau + T$

Since we made the simplifying assumption that the country suffers only one financial crisis, the maximization problem for $t > \tau + T$ is simply:

$$V(X_{\tau+T}, y^*_{\tau+T}) = \max_{c^*_t} \int_{\tau+T}^{\infty} \frac{c_t^{*\,1-\gamma}}{1-\gamma} e^{-r(t-(\tau+T))}\, dt \tag{6}$$

$$s.t.$$

$$\int_{\tau+T}^{\infty} c^*_t e^{-r(t-(\tau+T))}\, dt = X_{\tau+T} + \int_{\tau+T}^{\infty} y^*_t e^{-r(t-(\tau+T))}\, dt \tag{7}$$

This problem has the trivial solution:

$$c^*_t = rW^e_2 \qquad \tau + T < t < \infty \tag{8}$$

with

$$W^e_2 = X_{\tau+T} + \frac{y^*_{\tau+T}}{r-g}. \tag{9}$$

This constant can be interpreted as the "excess" wealth at date $\tau + T$ (that is, the wealth in excess of that which is needed to cover the reservation-consumption level κy_t).

2.3.2 Sudden Stop: $\tau \le t \le \tau + T$

With the continuation value function, $V(X_{\tau+T}, y^*_{\tau+T})$, from above, we can now write the maximization problem for the sudden stop phase:

$$V^{SS}(X_\tau, y^*_\tau) = \max_{c^*_t} \int_{\tau}^{\tau+T} \frac{c_t^{*\,1-\gamma}}{1-\gamma} e^{-r(t-\tau)}\, dt + e^{-rT} V(X_{\tau+T}, y^*_{\tau+T})$$

$$s.t.$$

$$\int_{\tau}^{\tau+T} c^*_t e^{-r(t-\tau)}\, dt = X_\tau + \int_{\tau}^{\tau+T} y^*_t e^{-r(t-\tau)}\, dt - e^{-rT} X_{\tau+T}$$

$$X_{\tau+T} \ge \overline{X}_{\tau+T}. \tag{10}$$

Since the country is growing and wishes to expand its consumption at a faster rate than the constraint allows, the sudden stop constraint, (10), is always binding (see the appendix for a formal proof):

$$X_{\tau+T} = \overline{X}_{\tau+T}. \tag{11}$$

Given this result, the optimization problem is straightforward. It is a deterministic consumption problem subject to a final wealth condition. A few steps of algebra show that:

$$c^*_t = \frac{r}{1-e^{-rT}} W^e_1, \qquad \tau < t < \tau + T \tag{12}$$

with

$$W_1^e = X_\tau + \frac{y_\tau^*(1 - e^{-(r-g)T})}{r-g} - e^{-rT}\overline{X}_{\tau+T}.$$

It is easy to see from these expressions that as $\overline{X}_{\tau+T}$ rises, c_t^* falls. The country has to cut back consumption during the crisis in order to satisfy a tighter sudden-stop constraint.

We are now ready to determine the sudden-stop value function:

$$\begin{aligned} V^{SS}(X_\tau, y_\tau^*) &= \int_\tau^{\tau+T} \frac{(rW_1^e)^{1-\gamma}}{1-\gamma} e^{-r(t-\tau)}\, dt + e^{-rT} \int_{\tau+T}^{\infty} \frac{(rW_2^e)^{1-\gamma}}{1-\gamma} e^{-r(t-(\tau+T))}\, dt & (13)\\ &= \frac{(rW_1^e)^{1-\gamma}}{1-\gamma}\left(\frac{1-e^{-rT}}{r}\right) + \frac{(rW_2^e)^{1-\gamma}}{1-\gamma}\left(\frac{e^{-rT}}{r}\right). & (14) \end{aligned}$$

It is easy to verify that our careful choice of the constraint pays off at this stage. The value function simplifies to:

$$\begin{aligned} V^{SS}(X_\tau, y_\tau^*) &= K\left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma} \\ &= K\, V(X_\tau, y_\tau^*) & (15) \end{aligned}$$

where $V(X_\tau, y_\tau^*)$ denotes the value function in the *absence* of a sudden-stop constraint and K is a constant given by:

$$K = (1-e^{-rT})^\gamma(1 - e^{-rT}\phi(1-e^{-(r-g)T}) - e^{-(r-g)T})^{1-\gamma} + e^{-rT}\left(\phi(1-e^{-(r-g)T}) + e^{gT}\right)^{1-\gamma}. \qquad (16)$$

That is, the dimension of the state space effectively is reduced from two, (X_τ, y_τ), to one, $\left(X_\tau + \frac{y_\tau^*}{r-g}\right)$. Moreover, *up to the constant* K, we have arrived at a value function that is identical to the one in the problem *without* sudden stops. This problem corresponds to the trivial case of an infinite horizon consumption-savings problem under certainty. Note that for $\gamma > 1$, which we assume throughout, $K > 1$ and increases with ϕ.[8]

While the particular simplicity of our formulae is due to stylized assumptions, the basic message is more general. The model also can be understood as an approximation to a potentially more complicated specification of constraints that result in higher marginal disutility of debt in the event of a crisis.

[8] Observe that for $\gamma > 1$ the function $\frac{Z^{1-\gamma}}{1-\gamma}$ is negative for all $Z > 0$. Thus $K > 1$ reflects that the constrained value function is lower than the continuation value function for the unconstrained problem. The reason we need $\gamma > 1$ is that the flow aspect of the constraint implies that having a higher X_τ does not relax the constraint one for one, because the lender does not reduce its demands by the same amount during the sudden stop. When γ is below one, this effect is strong enough to discourage saving for the crisis.

3 Precautionary Recessions

Let us now address our main concern and begin to characterize the country's optimization problem prior to the sudden-stop phase. Since we have assumed that there are no hedging instruments contingent on s or τ, the country's only mechanism for reducing the cost of a sudden stop is to cut consumption and borrowing before it takes place. We show that in addition to a precautionary savings result, the amount of self-insurance varies over time, because sudden stops have some elements of predictability in them. In particular, when the signal of a sudden stop deteriorates, the country falls into what can be labelled as a "precautionary recession:" that is, a sharp reduction in consumption to limit the cost of the potential sudden stop. Such behavior is widely observed in emerging market economies, where private decisions and macroeconomic policy tighten on the face of external risk.

In practice, such a problem is complex for many reasons, one of the most important being the uncertainty that surrounds the factors that trigger such crises. In order to isolate the main issues, we proceed in two steps. First, we study a case where there is a perfect (stochastic) signal: A sudden stop occurs when this signal hits a minimum threshold, $\underline{s}$, for the first time. Second, we add (local) uncertainty: while sudden stops still are more likely as the signal deteriorates, they can occur at any time.

3.1 Perfect Signal: The Threshold Model

In the threshold model, the dynamic programming problem is:

$$V(X_t, s_t, y_t^*) = \max_{c_u^*} E\left[\int_t^\tau \frac{c_u^{*\,1-\gamma}}{1-\gamma} e^{-r(u-t)}\,du + e^{-r(\tau-t)} V^{SS}(X_\tau, y_\tau^*)|F_t\right] \tag{17}$$

where

$$\tau = \inf\{(t : s \leq \underline{s}) \wedge \infty\} \tag{18}$$

and the evolution of (X_t, s_t, y_t) is given by:

$$\begin{aligned} dX_t &= (rX_t - c_t^* + y_t^*)\,dt && (19)\\ \frac{dy_t}{y_t^*} &= g\,dt && (20)\\ ds_t &= \mu dt + \sigma dB_t. && (21)\end{aligned}$$

The boundaries of the value function, $V(X_t, s_t, y_t)$, can be found readily. On one end, we showed in the previous section that:

$$V(X_\tau, \underline{s}, y_\tau^*) = K\left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma} \tag{22}$$

where $K > 1$ measures the intensity of the crisis and is given in (16).

On the other end, we show in the appendix that as s_t goes to infinity, the value function converges to the value function of the deterministic problem with no sudden stops:

$$\lim_{s\to\infty} V(X_\tau, s, y_\tau^*) = \left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma}. \tag{23}$$

That is, the value function becomes independent of the signal as the crisis event becomes less and less likely.

The value function V satisfies the following Hamilton-Jacobi-Bellman (henceforth HJB) equation:

$$0 = \max_{c_t^*} \left\{ \frac{c_t^{*1-\gamma}}{1-\gamma} - V_X c_t^* \right\} - rV + V_X\left(rX + y_t^*\right) + V_{y^*} g y_t^* + V_s \mu + \frac{1}{2} V_{ss}\sigma^2 \tag{24}$$

subject to the boundary conditions (22) and (23).

In the appendix, we show that the solution has the form:

$$V(X_t, s_t, y_t^*) = a(s_t)^\gamma \frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma} \tag{25}$$

for some twice differentiable function $a(s_t)$. The latter satisfies the boundary conditions:

$$\lim_{s\to\infty} a(s) = \left(\frac{1}{r}\right) \tag{26}$$

$$a(\underline{s}) = K^{1/\gamma}\left(\frac{1}{r}\right) \tag{27}$$

and solves an ordinary differential equation that can be obtained in three steps. First, carry out the maximization in (24) with respect to c_t^* to get:

$$c_t^* = \left(V_X\right)^{-1/\gamma}. \tag{28}$$

Second, substitute this into (24). Third, divide the resulting expression by $\frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$. These steps yield:

$$1 - ra + \mu a_s + \frac{1}{2}\sigma^2\left((\gamma-1)\frac{(a_s)^2}{a} + a_{ss}\right) = 0 \tag{29}$$

which can be solved numerically subject to (26) and (27). We show in the appendix that the solution to this ODE exists and is unique. We plot this function (multiplied by r) in the left panel of Figure 1. The function $a(s)$ is decreasing with respect to s. As the signal deteriorates toward $\underline{s}$, this function rises rapidly, reflecting the increasing value of wealth (and marginal wealth) as the sudden stop becomes more likely.

This setup allows for an explicit characterization of the optimal consumption policy in feedback form. Given the value function, it follows from (28) that the optimal consumption policy takes the form:

$$c_t^* = \frac{\left(X_t + \frac{y_t^*}{r-g}\right)}{a(s_t)}. \tag{30}$$

The effect of the signal can be seen clearly in this expression. As s rises, $a(s)$ falls toward its frictionless limit. Conversely, when the signal worsens, consumption falls for any given level of income and debt. This is precisely the precautionary recession mechanism: as the crisis becomes imminent, consumption falls in order to reduce external debt and hence exposure to the sudden stop.

Applying Ito's Lemma to the right hand side of (30), we obtain the country's (excess) consumption process:

$$\frac{dc_t^*}{c_t^*} = \frac{1}{2}(\gamma+1)\left(\frac{\sigma a_s}{a}\right)^2 dt - \frac{a_s}{a}\sigma dB_t. \tag{31}$$

This process reinforces the above conclusion and shows that consumption has a nontrivial diffusion term. Without hedging, consumption "picks up" the volatility and fulfills the function of a hedging strategy.

Up to the functions $\frac{a_s}{a}$, which can be evaluated numerically, the rest of the terms in this expression have a straightforward interpretation. The first term is positive and captures the effect of precautionary savings, whereas the second term $\left(-\frac{a_s}{a}\sigma\right)$ is also positive and captures the sensitivity of consumption to signal news. Positive news about the signal increases consumption whereas negative news decreases it. This is the outcome of the country's attempt to accumulate resources as the sudden stop becomes imminent. As can be seen in the right panel of Figure 1, which plots $\frac{a_s}{a}$, this effect becomes more intense as the signal deteriorates. That is, the sensitivity of consumption to news about the signal increases during downturns.

3.2 Imperfect Signal: The jump model

Countries do not have a perfect signal for their sudden stops. Variables such as terms-of-trade and the conditions of international financial markets raise the probability of a sudden stop, but it is never as stark as in the threshold model. In this section we capture this dimension of reality by making the trigger of a sudden stop a probabilistic function of the underlying signal.

The setup and optimization problem are exactly as in (17)-(21), with the exception of the stopping time, τ, in (18). Now the sudden stop is a Poisson Jump-Event with intensity:

$$\lambda(s_t) = e^{\alpha_0 - \alpha_1 s_t}, \qquad \alpha_1 \geq 0.$$

Thus, we replace (18) for:

$$\tau = \inf\{t \in (0,\infty) : \int_0^\tau \lambda(s_t)dt < z\}, \quad \lambda(s_t) = e^{\alpha_0 - \alpha_1 s_t}, \quad z \sim \exp(1).$$

Once the sudden stop takes place, events unfold exactly as in the threshold model. In particular, the value function still takes the form:

$$V^{SS}(X_\tau, y_\tau^*) = K\left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y_\tau^*}{r-g}\right)^{1-\gamma}}{1-\gamma}.$$

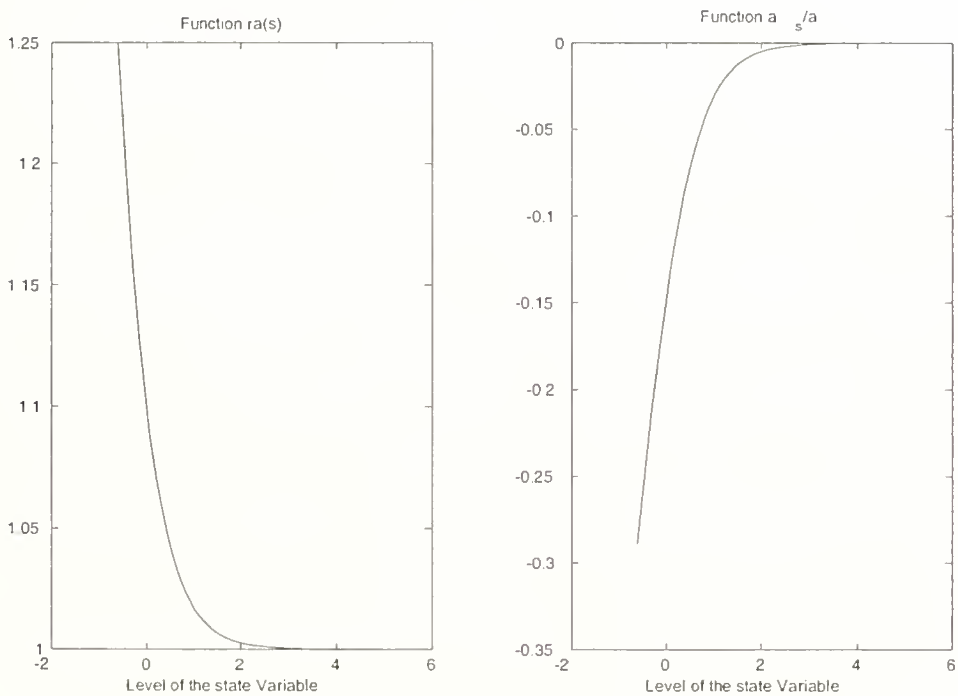

Figure 1: Functions $ra(s)$ (left panel) and $\frac{a_s}{a}$ (right panel). The parameters used for this example are: $\underline{s} = -0.6, s_0 = 0, r = 0.09, T = 1, \phi = 0.35e^{rT}, \sigma = 0.23, \gamma = 7, a = 0.03, \kappa = 0.8, y_0 = 1, X_0 = -0.5$.The drift of the state variable is set to 0.

The only difference is that V^{SS} now can be reached from any s for which $\lambda(s) > 0$, rather than from just $\underline{s}$. In this case the HJB equation for the value function is:

$$0 = \max_{c^*}\{\frac{c^{*1-\gamma}}{1-\gamma} - V_X c_t^*\} - rV + V_X(rX + y^*) + V_{y^*} y^* g + V_s \mu + \frac{1}{2}\sigma^2 V_{ss} + \lambda(s_t)\left[V^{SS} - V\right].$$

With essentially identical steps as in the previous subsection, we can show that our stylized framework still yields a simple solution of the form:

$$V = b(s_t)^{\gamma} \frac{\left(X_t + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}.$$

3.2.1 Precautionary Savings

Let us pause and focus on the case where sudden stops are totally unpredictable, $\alpha_1 = 0$. Since in this case $b(s)$ is no longer a function of the signal s, replacing the function V in the HJB equation yields a simple algebraic equation for b:

$$1 - rb + \frac{b}{\gamma}\lambda\left[\left(\frac{\underline{b}}{b}\right)^{\gamma} - 1\right] = 0.$$

where

$$\underline{b} \equiv K^{\frac{1}{\gamma}}\left(\frac{1}{r}\right).$$

It is now straightforward to obtain the (excess) consumption function from the envelope theorem:

$$c_t^* = \frac{\left(X_t + \frac{y_t^*}{r-g}\right)}{b}.$$

Applying simple differentiation to this expression, we obtain the country's (excess) consumption process:

$$\frac{dc_t^*}{c_t^*} = \frac{1}{\gamma}\lambda\left[\left(\frac{\underline{b}}{b}\right)^{\gamma} - 1\right] dt$$

Relative to the threshold model, there is a new drift term in the excess consumption process. This term reflects the additional precautionary savings attributable to the local uncertainty introduced by the strictly positive probability of a sudden stop taking place in the next instant. But, because this probability is not correlated with the signal s, there are no precautionary recessions. In this case, the pattern of saving for self-insurance is not a source of business cycles.

3.2.2 Precautionary Recessions

Let us now return to the general case, where $\alpha_1 > 0$. After a few simplifications, substituting the function V in the HJB equation yields the following ODE:

$$1 - rb + b_s\mu + \left[(\gamma - 1)\frac{(b_s)^2}{b} + b_{ss}\right]\frac{1}{2}\sigma^2 + \frac{b}{\gamma}\lambda(s_t)\left[\left(\frac{\underline{b}}{b}\right)^{\gamma} - 1\right] = 0, \quad (32)$$

which differs from (29) only in the last term.

The boundary conditions are also different. First, when s goes to infinity there is still a strictly positive probability of a crisis. Second, as conditions worsen, there is no equivalent to the threshold $\underline{s}$ where a crisis happens with probability one. Let

$$b^* = \lim_{s \to \infty} b(s).$$

$$b_* = \lim_{s \to -\infty} b(s).$$

b^* and b_* are given by:

$$\begin{aligned} b^* &= \frac{1}{r} \\ b_* &= K^{1/\gamma}\frac{1}{r}. \end{aligned}$$

Again, it is straightforward to obtain the (excess) consumption function from the envelope theorem:

$$c_t^* = \frac{\left(X_t + \frac{y_t^*}{r-g}\right)}{b(s_t)}. \tag{33}$$

Finally, applying Ito's lemma to the right hand side of (33), we obtain the country's (excess) consumption process:[9]

$$\frac{dc_t^*}{c_t^*} = \left[\frac{\gamma+1}{2}\left(\frac{\sigma b_s}{b}\right)^2 + \frac{1}{\gamma}\lambda(s_t)\left[\left(\frac{\underline{b}}{b}\right)^\gamma - 1\right]\right]dt - \frac{b_s}{b}\sigma dB_t \tag{34}$$

This case integrates the insights of the previous models. There is ongoing precautionary savings attributable to local uncertainty, but the amount varies with the signal. For the same reasons of the previous section, $\frac{b_s}{b}$ is strictly negative, so that the conclusions of the previous section also carry through here. A deteriorating signal increases the need to accumulate resources and accordingly makes consumption respond more to a one-standard-deviation increase in the signal by a factor of $-\frac{b_s}{b}\sigma$. Finally, the drift term includes a new precautionary term $\left(\frac{1}{\gamma}\lambda(s_t)\left[\left(\frac{\underline{b}}{b}\right)^\gamma - 1\right]\right)$, which we also found in the case $\alpha_1 = 0$, and captures precautioning against the Poisson jump event that can occur at any time. In the application section of the paper we quantify these effects in the context of Chile.

4 Aggregate Hedging

Precautionary savings and recessions are very costly and imperfect self-insurance mechanisms for smoothing the impact of a sudden stop. In this section, we enlarge the options of the country and allow it to hedge using derivatives and insurance contracts. Of course, the effectiveness of the hedging strategy depends on

[9] We are obviously focusing on a case where the jump has not yet taken place so that $dq = 0$, where q is the poisson event.

the contracts and instruments that are available to the country, how these contracts enter into the sudden stop constraint, and how accurate the crisis-signal is.

The issue of how the fresh funds relax the country's constraint is at the core of the current debate about optimal assistance mechanisms and is not well understood. We do not try to solve this debate but rather characterize optimal hedging strategies under different scenarios. We begin by exploring two polar cases: In the first case, the hedge cannot relax the constraint directly but only improves the initial conditions of the country once it hits the constraint. This would be the case, for example, when the resources from the hedge are used entirely to pay other lenders.[10] In the second case, the country can buy hedging that directly relaxes the sudden stop constraint. That is, each dollar received from the hedge can be used to fulfill the sudden stop constraint.

In this section, we isolate the above distinction and focus on the simpler threshold model. We do this for analytical tractability, because we can use complete-markets tools in this model that allow us to obtain closed-form solutions. We use these results as an approximation benchmark for the more realistic imperfect signal case, which is the focus of the empirical section.

4.1 Hedging Precautionary Recessions

Assume for the moment that the country has no mechanism of injecting net resources into the sudden stop constraint. However, the country faces complete hedging markets before the sudden stop arises. In this sense, we can interpret the sudden stop as a time when all financial markets close and the country is left only with its resources at the outset of the crisis: X_{τ^-}.

An alternative interpretation is that the hedge is used (crowded out) by existing lenders and the sudden-stop constraint remains unchanged, except for the positive effect of a reduced initial debt (hence, there is a reduction in the required balance of trade surplus).

Let us re-write the dynamic budget constraint for $t \geq 0$ as:

$$dX_t = (rX_t - c_t^* + y_t^*)dt + \pi_t dF_t$$

where dF_t denotes the profit/loss in the futures position and π_t is the number of future contracts.

Since under the assumption of zero risk premium and any constant convenience yield for copper, d, we have:

$$F_t = S_t e^{(r-d)(T-t)}$$

we can apply Ito's lemma to obtain:

$$dF_t = \sigma F_t dB_t.$$

[10] Recall that our model has no straight default. Implicitly, however, it does allow for limited rescheduling.

Defining the portfolio process as $p_t = \pi_t F_t$, we obtain the new dynamic budget constraint:

$$dX_t = (rX_t - c_t^* + y_t^*)dt + p_t \sigma dB_t. \tag{35}$$

Thus we modify the dynamic programming problem in (17) to allow for a hedging portfolio p_t (correspondingly, we refer to this hedging portfolio as p–hedging):

$$V(X_t, s_t, y_t^*) = \max_{c_u^*, p_u} E\left[\int_t^\tau \frac{{c_u^*}^{1-\gamma}}{1-\gamma} e^{-r(u-t)}\, du + e^{-r(\tau-t)} V^{SS}(X_\tau, y_\tau^*) | F_t\right] \tag{36}$$

where

$$\tau = \inf\{(t : s \le \underline{s}) \wedge \infty\}$$

and the evolution of (X_t, s_t, y_t) is now given by:

$$dX_t = (rX_t - c_t^* + y_t^*)\, dt + p_t \sigma dB_t \tag{37}$$

$$\frac{dy_t^*}{y_t^*} = g\, dt \tag{38}$$

$$ds_t = \mu dt + \sigma dB_t. \tag{39}$$

It turns out that the solution to this problem is simpler than the no-hedging problem because we can use well-known techniques from the complete markets case.[11] Following a derivation similar to the no-hedging case, one verifies that the value function of the problem in the presence of complete hedging is given by (see the appendix):

$$V(X_t, y_t^*, s_t) = a^p(s_t)^\gamma \frac{\left(X_t + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

with

$$a^p(s_t) = \frac{1}{r}\left(1 + e^{-\lambda_1 (s - \underline{s})}(K^{\frac{1}{\gamma}} - 1)\right) \tag{40}$$

$$\lambda_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2}.$$

Correspondingly, the consumption policy in feedback form is:

$$c_t^* = \frac{\left(X_t + \frac{y_t^*}{r-g}\right)}{a^p(s_t)}. \tag{41}$$

While it may appear from this expression that not much has changed with respect to the no-hedging case, this is not so. To see this, apply Ito's lemma to the right hand side and simplify (see the appendix), to obtain:

$$dc^* = 0.$$

That is, excess consumption is constant throughout the pre-sudden stop phase. *There are no more precautionary recessions,* and the signal does not affect consumption.

[11] This is possible because markets are complete at all dates but τ.

How can this be reconciled with the consumption expression in (41)? The answer is in the behavior of X_t. While in the no-hedging case X_t was simply made of riskless debt, it now includes a hedging portfolio component, p_t:

$$p_t = -\lambda_1 \left(\frac{e^{-\lambda_1 (s-\underline{s})}(K^{\frac{1}{\gamma}} - 1)}{1 + e^{-\lambda_1 (s-\underline{s})}(K^{\frac{1}{\gamma}} - 1)} \right) \left(X_t + \frac{y_t^*}{r-g} \right)$$

which once replaced in

$$dX_t = (rX_t - c_t^* + y_t^*)dt + p_t \sigma dB_t$$

implies that all the variation in precautioning is absorbed by X_t rather than by consumption.[12] The hedging portfolio, p, is always negative. Its absolute value is largest when the signal is at the trigger point $\underline{s}$ and goes to 0 when the signal goes to infinity. This means that the country is always shorting the asset that is perfectly correlated with the signal, and the amount of shorting rises as the signal worsens. The counterpart of this investment position is a reduction in the countries' external debt as the signal deteriorates.

However, this form of insurance will not entirely remove the consumption drop during the sudden stop. All the country can do is to arrive at the sudden stop with less debt, and hence to reduce the size of the trade surpluses it is required to have during the crisis. Because of this drop, the country still cuts consumption and borrowing throughout the pre-sudden stop phase for precautionary reasons. It is simply no longer state (signal)-dependent.

Optimal consumption before the sudden stop is constant at (see the appendix):

$$c^* = \frac{X_0 + \frac{y_0^*}{r-g}}{E\left[\int_0^\tau e^{-rt}\,dt + e^{-r\tau}\frac{K^{1/\gamma}}{r}\right]}.$$

Since $K > 1$, it clearly is lower than the level of consumption in a framework without sudden stops:

$$c^* < c^{*NSS} = r\left(X_0 + \frac{y_0^*}{r-g}\right).$$

4.2 Hedging Sudden Stops

Let us now assume that there is a second asset, H, with the property that it can be excluded from the sudden stop constraint and hence can be used directly to overcome the forced savings problem. This can be thought of as a credit line that does not crowd out alternative funding options (we refer to this case as $h-$hedging).

Formally, the problem becomes:

$$V(X_t, s_t, y_t^*) = \max_{c_u^*, p_u, \tilde{p}_u, dN_u} E\left[\int_t^\tau \frac{c_u^{*\,1-\gamma}}{1-\gamma} e^{-r(u-t)}\,du + e^{-r(\tau-t)} V^{SS}(X_\tau, H_t, y_\tau^*) | F_t\right] \quad (43)$$

[12] The appendix provides a formal proof of this statement, which follows steps similar to those in Karatzas and Wang (2000), and return to a fuller characterization in the implementation section.

where

$$\tau = \inf\{(t : s \leq \underline{s}) \wedge \infty\} \tag{44}$$

and the evolution of (X_t, H_t, s_t, y_t) is now given by:

$$\begin{aligned}
dX_t &= (rX_t - c_t^* + y_t^*)\,dt + p_t\sigma dB_t - \xi dN_t && (45)\\
dH_t &= rH_t + \widetilde{p_t}\sigma dB_t + dN_t && (46)\\
\frac{dy_t^*}{y_t^*} &= g\,dt && (47)\\
ds_t &= \mu dt + \sigma dB_t. && (48)\\
H_t &\geq 0, \quad \xi \geq 1 && (49)
\end{aligned}$$

Observe that we now have a second portfolio process $\widetilde{p_t}$ and an increasing process dN_t representing additions into the second type of asset. Moreover, every addition into H_t from X_t is accompanied by a markup fee, $\xi \geq 1$. Since the income flows arrive in the form of X, the presence of a markup would seem to imply that H_t is dominated by X_t. However, this is not the case precisely because H_t has the ability to relax the constraint: $X_{\tau+T} \geq \overline{X_\tau}$. In other words we assume that at time τ the holdings of the asset H_τ will be added to X_{τ^-} in order to relax the constraints as we explain below.

To be more precise, let us re-write the sudden stop constraint as:

$$X_{\tau+T} \geq X_{\tau^-}\left(\phi(1-e^{-(r-g)T}) + e^{gT}\right) + \phi\frac{y_\tau^*}{r-g}(1-e^{-(r-g)T}) \stackrel{d}{=} \overline{X_\tau}^-, \quad 0 \leq \phi < e^{rT}. \tag{50}$$

Notice that the constraint now reads X_{τ^-} rather than X_τ. This is because the country can use its holdings of H_τ to relax the constraint directly, so that we can write:

$$X_{\tau^+} = X_{\tau^-} + H_\tau. \tag{51}$$

It follows that the constraint will be non-binding if

$$H_\tau \geq h^*\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right) \tag{52}$$

where

$$h^* \equiv \left(\phi(1-e^{-(r-g)T}) + e^{gT} - 1\right). \tag{53}$$

To see why this is so, note that for $H_\tau = h^*\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)$ at time τ^+ total resources become:

$$\begin{aligned}
X_{\tau^+} &= X_{\tau^-} + H_\tau\\
&= X_{\tau^-}\left(\phi(1-e^{-(r-g)T}) + e^{gT}\right) + \phi\frac{y_\tau^*}{r-g}(1-e^{-(r-g)T}) + \left(e^{gT}-1\right)\frac{y_\tau^*}{r-g}\\
&= \overline{X_\tau}^- + \frac{y_{\tau+T}^*}{r-g} - \frac{y_\tau^*}{r-g}.
\end{aligned}$$

However, as we established in equation (2), the unconstrained solution automatically satisfies the relation:

$$X_{\tau^+} + \frac{y_\tau^*}{r-g} = X_{\tau+T} + \frac{y_{\tau+T}^*}{r-g},$$

which implies that $X_{\tau+T} \geq \overline{X_\tau}^-$, i.e. the constraint is satisfied automatically for $H_\tau \geq h^* \left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)$.

A modification of the martingale approach of Karatzas and Wang (2000) is particularly well suited to finding and characterizing the solution $< c_t^*, p_t, \widetilde{p}_t, X_t, H_t, dN_t >$ in this context. Applying Ito's lemma to $e^{-rt}X_t$ and $e^{-rt}H_t$, and because we have a bounded portfolio and a stopping time that is finite almost surely, for any feasible consumption-portfolio plan we get:

$$\begin{aligned} E(e^{-r\tau}X_\tau) &= X_0 + E\left(\int_0^\tau (y_t^* - c_t^*)\, e^{-rt}\, dt - \xi \int_0^\tau e^{-rt} dN_t\right) \\ E(e^{-r\tau}H_\tau) &= H_0 + E\left(\int_0^\tau e^{-rt} dN_t\right). \end{aligned}$$

Combining the above two equations we get:

$$E(e^{-r\tau}(X_\tau + \xi H_\tau)) = X_0 + \xi H_0 + E\left(\int_0^\tau (y_t^* - c_t^*)\, e^{-rt}\, dt\right)$$

We will assume that at time 0 the country starts with $H_0 = 0$, so that we end up with:

$$E(e^{-r\tau}(X_\tau + \xi H_\tau)) = X_0 + E\left(\int_0^\tau (y_t^* - c_t^*)\, e^{-rt}\, dt\right). \tag{54}$$

An interesting interpretation of this equation is as follows: Suppose the country is offered a contingent credit line that it can structure as it desires. I.e., it can choose H_τ "state by state." But assume too that it pays a "markup," ξ, on this credit line. Then the price of the credit line is:

$$P_{ccl} = \xi E(e^{-r\tau}H_\tau).$$

With this definition we can rewrite (54) as:

$$E(e^{-r\tau}X_\tau) = (X_0 - P_{ccl}) + E\left(\int_0^\tau (y_t^* - c_t^*)\, e^{-rt}\, dt\right)$$

which is the standard budget constraint (e.g. of Section 4.1.) after subtracting the initial cost of the contingent credit line from X_0.

From now on, it will prove useful to express the amount of the credit line as a fraction of $\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)$ so that:

$$h_\tau = \frac{H_\tau}{\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)}.$$

For values of $h < h^*$ the constraint will be binding and consumption will drop during the sudden stop. It is not difficult to show that in this case consumption after the sudden stop ($t > \tau + T$) is equal to:

$$c_{\tau+T}^* = r\,(1 + h^*)\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right).$$

which follows from (4) and (8).

Given X_{τ^-}, this is the same quantity as in the no-hedging case because the sudden stop constraint binds as long as $h < h^*$. However, the effect of the credit line is to raise consumption during the sudden stop phase ($\tau < t < \tau + T$) since H_τ is entirely used during this period . Consumption during the crisis is now equal to:

$$c^*_{\tau+} = \frac{r}{1-e^{-rT}}\left(X_{\tau^-} + \frac{y^*_\tau}{r-g}\right)(1 + h - e^{-rT}\phi(1-e^{-(r-g)T}) - e^{-(r-g)T})$$

which for $h = h^*$ rises to

$$r(1+h^*)\left(X_{\tau^-} + \frac{y^*_\tau}{r-g}\right) = c^*_{\tau+T}.$$

Accordingly, the value function at time τ is:

$$V^{SS} = C\frac{\left(X_{\tau^-} + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} \tag{55}$$

$$C = \left(\frac{1}{r}\right)^\gamma \left[(1+h^*)^{1-\gamma}1_{\{h=h^*\}} + K_1 1_{\{h<h^*\}}\right]$$

where:[13]

$$K_1 = (1-e^{-rT})^\gamma(1+h-e^{-rT}(1+h^*))^{1-\gamma} + e^{-rT}(1+h^*)^{1-\gamma} < K.$$

Now the optimization problem has the same form as before except that the intertemporal constraint takes into account the fact that the consumer essentially faces two types of assets, X and H.

Adopting the Cox-Huang (1989) methodology and its application to problems involving a random stopping time in Karatzas and Wang (2000), we are able to reduce the problem to the following static problem:

$$\min_k \max_{c_t, X_\tau, H_\tau} E\left[\int_0^\tau \frac{c_t^{*1-\gamma}}{1-\gamma}e^{-rt}\,dt + e^{-r\tau}V^{SS}(X_\tau, H_\tau) - k\left(\int_0^\tau c_t e^{-rt}\,dt + e^{-r\tau}(X_\tau + \xi H_\tau)\right)\right]$$

$$s.t.\ E\left(\int_0^\tau c^*_t e^{-rt}\,dt + e^{-r\tau}(X_\tau + \xi H_\tau)\right) = X_0 + E\left(\int_0^\tau y^*_t e^{-rt}\,dt\right)$$

where V^{SS} is given by (55).

The presence of dynamically complete markets allows us to reduce the problem to an essentially static problem. Parties can contract on the payoffs to be transferred "state by state." In other words, the objective inside the brackets is maximized state by state and the optimal payoff is replicated dynamically.

It is easy to show that in this framework one can derive optimal consumption to be:

$$c^*_t = k^{-\frac{1}{\gamma}}, t \in (0, \tau).$$

[13] It might seem puzzling why $C = \left(\frac{1}{r}\right)^\gamma (1+h^*)^{1-\gamma}$ when $h = h^*$ and not $C = \left(\frac{1}{r}\right)^\gamma$. The reason is that total resources at time τ^+ are now given by $(1+h^*)\left(X_{\tau^-} + \frac{y^*_\tau}{r-g}\right)$. Actually one can easily show in the framework of the threshold model that in the absence of a markup, it will be the case that $(1+h^*)\left(X_{\tau^-} + \frac{y^*_\tau}{r-g}\right) = X^{unc}_\tau + \frac{y^*_\tau}{r-g}$, where X^{unc}_τ denotes the level of X_τ in the complete absence of sudden stops.

where k is a constant that is determined in such a way that the intertemporal budget constraint is satisfied.

The crucial step is the maximization of the problem involving the continuation value function at time τ. In the appendix we show that the solution to this problem is:

$$\begin{aligned} X_{\tau-} &= \frac{k^{-\frac{1}{\gamma}}}{r(1+h^*)}\Gamma^{-\frac{1}{\gamma}} - \frac{y_\tau^*}{r-g} \\ h^{opt} &= h^* - (1-e^{-rT})(1+h^*)\left[1-\left(\frac{\Gamma}{\xi}\right)^{\frac{1}{\gamma}}\right] \end{aligned}$$

with

$$0 < \Gamma \equiv \xi - \frac{(\xi - 1)}{e^{-rT}(1+h^*)} < 1.$$

Notice that as $\xi \rightarrow 1$, $h \rightarrow h^*$. As might be expected in this case, where it costs nothing to avoid the constraint, the country will optimally choose $h = h^*$. If $\xi > 1$ it will be true that $h < h^*$ and some consumption adjustment during the sudden stop will be required.

Because of the homotheticity of the problem, the optimal credit line ratio does not depend on the level of initial wealth. To complete the solution to the overall problem, let us return to the time 0 budget constraint and combine everything to solve for c^*:

$$\begin{aligned} c^* &= \left[E\left[\int_0^\tau e^{-rt}\,dt\right] + \frac{E\left[e^{-r\tau}\right]}{r}\frac{(1+h^{opt}\xi)\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)}\right]^{-1}\left(X_0 + \frac{y_0^*}{r-g}\right) = \\ &= \left[\frac{1-e^{-\lambda(s-\underline{s})}}{r} + \frac{e^{-\lambda(s-\underline{s})}}{r}\frac{(1+h^{opt}\xi)\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)}\right]^{-1}\left(X_0 + \frac{y_0^*}{r-g}\right) \end{aligned}$$

where as usual:

$$\lambda = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} \tag{56}$$

The credit line reduces the magnitude of the sudden stop, which translates into higher pre-sudden stop consumption. However, since $h^{opt} < h^*$, there is still scope for hedging; p_t takes the slack and eliminates the remaining need for a precautionary recession. The economy still suffers through the sudden stop, but significantly less than without hedging.

4.3 Imperfect Signal

In general, crises will be correlated with a signal but not perfectly. This complicates the p-insurance case but the nature of the solution is similar to that of the threshold model. On the other hand, the fact that τ can occur for any s, considerably complicates the h-hedging case. In this section we develop the former case to contrast it with the perfect-signal scenario. We return to the $h-$hedging case in the empirical section.

The steps of the derivation are similar to those in the threshold model, and we relegate them to the appendix. The optimal (excess) consumption and portfolio policies become:

$$c_t^* = \frac{\left(X_t + \frac{y_t^*}{r-g}\right)}{b^H(s_t)} \tag{57}$$

$$p_t = \frac{b_s^H(s_t)\left(X_t + \frac{y_t^*}{r-g}\right)}{b^H(s_t)} \tag{58}$$

Unfortunately the function $b^H(s)$ can no longer be characterized in closed form, but it can be computed numerically.

Applying Ito's lemma to the right hand side of (57), we obtain the process:

$$\frac{dc_t^*}{c_t^*} = \left(\frac{\lambda(s_t)}{\gamma}\left[\left(\frac{b}{b^H(s_t)}\right)^\gamma - 1\right]\right) dt. \tag{59}$$

Comparing this expression to that of the unhedged case in (34), shows that the possibility of hedging eliminates the diffusion term. However, unlike the hedged case in the threshold model, there is still pre-sudden-stop precautionary savings and these fluctuate over time as the signal changes. This happens because the sudden-stop jump is not directly contractible.

5 An Illustration: The Case of Chile

In this section we illustrate our main results through an application to the case of Chile. This is a good case study since Chile is an open economy, with most of its recent business cycle attributable to capital flows' volatility. Moreover, the price of copper (its main export) is an excellent indicator of investor attitude toward Chile.

5.1 Calibration

While our purpose here is only illustrative, and hence our search for parameters is rather informal, we spend some time describing our estimation/calibration of the key function $\lambda(s)$.

5.1.1 Estimating $\lambda(s)$

Sudden stops are very severe but rare events. This makes it hard to estimate $\lambda(s)$ with any precision. However, we still highlight our procedure because it provides a good starting point for an actual implementation. Similarly, a key issue is determining the components of the signal, s. Given the limited goal of this section, we use only the logarithm of the price of copper. In a true implementation it also would be worth including some global risk-financial indicator, such as the U.S. high yield spread or the EMBI+, and removing slow moving trends from the price of copper.[14]

With these caveats behind us, let us describe the reduced-form Markov regime switching model we use to estimate $\lambda(s)$. Unlike the standard regime-switching model, ours has time-varying transition probabilities. Our left-hand side variable is aggregate demand, Y_t, which we assume to be generated by the process:[15]

$$Y_t = \mu_0 + 1_{\{z_t=1\}}\mu_1 + \sigma_{z_t}\varepsilon, \quad \varepsilon \sim N(0,1), \mu_1 < 0.$$

The growth rate of Y_t depends on an unobserved regime z_t that takes values 0 and 1. When the value of the regime is 0, the regime is "normal" and growth is just a normal variable with mean μ_0 and variance σ_0^2. Otherwise, the regime is "abnormal" and growth has a lower mean, $\mu_0 + \mu_1$, and a variance σ_1^2. The transition matrix between the two states is assumed to be the following:

$$\begin{array}{ll} \Pr(z_{t+1} = 0 | z_t = 0, X_{s,s\in(t,t+1)}) = \exp^{-\int_t^{t+1}\lambda(x_u)du} = p_{00} & \Pr(z_{t+1} = 1 | z_t = 0, X_{s,s\in(t,t+1)}) = p_{01} \\ \Pr(z_{t+1} = 0 | z_t = 1, X_{s,s\in(t,t+1)}) = p_{10} & \Pr(z_{t+1} = 1 | z_t = 1, X_{s,s\in(t,t+1)}) = q \end{array}$$

where $p_{01} = 1 - p_{00}$, $p_{10} = 1 - q$, and X_t stands for the regressors that enter the hazard function (the logarithm of the price of copper in this application).[16] For the purpose of estimating $\lambda(s)$, we use annual GDP data starting from 1976 to 1999 and monthly copper data for the same period (source, International Financial Statistics).[17]

For estimation, we use a Bayesian approach that is suitable for the very few datapoints that we have. Bayesian inference seems natural in this context, since it allows exact statements that do not require asymptotic justification. To estimate the parameters of interest (namely (α_0, α_1)) we use a multimove Gibbs Sampler as described in Kim and Nelson (1998,1999) and is based on the filtering algorithm of Hamilton (1989). For details we refer the reader to these references. The basic idea is to augment the parameter

[14] Removing the slow moving trend not only seems to improve the fit but also is a key ingredient in designing long run insurance and hedging contracts. Few investors/insurers are likely to be willing to hold long-term risk on a variable that may turn out to be non-stationary.

[15] All aggregate quantity data are highly correlated during large crises; hence the particular series used is immaterial for our purposes.

[16] Strictly speaking the quantity $\exp^{-\int_t^{t+1}\lambda(x_u)du}$ is unobservable, since we cannot observe the continuous path of copper, only discrete points. However, we can obtain copper data at reasonable high frequencies so that we can safely ignore this issue and calculate the integral by a Rieman sum as

$$\exp^{-\int_t^{t+1}\lambda(x_u)du} \approx \exp^{-\sum\lambda(x_u)\Delta u}$$

[17] There is a caveat here. Our extended-sample starts from 1972, but all the years up to 1976 are part of a deep contraction due to political turmoil combined with extremely weak external conditions. However, since the extended-sample starts during abnormal years these do not influence the estimation of $\lambda(s)$ (which is estimated from the transitions from normal to abnormal states). It is in this sense that the sample relevant for the estimation of the latter starts in 1976. For details see below.

set by treating the unobserved states as parameters. Then we fix a draw from the posterior distribution of $(\mu_0, \mu_1, \sigma_0, \sigma_1, q, \alpha_0, \alpha_1)$ and conditional on these parameters we draw from the posterior distribution of the states in a sigle step as described in Kim and Nelson (1999). Given the draw from the posterior distribution of the states we sample the conditional means μ_0, μ_1 using a conjugate normal / trucated normal prior which leads to a normal / trucated normal posterior.[18] Conditional on the states and the draw from (μ_0, μ_1) se sample from the posterior distribution of (σ_0, σ_1) using inverse gamma priors which lead to posteriors in the same class. Similarly, fixing the states and the draw from $(\mu_0, \mu_1, \sigma_0, \sigma_1)$ we draw q using a conjugate beta prior leading to conjugate beta posterior. The sampling of (α_0, α_1) presents the difficulty that there does not seem to be a natural conjugate prior to use and thus we use a Metropolis Hastings-Random walk-accept-reject step where we sample from a bivariate normal centered at the previous iterations' estimate as described in Robert and Casella (1999). This provides us with a new draw form $(\mu_0, \mu_1, \sigma_0, \sigma_1, q, \alpha_0, \alpha_1)$ and conditional on this new draw we can iterate the algorithm by filtering the states again, based on the new draw etc. It is then a standard result in Bayesian computation that the stationary distribution of the parameters sampled with these procedure (treating the unobserved states as parameters too) coincides with the posterior distribution of the parameters.[19]

A first output of this procedure is Figure 2 which plots the probability of being in an abnormal state.The model recognizes roughly three years out of the 24 as abnormal years: the early 1980s and the recent episode following the Asian/Russian crises. The early 1980s episode corresponds to a devastating debt-crisis, and was significantly more severe than the recent one. In fact, the recent episode appears to be a mix of a milder sudden stop and a precautionary recession.

An interesting observation about these probabilities is that they allow us to identify the abnormal regimes with great confidence. To improve the tightness of the posterior confidence intervals concerning the parameters of interest (α_0, α_1) we observe that conditional on the states the (log-) likelihood function becomes:

$$\sum_{z_i=0} [\chi_i \log(p_{00}(\alpha_0, \alpha_1)) + (1 - \chi_i) \log(p_{01}(\alpha_0, \alpha_1))]$$

where χ_i takes the value 1 if there is no transition to state 1, and takes the value 0 otherwise. It is interesting to notice that the likelihood for (α_0, α_1) depends on the data only through the filtered states. In other words

[18] For details see Albert and Chib (1993).

[19] To reduce computational time and satisfy the technical conditions required for the applicabibity of the Gibbs Sampler, we used proper priors for (μ_0, μ_1) and (α_0, α_1) and improper priors for the rest of the parameters. The proper prior for (μ_0, μ_1) was Normal / Truncated normal with means (0,0) and a diagonal covariance matrix. The standard deviations where chosen to be roughly 5 times the range of the sample, so that the priors had effectively no influence on the estimation. Similarly for (α_0, α_1) we used independent normal priors centered at 0 with a standard deviation of 10. Even when we experimented with more diffuse priors the algorithm produced virtually identical results but computational time was significantly increased, because convergence was significantly slower. Most importantly though, for the results that we report in Table 1 and that we use in the calibration exercise we used completely flat priors.

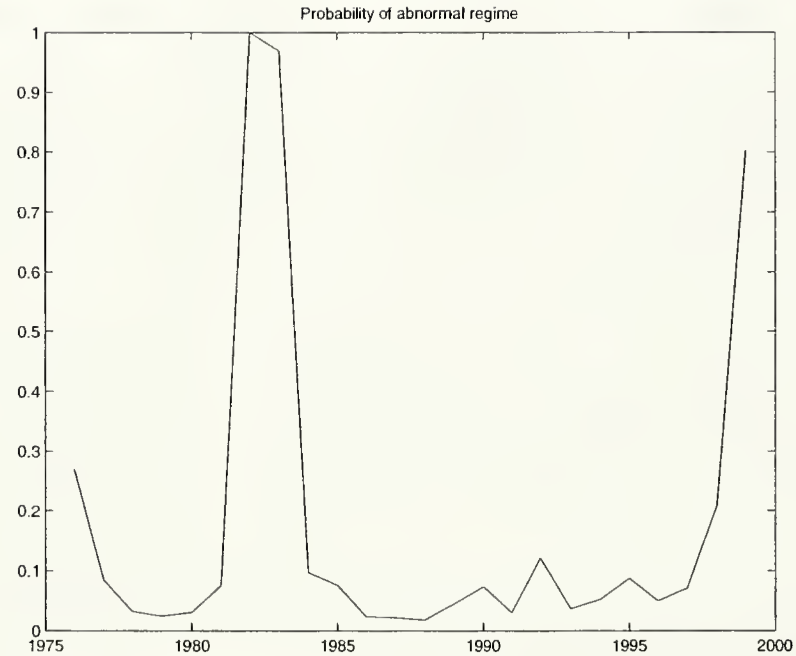

Figure 2: Probability of abnormal regime

all other parameters of the model including the GDP-data are relevant for our purpose of estimating (α_0, α_1) only to the extent that they influence the identification of the states. In other words if we were to condition directly on the states we would be able to get rid of the noise introduced by filtering.

Given the few data points and the quite clear identification of the abnormal states we report directly the posterior distribution of (α_0, α_1) conditional on the early 80's and 1999 being the abnormal states. we report the posterior distributions of the parameters (α_0, α_1) conditional on the states that are identified as abnormal.[20] Table 1 reports these results.

Correspondingly, in what follows we use $\alpha_0 = 2.5, \alpha_1 = 5.2$ as our benchmark values for the function $\lambda(s)$.

5.1.2 Other parameters

We calibrate ϕ and T to generate the average cumulative consumption drop caused by the sudden stops in the sample, which is about 8-10% of GDP. One such configuration is $T = 1$ and $\phi = 0.35e^{rT}$.

The parameters κ and γ are calibrated to generate reasonable amounts of steady-state debt (and hence

[20] I.e., states that have a posterior probability of being abnormal above 0.75. The conditioning is done in order to increase the precision of the estimates and seems to be warranted just by a visual inspection of Figure 2.

Posterior Distributions							
	mean	std. Deviation	10	25	50	75	90
α_0	-2.522	1.067	-3.826	-3.027	-2.316	-1.789	-1.420
α_1	-5.170	3.387	-9.259	-6.802	-4.708	-3.040	-1.515

Table 1: Posterior Distributions of the Parameters (α_0, α_1) Conditional on the States that are Identified as Abnormal. 10, 25 etc. refer to the respective quantiles.

reasonable amounts of insurance needs) together with significant precautionary fluctuations (see below, in particular, to explain much of the precautionary recession experienced by Chile at the early stages of the 1998-9 episode). For this, we set $\kappa = 0.8$ and $\gamma = 7$.

The interest rate r is set to 0.09 and the growth rate to 0.03. The latter is a constant-rate approximation to a path that grows significantly faster than that level for a few years, while the country is catching up, and then decelerates below that level forever. We normalize initial GDP (y_0) to 1, and set the initial debt-to-GDP ratio, X_0, to 0.5.

Finally, we approximate the process for copper by a driftless Brownian motion with constant volatility σ, which we estimate using monthly data from 1972 to 1999 (source, IFS). We find a value of roughly $\sigma = 0.23$ and normalize the initial value of s to 0.

5.2 Aggregate Hedging

Recall that our concern in this economy is to stabilize both precautionary recessions and sudden stops. Let us start with the former.

5.2.1 Hedging Precautionary Recessions

Evaluating at $\frac{c_t}{y_t} \approx 1$, and $s = 0$ in the calculation of $\frac{b_s}{b}$, we can approximate the volatility of log consumption in our model before the sudden stop takes place by:

$$-(1-\kappa)\frac{b_s}{b}\sigma \approx 0.01.$$

That is, despite the absence of income fluctuations in our model, fluctuations in precautionary behavior can account for about a fourth of consumption volatility in the regimes that are characterized as normal by our algorithm. More importantly, the contribution of precautionary fluctuations is particularly relevant near sudden stops, as shown by Figure 3.

Panel (a) in this figure shows a random realization of the path s that runs for nearly eight years before a sudden stop takes place. The main features of this path are not too different from the realization of the

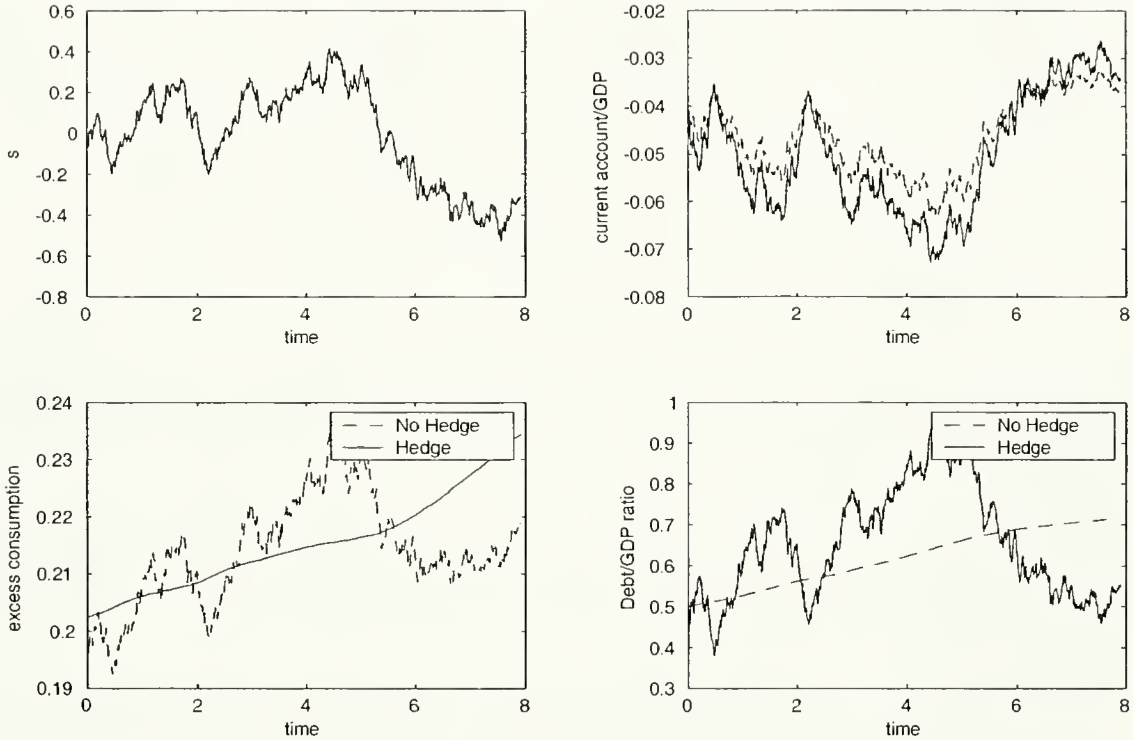

Figure 3: p-Hedging for the Chilean case

price of copper during the 1990s. In particular, the large rise in s near the middle of the path followed by a sharp decline toward the end of the period is reminiscent of the path of the price of copper from 1996 to 2000.

The dashed line in the bottom left panel (b) illustrates the corresponding path of (excess) consumption generated by our model without hedging. The drift in the process is due to average precautionary savings. More interestingly, one can clearly see the precautionary recessions caused by the decline in s. The dashed line in the top right panel (c) shows the impact of the latter on the current account. In particular, the sharp decline in the deficit in the current account from 6 to about 3.5 percent of GDP toward the end of the sample. This suggests that about half of the current account adjustment observed in Chile during the 1998-9 crisis can be accounted by *optimal* precautionary behavior in the absence of hedging. The rest of the adjustment could be accounted for by excess adjustment (some have argued that the central bank contracted monetary policy excessively during this episode) and by the partial sudden stop itself (the "sovereign" spread tripled during this episode).

Now, let the country hedge by shorting copper futures, which do not relax the sudden stop directly but only through their effect on X_τ (p-hedging). The solid lines in panels (b) to (d) show the paths corresponding to the dashed paths of the unhedged economy. Panel (b) clearly shows that hedging, even of this very limited kind, virtually eliminates precautionary fluctuations. Interestingly, panel (c) shows that the insulation of consumption from precautionary cycles is not done through a smoothing of the current account, which looks virtually identical in the hedged and unhedged economies. The difference comes from the fact that the improvement in the current account as s deteriorates comes from hedge-transfers in favor of the country. The latter are reflected in a sharp decline in external debt as s worsens (panel (d)).

As a practical matter, it is important to point out that the hedge-ratios required to achieve this success are very large. The size of the implied portfolios can be calculated as (evaluating at $s = 0$):

$$\frac{b_s}{b}(X_0 + (1-\kappa)\frac{y_t^*}{r-g}) \approx -0.56.$$

That is, the notional amount in the futures position would have to exceed 50 percent of GDP. This is very large when compared with the existing futures markets for copper, pointing to the need to develop these or related markets for sudden-stop-insurance purposes. However, it is not a disproportionate number when compared with the very large and much costlier practice of accumulating international reserves (around 30 percent of GDP).

Finally, and as indicated in section 4, this strategy — if subject to full crowding out— is not very effective in smoothing the sudden stop itself. Our estimates show that in case of a sudden stop, consumption with hedging of this sort exceeds consumption in the unhedged case only by about 2% for consumption drops of roughly 8-10%.

5.2.2 Hedging Sudden Stops

Let us now analyze the opposite extreme and assume that hedging directly relaxes the sudden stop constraint (h-hedging). Full insurance in this case requires a transfer in the case of a sudden stop of:

$$h^* \left(X_0 + \frac{(1-\kappa)y_0}{r-g}\right) = \left(\phi(1-e^{-(r-g)T}) + e^{gT} - 1\right)\left(X_0 + \frac{(1-\kappa)y_0}{r-g}\right) \approx 0.15.$$

Importantly, note that 15 percent of GDP exceeds the 8-10 percent we calibrated the decline in consumption to be in the case of a sudden stop. This is because 15 percent covers not only the drop in consumption in case of a sudden stop but also the precautionary savings and recession that precedes the sudden stop.

In order to calculate the NPV of such a claim, we must multiply the number above by $\mathrm{E}[e^{-r\tau}]$. Since the latter depends on the initial value of the signal s and on the estimates of α's, which are very imprecise, we report in Table 2 the value of $\mathrm{E}[e^{-r\tau}]$ for different combinations of these parameters.

Using the benchmark case, we see that "fairly" priced full insurance would cost the country about 8 percent of GDP (0.15 times 0.536) if contracted when the price of copper is at "normal" levels and about 5

	(2.5,5.2)	(5,5.2)	(2.3,0)
$s = 0$	0.536	0.247	0.517
$s = 1.5\sigma$	0.323	0.139	0.523

Table 2: Values of $\mathrm{E}[e^{-r\tau}]$ for Different Combinations of (α_0, α_1).

percent (0.15 times 0.323) when at very high levels. While these amounts are small when compared with the cost of sudden stops and precautionary recessions, they still involve amounts which are probably too large for these countries to undertake, even if such markets existed.

Fortunately, much of the advantage of insurance can be obtained with significantly smaller amounts of insurance. In what follows, we take h as given and solve:

$$\max_{c_s^*, X_\tau} \mathrm{E}\left[\int_0^\tau e^{-rs}\frac{c_s^{*1-\gamma}}{1-\gamma}ds + e^{-r\tau}V^{SS}(X_{\tau^-})\right]$$

$$s.t.$$

$$dX_t = (rX_t - c_t^* + y_t^*)\,dt$$

$$X_{0^+} = X_{0^-} - P_0$$

$$P_0 = E\left[e^{-r\tau}h(X_{\tau^-} + \frac{y_\tau^*}{r-g})\right].$$

In other words we assume that the country starts with an initial debt X_{0^-}, and purchases insurance costing P_0 which is priced fairly and denominated as a fraction $h < h^*$ of the quantity $X_{\tau^-} + \frac{y_\tau^*}{r-g}$. The setup is common knowledge to both the borrower and the lender, so the lender prices this claim understanding the subsequent optimization problem of the borrower and accordingly the resulting path of $X_{\tau^-} + \frac{y_\tau^*}{r-g}$. Using this particular type of contingent claim results in a problem that we can easily solve numerically with the tools we developed in the previous sections, plus a fixed point problem for the initial price of the insurance, P_0.[21] The results are summarized in Table 3.

The first row shows P_0, the second c_0^*, while the third row shows the maximum value attained by excess consumption along the path before the sudden stop. The next three rows show consumption right before the sudden stop takes place, during the sudden stop, and after the crisis, respectively. The final row describes the standard deviation of $c_{\tau^+}^*$ for a sample of 1000 simulations. The first three columns present values for $s_0 = 0$, while the last three columns do the same for $s_0 = 1.5\sigma$.

There are three main lessons to be learned from this table. First, clearly much of the cost of a lack of insurance is paid for with the large precautionary behavior that the economy needs to undertake without

[21] Even though the contingent claim signed by the borrower and the lender seems complicated, it isn't since $X_{\tau^-} + \frac{y_\tau^*}{r-g}$ is not varying very much and can be approximated reasonably well by $X_0 + \frac{y_0^*}{r-g}$, especially for large h.

s_0 :	0			1.5σ		
h/h^*:	0	0.4	0.8	0	0.4	0.8
Initial Cost	0.000	0.032	0.063	0.000	0.019	0.039
c_0^*	0.198	0.243	0.252	0.213	0.247	0.252
c_{max}^*	0.226	0.248	0.253	0.243	0.252	0.254
$c_{\tau-}^*$	0.212	0.244	0.252	0.224	0.247	0.253
Standard Deviation of $c_{max}^* - c_{\tau-}^*$	0.010	0.002	0.002	0.011	0.002	0.001
$c_{\tau+}^*$	0.125	0.175	0.229	0.131	0.177	0.230
$c_{\tau+T}^*$	0.300	0.269	0.259	0.313	0.271	0.260
Standard Deviation of $c_{\tau+}^*$	0.013	0.002	0.001	0.014	0.002	0.001

Table 3: $h-$Hedging

insurance. The level dimension of this effect can be seen in the value of c_0^*, while the cyclical component can be seen in the difference between $c^*{}_{max} - c_{\tau^-}^*$. Second, much of the precautionary costs can be removed with very limited amounts of insurance; a value of $h = 0.4$ does nearly the same as one of 0.8. Third, the sudden stop itself is significantly harder to smooth, but it is still possible to make a difference with h significantly less than h^*.

5.2.3 Asymmetric Beliefs and Contingent Credit Lines

Before concluding, we discuss an important practical issue. Sudden stops are not entirely exogenous to the country's actions and there is significant asymmetric information about these actions between borrowers (the country) and lenders. Suppose then that the financial markets overstate (from the viewpoint of the country) the constant part of the hazard, α_0. In particular, we assume that the lender takes this constant to be α_0^L. The borrower on the other hand takes this number to be $\alpha_0^B << \alpha_0^L$. Let the expectation operators of the borrower and the lender be E^{L} and E^{B}, respectively, so that the problem becomes:

$$\max_{c_s^*, X_\tau} E^B \left[\int_0^\tau e^{-rs} \frac{c_s^{*1-\gamma}}{1-\gamma} ds + e^{-r\tau} V^{SS}(X_{\tau^-}) \right]$$

$$s.t.$$

$$dX_t = (rX_t - c_t^* + y_t^*)\, dt$$

$$X_{0^+} = X_{0^-} - P_0$$

$$P_0 = E^L \left[e^{-r\tau} h(X_{\tau^-}^{opt} + \frac{y_\tau^*}{r-g}) \right].$$

In this case the country obviously will find the price of the insurance "unfairly" high. One way to reduce

		uncontingent		w=3	
h/h^*:	0	0.4	0.8	0.4	0.8
Initial Cost	0.000	0.032	0.063	0.009	0.020
c_0^*	0.219	0.247	0.250	0.238	0.246
$c_{\tau-}^*$	0.219	0.246	0.251	0.245	0.255
$c_{\tau+}^*$	0.128	0.175	0.228	0.155	0.187
Standard Deviation of $c_{\tau+}^*$	0.009	0.002	0.001	0.016	0.026

Table 4: $h-$Hedging with asymmetric Beliefs

the extent of this problem is for the country to add another contingency to the credit line. It is clear that in this case the country would benefit from making the line contingent on the value of s. Since the country sees crises not related to s as much less likely than the markets – for example, it may be certain that it will not run up fiscal deficits — it is optimal for it not to pay for insurance in states when s is high. Let us capture this feature by assuming that the borrower and the lender enter an agreement whereby the lender agrees to pay:

$$f(s_\tau)h\left(X_{\tau-}^{opt} + \frac{y_\tau^*}{r-g}\right)$$

to the borrower and the fraction of h paid out depends on s_τ:

$$f(s_\tau) = \frac{\exp(-w(s_\tau + 2\sigma))}{1 + \exp(-w(s_\tau + 2\sigma))}$$

Obviously $0 < f(s_\tau) < 1$. Taking $\alpha_0^L = 2.5$, $\alpha_0^B = 5$ and $w = 3$, i.e. a claim that pays quite steeply when and only when s_τ is very low (2 standard deviations below 0), the results are reported in the first half of Table 4, and contrasted with the case without the additional contingency.

By adding the additional state contingency, a country that is confident of its "good behavior" is able to lower the price of the claim without incurring a very significant rise in risk exposure.

6 Final Remarks

In this paper we characterized many aspects of sudden stops, precautionary recessions, and the corresponding aggregate hedging strategies. For this, we built a model simple enough to shed analytical light on some of the key issues but realistic enough to provide some quantitative guidance.

We showed that even after removing all other sources of uncertainty, the combination of infrequent sudden stops and the mere anticipation of them has the potential to explain a large share of the volatile business cycle experienced by emerging market economies. This important source of volatility could be overcome with suitable aggregate hedging strategies, but these would require developing large new financial markets.

In our application, we hinted at one aspect of the insurance arrangements that could facilitate the development of such markets by reducing the inherent asymmetric information problems. We argued for credit lines and financial instruments being contingent on indicators that are exogenous to the country. Thus, for example, Chile could remove many of the signaling problems it fears from the IMF's Contingent Credit Lines by adding a clause linking the size of the line to the price of copper. Mexico could do the same by indexing its line to the price of oil and US GDP, and so on.

There are several aspects of an aggregate risk management strategy that we left unexplored. In particular, we did not model the maturity structure of debt and how this changes with the signal, s. Similarly, we did not discuss accumulation of international reserves, as these are dominated by our credit lines. However, building reserves financed with long-term debt may partially substitute for overpriced credit lines. In order to address these issues properly, we also need to have a better understanding of the behavior of the supply side as the signal worsens. Of course, a country would want to postpone accumulating reserves and borrowing long term until the crisis is imminent, but it is highly unlikely that the lenders will go along with this strategy. We intend to explore these dimensions in future work.

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7 Appendix

7.1 Propositions and Proofs for section 2.3.

Proposition 1 *The optimal solution of the τ period problem is:*

$$\begin{aligned} c_t^{*opt} &= (k_1)^{-\frac{1}{\gamma}}, \quad \tau < t < \tau + T & (60) \\ X_{\tau+T} &= \max\{\overline{X}, \frac{(k_1)^{-\frac{1}{\gamma}}}{r} - \frac{y_\tau^* e^{gT}}{r-g}\} & (61) \end{aligned}$$

and k_1 is determined as the (unique) solution to:

$$X_\tau + \frac{y_\tau^*(1 - e^{-(r-g)T})}{r-g} = \frac{(k_1)^{-\frac{1}{\gamma}}(1 - e^{-rT})}{r} + e^{-rT} X_{\tau+T} \tag{62}$$

Moreover the maximum in (61) is always given by $\overline{X}$. That is, the constraint is always binding. Combining the above statements, we have that the optimal consumption-wealth pair is given by (12) and (11).

Proof. To establish the first assertion we need to show that the proposed policy satisfies (we normalize $\tau = 0$ and thus $y_\tau e^{gT} = y_T$ without loss of generality)

$$\int_0^T e^{-ru}\frac{(c_u^{*opt})^{1-\gamma}}{1-\gamma}du + e^{-rT}V(X_T^{opt} + \frac{y_T^*}{r-g}) \geq \int_0^T e^{-ru}\frac{c_u^{*1-\gamma}}{1-\gamma}du + e^{-rT}V(X_T + \frac{y_T^*}{r-g})$$

for any other admissible policy pair c_u^*, X_T^*, for $0 < u < T$. By feasibility we have

$$\begin{aligned} X_0 + \int_0^T e^{-ru} y_u^* du &\geq \int_0^T e^{-ru} c_u^* du + e^{-rT} X_T \\ X_T &\geq \overline{X} \end{aligned}$$

where we can focus without loss of generality on deviations that satisfy the first equation as an equality. By concavity of the utility function and the terminal Value function it follows that:

$$\begin{aligned} &\int_0^T e^{-ru}\frac{c_u^{*1-\gamma}}{1-\gamma}du + e^{-rT}V(X_T + \frac{y_T^*}{r-g}) \\ &-\left(\int_0^T e^{-ru}\frac{(c_u^{*opt})^{1-\gamma}}{1-\gamma}du + e^{-rT}V(X_T^{opt} + \frac{y_T^*}{r-g})\right) \end{aligned}$$

$$\begin{aligned} &= \int_0^T e^{-ru}\frac{c_u^{*1-\gamma} - (c_u^{*opt})^{1-\gamma}}{1-\gamma}du + e^{-rT}\left(V(X_T + \frac{y_T^*}{r-g}) - V(X_T^{opt} + \frac{y_T^*}{r-g})\right) \\ &\leq \int_0^T e^{-ru}\left(c_u^{*opt}\right)^{-\gamma}(c_u^* - c_u^{*opt})du + e^{-rT}V'(X_T^{opt} + \frac{y_T^*}{r-g})(X_T - X_T^{opt}) \end{aligned}$$

Now we need to distinguish between two cases: a) if $X_T^{opt} > \overline{X}$ then $V'^{opt} = k_1 = (c_u^{*opt})^{-\gamma}$ for a constant k_1 such that the budget constraint (62) is satisfied and the result follows upon substituting in the last expression:

$$k_1\left(\int_0^T e^{-ru}(c_u^* - c_u^{*opt})du + e^{-rT}(X_T - X_T^{opt})\right) = 0$$

since both policies satisfy:

$$X_0 + \int_0^T e^{-ru} y_u^* du = \int_0^T e^{-ru} c_u^* du + e^{-rT} X_T$$

b)When $X_T^{opt} = \overline{X}$, it is still the case that $(c_u^{*opt})^{-\gamma} = k_1$, and because the alternative candidate policy is feasible, it also satisfies $X_T \geq \overline{X} = X_T^{opt}$. Because of concavity of V this also implies $V'(\overline{X} + \frac{y_T^*}{r-g}) < k_1$. These two arguments can be combined to get:

$$\begin{aligned} \int_0^T e^{-ru} \left(c_u^{*opt}\right)^{-\gamma} (c_u^* - c_u^{*opt}) du + e^{-rT} V'(\overline{X} + \frac{y_T^*}{r-g})(X_T - X_T^{opt}) &= \\ \int_0^T e^{-ru} k_1 (c_u^* - c_u^{*opt}) du + e^{-rT} V'(\overline{X} + \frac{y_T^*}{r-g})(X_T - X_T^{opt}) &< \\ \int_0^T e^{-ru} k_1 (c_u^* - c_u^{opt}) du + e^{-rT} k_1 (X_T - X_T^{opt}) &= 0 \end{aligned}$$

This verifies the optimality of the proposed policy.

For the second assertion, that the constraint is always binding, the argument runs as follows. Suppose otherwise. Then k_1 is given as:

$$\frac{(k_1)^{-\frac{1}{\gamma}}}{r} = X_\tau + \frac{y_\tau^*}{r-g}$$

Under our (counterfactual) assumption we should have:

$$X_\tau \left(\phi(1 - e^{-(r-g)T}) + e^{gT}\right) + \phi \frac{y_\tau^*}{r-g}(1 - e^{-(r-g)T}) \leq X_\tau + \frac{y_\tau^*(1 - e^{gT})}{r-g}$$

or

$$X_\tau(1 - \phi(1 - e^{-(r-g)T}) - e^{gT}) \geq -\frac{y_\tau^*}{r-g}(1 - \phi(1 - e^{-(r-g)T}) - e^{gT})$$

or

$$X_\tau \leq -\frac{y_\tau^*}{r-g} \tag{63}$$

since

$$(1 - \phi(1 - e^{-(r-g)T}) - e^{gT}) < 0$$

But (63) contradicts non-negativity of consumption when combined with the transversality condition $\lim_{t\to\infty} e^{-rt} X_t = 0$. ■

7.2 Propositions and Proofs for section 3.1

In this section we proof all the steps of the perfect signal case. The proofs of the imperfect signal case follow the same steps but has a few steps which are much harder to proof, hence we only validate them though our numerical simulations.

The proof consists of the following steps. First we establish rigorously the boundary conditions that the value function should satisfy. Then we prove the existence of a unique solution to the ODE presented in the text subject to the required boundary conditions. Then we verify the "tail" condition on the value function that we use in order to apply a (classical) verification Theorem along the lines of Fleming and Soner (1993) p.172.

Proposition 2 *The value function of the problem with sudden stops for $s > \underline{s}$ satisfies:*

$$\lim_{s\to\infty} V(X_\tau, s, y^*_\tau) = \left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} \tag{64}$$

Proof. We focus on the $\gamma > 1$ case. The proof proceeds in two steps and does not depend on whether we allow hedging or not (i.e., whether we require that the portfolio $p = 0$ or whether p is determined as part of the optimization problem). First one obtains an upper bound on the value function of the problem with sudden stops, which in this case is naturally given by the value function of the problem in the absence of sudden stops:

$$V^{NSS} = \left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

To see why this is an upper bound notice that the following inequality holds for any feasible policy:

$$E\left[\int_t^\tau e^{-r(u-t)} \frac{(c^*_u)^{1-\gamma}}{1-\gamma} du + 1\{\tau < \infty\} e^{-r(\tau-t)} K \left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} | F_t\right]$$

$$\begin{aligned}
\leq \;& E\left[\int_t^\tau e^{-r(u-t)} \frac{(c^*_u)^{1-\gamma}}{1-\gamma} du + 1\{\tau < \infty\} e^{-r(\tau-t)} \left(\frac{1}{r}\right)^\gamma \frac{\left(X_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} | F_t\right] = \\
& E\left[\int_t^\tau e^{-r(u-t)} \frac{(c^*_u)^{1-\gamma}}{1-\gamma} du + 1\{\tau < \infty\} e^{-r(\tau-t)} V^{NSS}(X_\tau, y^*_\tau) | F_t\right]
\end{aligned}$$

since for $\gamma > 1$, $\frac{\left(X_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} < 0$. This holds true for any feasible consumption /portfolio policies and thus, when one evaluates this inequality at the optimal plan c^{*opt}_u, p^*_u:

$$\begin{aligned}
& V(X_t, s_t, y^*_t) = \\
& E\left[\int_t^\tau e^{-r(u-t)} \frac{(c^{*opt}_u)^{1-\gamma}}{1-\gamma} du + 1\{\tau < \infty\} e^{-r(\tau-t)} K \left(\frac{1}{r}\right)^\gamma \frac{\left(X^{opt}_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} | F_t\right] \leq \\
& E\left[\int_t^\tau e^{-r(u-t)} \frac{(c^{*opt}_u)^{1-\gamma}}{1-\gamma} du + 1\{\tau < \infty\} e^{-r(\tau-t)} \left(\frac{1}{r}\right)^\gamma \frac{\left(X^{opt}_\tau + \frac{y^*_\tau}{r-g}\right)^{1-\gamma}}{1-\gamma} | F_t\right]
\end{aligned}$$

$$\leq \max_{c_u^*, p_u} E\left[\int_t^{\tau} e^{-r(u-t)}\frac{(c_u^*)^{1-\gamma}}{1-\gamma}du + 1\{\tau<\infty\}e^{-r(\tau-t)}V(X_\tau, y_\tau^*)|F_t\right] = V^{NSS}(X_t, y_t^*) =$$

$$\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_t+\frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

The last equality follows from the principle of dynamic programming. Since this holds for all s_t it is also true as $s_t \to \infty$. This establishes the upper bound. The lower bound is established upon observing that the consumption /portfolio policy in the absence of sudden stops c^{NSS}, p^{NSS} (=0) is still a feasible consumption /portfolio plan in the presence of sudden stops (since it satisfies the intertemporal budget constraint and also $X_t > -\frac{y_t^*}{r-g}$ for all t). It is also within the class of policies that mandate $p_u = 0$. Accordingly we have the two inequalities:

$$V^{SS,p=0}(X_t, s_t, y_t^*) \geq E\left[\int_t^{\tau} e^{-r(u-t)}\frac{\left(c_u^{*NSS}\right)^{1-\gamma}}{1-\gamma}du + e^{-r(\tau-t)}1\{\tau<\infty\}K\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_\tau^{NSS}+\frac{y_\tau^*}{r-a}\right)^{1-\gamma}}{1-\gamma}|F_t\right]$$

$$V^{SS,\ p=p^{opt}}(X_t, s_t, y_t^*) \geq E\left[\int_t^{\tau} e^{-r(u-t)}\frac{\left(c_u^{*NSS}\right)^{1-\gamma}}{1-\gamma}du + e^{-r(\tau-t)}1\{\tau<\infty\}K\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_\tau^{NSS}+\frac{y_\tau^*}{r-a}\right)^{1-\gamma}}{1-\gamma}|F_t\right]$$

However by standard arguments it is easy to show that as $s \to \infty$, $\Pr(\tau < Q) \to 0$ for all finite Q, and thus

$$\lim_{s\to\infty} E\left[\int_t^{\tau} e^{-r(u-t)}\frac{\left(c_u^{*NSS}\right)^{1-\gamma}}{1-\gamma}du + e^{-r(\tau-t)}K1\{\tau<\infty\}\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_\tau^{NSS}+\frac{y_\tau^*}{r-a}\right)^{1-\gamma}}{1-\gamma}|F_t\right] =$$

$$= E\left[\int_t^{\infty} e^{-r(u-t)}\frac{\left(c_u^{*NSS}\right)^{1-\gamma}}{1-\gamma}du|F_t\right] = V^{NSS}(X_t, y_t^*) =$$

$$\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_t+\frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

Since the upper and the lower bound coincide to V^{NSS} as $s \to \infty$, the claim is established. ■

Before we can invoke a standard verification Theorem we establish the existence of a solution to the following ODE:

$$\gamma C^{\frac{\gamma-1}{\gamma}} - \gamma r C + \mu C_s + \frac{1}{2}C_{ss}\sigma^2 = 0 \tag{65}$$

Even though this second order non-linear ODE does not seem to have any closed form solution it is not hard to establish that it has a unique solution that satisfies the required boundary conditions. This is done in the following proposition:

Proposition 3 *The ODE (65) has a unique solution satisfying the boundary conditions (26), (27), with the change in variables $a^\gamma = C$.*

Proof. The steady state of this 2nd order nonlinear ODE can be obtained at once as:

$$C^{SS} = \left(\frac{1}{r}\right)^\gamma \tag{66}$$

The Theorem now basically follows from the stable manifold Theorem upon reformulating the system as a system of two first order ODE's, realizing that the system has one positive and one negative eigenvalue and applying the stable manifold Theorem (see e.g. Verhulst(2000) p.33.) The argument is particularly easy and thus we do not provide any details apart from the fact that the linearized system has one positive and one negative eigenvalue. Indeed, one can approximate the solution by means of a linear approximation around the steady state value to get:

$$\gamma \left(C^{SS}\right)^{\frac{\gamma-1}{\gamma}} + (\gamma - 1)\left(C^{SS}\right)^{-\frac{1}{\gamma}}\left(C - C^{SS}\right) - r\gamma C + \mu C_s + \frac{1}{2}C_{ss}\sigma^2 = 0 \tag{67}$$

The two eigenvalues of the characteristic polynomial of this equation are:

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 + 2\sigma^2 r}}{\sigma^2} \tag{68}$$

Obviously one eigenvalue is positive, one is negative and this establishes the claim after a few straightforward steps and the use of the stable manifold Theorem. ■

Remark 4 *A simple reformulation of the ODE (65), yields the function $a(s)$ that we use in the text and is obtained by defining:*

$$C = a^\gamma$$

and rewriting the ODE as:

$$\gamma a^{\gamma-1} + -\gamma r a^\gamma + \mu\gamma a^{\gamma-1} a_s + \frac{1}{2}\sigma^2\left(\gamma(\gamma-1)a^{\gamma-2}(a_s)^2 + \gamma a^{\gamma-1}a_{ss}\right) = 0$$

or:

$$1 - ra + \mu a_s + \frac{1}{2}\sigma^2\left((\gamma-1)\frac{(a_s)^2}{a} + a_{ss}\right) = 0 \tag{69}$$

subject to the boundary constraints:

$$a(\underline{s}) = K^{\frac{1}{\gamma}}\left(\frac{1}{r}\right) \tag{70}$$

$$\lim_{s\to\infty} a(s) = \left(\frac{1}{r}\right) \tag{71}$$

The last step in verifying the fact that the conjectured Value function is indeed the Value function for the problem at hand is to invoke a verification Theorem along the lines of Fleming and Soner (1993) p. 172. Before we do that we prove an almost obvious Lemma that is of independent interest:

Lemma 1 *Consider a consumption policy that has the feedback form:*

$$c_t^* = A(s_t)\left(X_t + \frac{y_t^*}{r-g}\right), \quad K^{-1/\gamma} r \le A(s_t) \le r \quad \forall s_t \in (-\underline{s}, +\infty)$$

Then $X_t \ge X_t^{NSS}$, *where* X_t^{NSS} *is the asset process that results when one uses the optimal consumption policy in the absence of sudden stops:*

$$c_t^{NSS} = r\left(X_t^{NSS} + \frac{y_t^*}{r-g}\right)$$

Proof. This result is trivial. One solves for the asset process that results from the two policies to find that

$$X_t^{NSS} = X_0 - \frac{y_0^*}{r-g}\left(e^{gt} - 1\right)$$

The equivalent calculation for the optimal policy c_t conditional on a path of s_t gives:

$$\begin{aligned} d\left(e^{-\left(rt - \int_0^t A(s_u)du\right)} X_t\right) &= -(r - A(s_t))e^{-\left(rt - \int_0^t A(s_u)du\right)} X_t + e^{-\left(rt - \int_0^t A(s_u)du\right)} dX_t = \\ &= e^{-\left(rt - \int_0^t A(s_u)du\right)} \left[y_0^* \frac{r - A(s_t) - g}{r-g} e^{gt}\right] \end{aligned}$$

Now integrating both sides and rearranging gives:

$$\begin{aligned} X_t &= X_0 + \frac{y_0^*}{r-g} e^{\left(rt - \int_0^t A(s_u)du\right)} \left(\int_0^t e^{-\left((r-g)i - \int_0^i A(s_u)du\right)}(r - A(s_i) - g)di\right) = \\ &X_0 - \frac{y_0^*}{r-g}\left(e^{gt} - e^{\left(rt - \int_0^t A(s_u)du\right)}\right) \end{aligned}$$

The result now follows since $A(s_t) \le r \quad \forall s_t \in (\underline{s}, +\infty)$. ■

To invoke a verification Theorem we finally need to show that:

Lemma 2 *Assume* $\left(\frac{1}{r}\right)^\gamma \le C(s_t) \le K\left(\frac{1}{r}\right)^\gamma \; \forall s_t \in (\underline{s}, +\infty)$. *Then*

$$\lim_{t\to\infty} E\left[e^{-rt}1\{\tau > t\}C(s)\frac{\left(X_t^{opt} + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\right] = 0$$

Proof. For any t it is the case that

$$\begin{aligned} E\left[e^{-rt}1\{\tau > t\}C(s)\frac{\left(X_t^{opt} + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\right] &\ge \\ E\left[e^{-rt}1\{\tau > t\}K\left(\frac{1}{r}\right)^\gamma \frac{\left(X_t^{opt} + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\right] &\ge \\ KE\left[e^{-rt}1\{\tau > t\}\left(\frac{1}{r}\right)^\gamma \frac{\left(X_t^{NSS} + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\right] &\to 0 \end{aligned}$$

The first inequality follows from the assumption and the fact that $\frac{\left(X_t^{opt}+\frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$ is a negative number. The second from the previous lemma and the monotonicity of the value function and the last limit follows from the fact that one can trivially show that in the standard model without sudden stops $e^{-rt}\left(\frac{1}{r}\right)^{\gamma}\frac{\left(X_t^{NSS}+\frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\rightarrow 0$ ■

The final step is to prove that any solution to $a(s)$ (respectively $C(s)$) will be bounded. This can be done in a rather straightforward manner:

Proposition 5 *The function $a(s)=C^{\frac{1}{\gamma}}$, is a decreasing function that stays between $\frac{1}{r}K^{1/\gamma}$ and $\frac{1}{r}$.*

Proof. We will derive the result by means of two contradictions. Suppose $a(s)$ solves (69) subject to the boundary conditions (70),(71). To establish the claim we just need to show that $a(s)$ is everywhere decreasing. (If it exited the "band" $[\frac{1}{r},\frac{1}{r}K^{1/\gamma}]$ it would have to have at least one section where it would be increasing). To establish that it is decreasing, we are going to establish 2 contradictions by studying points that could be local maxima or minima. So suppose indeed that there exists one point s^* that is a local extremum. Then $a_s(s^*)=0$ and accordingly we have 2 cases:

1) $1-ra<0$. Then it has to be true (in order to satisfy the ODE) that $a_{ss}>0$, so that s^* would have to be a local minimum. But in order for $a(s)$ to satisfy the Boundary condition at infinity there would have to exist another $s^{**}>s^*$, that would have to be local maximum, which is impossible since it would still be in the region where $1-ra<0$. (Since -necessarily- $a(s^*)<a(s^{**})$).

2) $1-ra>0$. Then it has to be true that $a_{ss}<0$, so that s^* would have to be a local maximum. But in order to satisfy the boundary condition at infinity there would have to be a local minimum s^{**} to the right of s^* satisfying $a(s^{**})<a(s^*)$ and $a_{ss}(s^{**})>0$ which is impossible. ■

Now one can apply a standard verification Theorem and verify that indeed the proposed function V is the Value function for the problem.

The following Lemma derives the properties of the consumption process:

Lemma 3 *The optimal consumption process satisfies:*

$$\frac{dc_t^*}{c_t^*}=\frac{1}{2}(\gamma+1)\left(\frac{\sigma a_s}{a}\right)^2 dt-\frac{a_s}{a}\sigma dB_t$$

Proof. Straightforward application of Ito's Lemma to:

$$c_t^*=\frac{\left(X_t+\frac{y_t^*}{r-g}\right)}{a(s_t)}$$

along with equation (69.) ■

7.3 Propositions and Proofs for section 4.1.

We first show that the excess consumption process,c_t^*, is constant. There are two ways to see this. The first one is to apply Ito's lemma to c_t^* and observe that both parts of the resulting Ito process (informally speaking the dt and the dB_t terms) are identically zero. Another more direct way is to use the martingale methodology developed by Karatzas and Wang (2000) to deal with this problem along with standard formulas for distributions of hitting times of Brownian Motion. This approach is particularly appealing in the context of this section because of the presence of complete markets (up to the point where the sudden stop takes place).

We start with the first approach. We have the following result:

Lemma 4 *Under perfect hedging "excess" consumption, c_t^*, is constant for $t < \tau$*

Proof. To simplify notation somewhat we define

$$a^p(s_t) = \frac{1}{r}\left(e^{-\lambda_1(s-\underline{s})}(K^{\frac{1}{\gamma}} - 1) + 1\right) \tag{72}$$

and observe that this function solves the ordinary differential equation:

$$\frac{1}{2}\sigma^2 a^p_{ss} + \mu a^p_s - ra^p + 1 = 0 \tag{73}$$

The consumption policy can then be reexpressed by means of (41) and (72) as

$$c_t^* = \frac{1}{a^p(s_t)}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)$$

and the portfolio policy as:

$$p_t = \frac{a_s}{a}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)$$

Applying Ito's Lemma to the right hand side of this expression gives:

$$dc^* = C_1 dt + C_2 dB_t$$

where

$$\begin{aligned} C_1 \quad = \quad & -\frac{1}{a^p}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)\left(\frac{a^p_s}{a^p}\mu - \left(\frac{a^p_s\sigma}{a^p}\right)^2 + \frac{a^p_{ss}}{a^p}\frac{\sigma^2}{2}\right) + \\ & +\frac{1}{a^p}\left(rX_t - \frac{1}{a^p}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right) + (1-\kappa)\,y_t + g\frac{y_t(1-\kappa)}{r-g}\right) + \\ & -\frac{1}{a^p}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)\left(\frac{\sigma a^p_s}{a^p}\right)^2 \\ = & -\frac{1}{(a^p)^2}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)\left(\frac{\sigma^2}{2}a^p_{ss} + a^p_s\left(r - \frac{1}{2}\sigma^2\right) - ra^p + 1\right) = 0 \end{aligned}$$

and

$$C_2 = -\frac{1}{a^p}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)\left(\frac{\sigma a_s^p}{a^p}\right) + \frac{1}{a^p}\left(X_t + \frac{y_t(1-\kappa)}{r-g}\right)\left(\frac{\sigma a_s^p}{a^p}\right) = 0$$

■

In other words perfect hedging leads to complete smoothing in our example, despite the possibility of a fall in consumption when the state variable crosses the critical level.

Let us prove the same result with the martingale methodology developed by Cox and Huang (1989) and Karatzas, Lechoszky and Shreve (1987), as it will facilitate the proofs later on. For our exposition we will be using the results in Karatzas and Wang (2000).

Lemma 5 *Without loss of generality, take $t = 0$. The optimal policy is given as:*

$$\begin{aligned} c_u^* &= (k^{-1/\gamma})1\{u < \tau\} \\ X_\tau &= \left(\frac{1}{r}\left(\frac{k}{K}\right)^{-1/\gamma} - \frac{y_\tau^*}{r-a}\right) \end{aligned}$$

where k solves the intertemporal budget constraint:

$$X_0 + \frac{y_0^*}{r-g} = k^{-1/\gamma}\left(E\int_0^\tau e^{-ru}du + e^{-r\tau}\frac{K^{1/\gamma}}{r}1\{\tau < \infty\}\right)$$

or

$$k = \left(\frac{X_0 + \frac{y_0^*}{r-g}}{\left(E\int_0^\tau e^{-ru}du + e^{-r\tau}\frac{K^{1/\gamma}}{r}1\{\tau < \infty\}\right)}\right)^{-\gamma}$$

Proof. We provide only a sketch of the proof. The reader is referred to Karatzas and Wang (2000) for details.[22]

The objective is:

$$\max_{c_u^c, p_u} E\left(\int_0^\tau \frac{c_u^{*1-\gamma}}{1-\gamma}e^{-ru}du + e^{-r\tau}V(X_\tau, y_\tau^*)1\{\tau < \infty\}\right)$$

s.t.

$$dX_t = (rX_t - c_t^* + y_t^*)dt + \sigma p_t dB_t$$

and the rest of the evolution equations remain unchanged.

We proceed to show formally that the proposed strategy is optimal borrowing from Karatzas and Wang (2000). Let us denote for an alternative admissible strategy (c^*, X_τ^*) satisfying the intertemporal budget constraint with equality the Value of the objective:

$$J(c,p) = E\left(\int_0^\tau \frac{c_u^{*1-\gamma}}{1-\gamma}e^{-ru}du + e^{-r\tau}V(X_\tau, y_\tau^*)1\{\tau < \infty\}\right)$$

[22] Karatzas and Wang (2000) do not strictly cover the case we are considering here, since our problem is on an infinite horizon. Even though it seems easy to extend their approach to cover our case too, we refrain from doing so and just note that the techniques used for the limited hedging case at the beginning of section 4 along with the observation about the optimal policy and the differentiability of the a^H−function are enough to apply a standard verification Theorem.

where

$$V(X_\tau, y_\tau) = K\left(\frac{1}{r}\right)^\gamma \frac{\left(X_t + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

Let $\widetilde{f}$ denote the Fenchel - Legendre Transform of the function f:

$$\widetilde{f}(k) = \max_x f(x) - kx$$

so that this maximization e.g. for

$$\frac{\widetilde{c_u^{*1-\gamma}}}{1-\gamma} e^{-ru} = \max_x e^{-ru}\left(\frac{c_u^{*1-\gamma}}{1-\gamma} - kc_u^*\right)$$

yields:

$$c_u^{*opt} = (k^{-1/\gamma})1\{u < \tau\}$$

and similarly for X_τ.

Then it is the case that:

$$\begin{aligned}
J(c,p) &= E\left(\int_0^\tau \frac{c_u^{*1-\gamma}}{1-\gamma} e^{-ru} du + e^{-r\tau} V(X_\tau, y_\tau^*) 1\{\tau < \infty\}\right) \leq \\
&\leq E\left(\int_0^\tau \frac{\widetilde{c_u^{*1-\gamma}}}{1-\gamma} e^{-ru} du + e^{-r\tau} \widetilde{V(X_\tau, y_\tau^*)} 1\{\tau < \infty\}\right) + \\
&\quad +k\left(E\int_0^\tau e^{-ru} c_u^* du + e^{-r\tau} X_\tau 1\{\tau < \infty\}\right) \\
&\leq E\left(\int_0^\tau \frac{\widetilde{c_u^{*1-\gamma}}}{1-\gamma} e^{-ru} du + e^{-r\tau} \widetilde{V(X_\tau, y_\tau)} 1\{\tau < \infty\}\right) + k\left(X_0 + \frac{y_0^*}{r-g}\right) = \\
&= E\left(\int_0^\tau \frac{(c_u^{*opt})^{1-\gamma}}{1-\gamma} e^{-ru} du + e^{-r\tau} V(X_\tau^{opt}, y_\tau^*) 1\{\tau < \infty\}\right) + \\
&\quad +k\left[\left(X_0 + \frac{y_0^*}{r-g}\right) - \left(X_0 + \frac{y_0^*}{r-g}\right)\right] \\
&= E\left(\int_0^\tau \frac{(c_u^{*opt})^{1-\gamma}}{1-\gamma} e^{-ru} du + e^{-r\tau} V(X_\tau^{opt}, y_\tau^*) 1\{\tau < \infty\}\right)
\end{aligned}$$

This verifies optimality of the strategy. We will now use the optimal strategy to obtain the Value function using well known formulas about the first hitting times of Brownian Motions.

$$V(X_0 + \frac{y_0^*}{r-g}) =$$

$$
\begin{aligned}
&= \left(E\int_0^\tau \frac{k^{-(1-\gamma)/\gamma}}{1-\gamma}e^{-ru}du + e^{-r\tau}\frac{k^{-(1-\gamma)/\gamma}}{1-\gamma}\left(\frac{K^{1/\gamma}}{r}\right)^\gamma\left(\frac{K^{1/\gamma}}{r}\right)^{1-\gamma}1\{\tau<\infty\}\right) = \\
&= \frac{\left(X_0+\frac{y_0^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\left(E\int_0^\tau e^{-ru}du + e^{-r\tau}\frac{K^{1/\gamma}}{r}1\{\tau<\infty\}\right)^\gamma = \\
&= \frac{\left(X_0+\frac{y_0^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\left(\frac{1}{r}\left(1-e^{-\lambda_1(s-\underline{s})}\right) + e^{-\lambda_1(s-\underline{s})}\frac{K^{1/\gamma}}{r}1\{\tau<\infty\}\right)^\gamma = \\
&= \frac{\left(X_0+\frac{y_0^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\left(\frac{1}{r}\right)^\gamma\left(1-e^{-\lambda_1(s-\underline{s})}+e^{-\lambda_1(s-\underline{s})}K^{\frac{1}{\gamma}}\right)^\gamma = \\
&= \frac{\left(X_0+\frac{y_0^*}{r-g}\right)^{1-\gamma}}{1-\gamma}\left(\frac{1}{r}\right)^\gamma\left(e^{-\lambda_1(s-\underline{s})}(K^{\frac{1}{\gamma}}-1)+1\right)^\gamma
\end{aligned}
$$

The only important step in this straightforward derivation is going from the second to the third line. This involves a well known formula which can be found e.g. in Harrison (1985) p. 47 ■

7.4 Proofs and Propositions for section 4.2

Proposition 6 *For* $1<\xi<\xi^*$ *,where:*

$$
\xi^* \stackrel{d}{=} \frac{1}{1+e^{-rT}(1+h^*)\left[\left(1-\frac{h^*}{(1+h^*)(1-e^{-rT})}\right)^\gamma-1\right]}
$$

the solution to the problem (74) is given by:

$$
c_t^* = k^{-\frac{1}{\gamma}}, t\in(0,\tau)
$$

$$
\begin{aligned}
X_{\tau-} &= \frac{k^{-\frac{1}{\gamma}}}{r(1+h^*)}\Gamma^{-\frac{1}{\gamma}} - \frac{y_\tau^*}{r-g} \\
h^{opt} &= h^* - (1-e^{-rT})(1+h^*)\left[1-\left(\frac{\xi}{\Gamma}\right)^{-\frac{1}{\gamma}}\right]
\end{aligned}
$$

Γ is defined by

$$
\Gamma \stackrel{d}{=} \frac{1-\xi}{e^{-rT}(1+h^*)}+\xi
$$

and h^* is given in the text.

$k^{-\frac{1}{\gamma}}$ is given by:

$$
k^{-\frac{1}{\gamma}} = \left[E\left[\int_0^\tau e^{-ru}du\right] + \frac{E\left[e^{-r\tau}\right]}{r}\frac{(1+h^{opt}\xi)\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)}\right]^{-1}\left(X_0+\frac{y_0^*}{r-g}\right)
$$

and since:

$$
E\left[e^{-r\tau}\right] = e^{-\lambda(s-\underline{s})}
$$

where λ is given in (56), we get finally that:

$$c_t^* = k^{-\frac{1}{\gamma}} = \left[\frac{1-e^{-\lambda(s-\underline{s})}}{r} + \frac{e^{-\lambda(s-\underline{s})}}{r}\frac{(1+h^{opt}\xi)\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)}\right]^{-1}\left(X_0 + \frac{y_0^*}{r-g}\right), t \in (0,\tau)$$

Proof. Adopting the Cox-Huang (1989) methodology and its application to problems involving a random stopping time Karatzas-Wang (2000) we are able to reduce the problem to the following static problem:

$$\min_k \max_{c_u^*, X_\tau, H_\tau} E\left[\int_0^\tau e^{-ru}\frac{c_u^{*1-\gamma}}{1-\gamma}du + e^{-r\tau}V^{SS}(X_\tau, R_\tau) - k\left(\int_0^\tau e^{-ru}c_u + e^{-r\tau}(X_\tau + \xi H_\tau)\right)\right]$$

where V^{SS} is given by (55). By maximizing the objective inside the integral one can derive optimal consumption to be:

$$c_t^* = k^{-\frac{1}{\gamma}}, t \in (0,\tau)$$

The crucial step is the maximization of the problem involving the continuation value function at time τ:

$$\max_{H_t, X_t}\left[V^{SS}(X_\tau, H_\tau) - k(X_\tau + \xi H_\tau)\right] \tag{74}$$

where V^{SS} is given above. Assuming for the moment that

$$H_\tau < \xi^*\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)$$

we can solve the problem (74) using (55), to get:

$$\begin{aligned}\left(\frac{1}{r}\right)^\gamma\left[(1-e^{-rT})^\gamma B\left(B\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right) + H_\tau\right)^{-\gamma} + e^{-rT}(1+h^*)^{1-\gamma}\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)^{-\gamma}\right] &= k \\ \left(\frac{1}{r}\right)^\gamma(1-e^{-rT})^\gamma\left(B\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right) + H_\tau\right)^{-\gamma} &= k\xi\end{aligned}$$

where:

$$B = 1 - e^{-rT}(1+h^*)$$

Let us define

$$k_1 = k\left(\frac{1}{r}\right)^{-\gamma}$$

so that we arrive at:

$$k_1\xi B + e^{-rT}(1+h^*)^{1-\gamma}\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)^{-\gamma} = k_1$$

or:

$$\left(X_{\tau^-} + \frac{y_\tau^*}{r-g}\right)^{-\gamma} = \frac{k_1(1-\xi B)}{e^{-rT}(1+h^*)^{1-\gamma}}$$

or

$$
\begin{aligned}
X_{\tau^-} + \frac{y_\tau^*}{r-g} &= \left(\frac{k_1(1-\xi B)}{e^{-rT}\left(1+h^*\right)^{1-\gamma}}\right)^{-\frac{1}{\gamma}} = \\
&= \left(\frac{k_1(1-\xi(1-e^{-rT}(1+h^*)))}{e^{-rT}\left(1+h^*\right)\left(1+h^*\right)^{-\gamma}}\right)^{-\frac{1}{\gamma}} = \\
&= \frac{k_1^{-\frac{1}{\gamma}}}{(1+h^*)}\left[\frac{1-\xi}{e^{-rT}\left(1+h^*\right)} + \xi\right]^{-\frac{1}{\gamma}}
\end{aligned}
$$

It is an interesting observation (that we will use later) to note that since $\xi > 1, \xi B < 1$ we get:

$$
\Gamma \overset{d}{=} \frac{1-\xi}{e^{-rT}\left(1+h^*\right)} + \xi \tag{75}
$$

and we have that $0 < \Gamma < 1, \Gamma^{-\frac{1}{\gamma}} > 1$.

Now the holdings of the second (hedging) asset are determined as:

$$
\begin{aligned}
(1-e^{-rT})^\gamma \left(B\left(\frac{k_1(1-\xi B)}{e^{-rT}\left(1+h^*\right)^{1-\gamma}}\right)^{-\frac{1}{\gamma}} + H_\tau\right)^{-\gamma} &= k_1\xi \\
\left(B\left(\frac{k_1(1-\xi B)}{e^{-rT}\left(1+h^*\right)^{1-\gamma}}\right)^{-\frac{1}{\gamma}} + H_\tau\right)^{-\gamma} &= \frac{k_1\xi}{(1-e^{-rT})^\gamma}
\end{aligned}
$$

$$
H_\tau = \left(\frac{k_1\xi}{(1-e^{-rT})^\gamma}\right)^{-\frac{1}{\gamma}} - B\left(\frac{k_1(1-\xi B)}{e^{-rT}\left(1+h^*\right)^{1-\gamma}}\right)^{-\frac{1}{\gamma}}
$$

To get the ratio

$$
h = \frac{H_\tau}{X_{\tau^-} + \frac{y_\tau^*}{r-g}}
$$

we combine the preceding equations to get:

$$
h = (1-e^{-rT})(1+h^*)\left(\frac{\xi}{\Gamma}\right)^{-\frac{1}{\gamma}} - \left(1-e^{-rT}(1+h^*)\right)
$$

We first verify that indeed $h < h^*$. This is true since:

$$
\begin{aligned}
h &= (1-e^{-rT})(1+h^*)\left(\frac{\xi}{\Gamma}\right)^{-\frac{1}{\gamma}} - \left(1+h^* - e^{-rT}(1+h^*)\right) + h^* \\
&= (1-e^{-rT})(1+h^*)\left[\left(\frac{\xi}{\Gamma}\right)^{-\frac{1}{\gamma}} - 1\right] + h^* < h^*
\end{aligned}
$$

since:

$$
\frac{\xi}{\Gamma} = \frac{\xi}{\frac{1-\xi}{e^{-rT}(1+h^*)} + \xi} = \frac{1}{\frac{\frac{1-\xi}{\xi}}{e^{-rT}(1+h^*)} + 1} > 1
$$

(since $\xi > 1$) and thus $(1-e^{-rT})(1+h^*)\left[\left(\frac{\xi}{\Gamma}\right)^{-\frac{1}{\gamma}} - 1\right] < 0$.

Finally we want to provide conditions in order to exclude that h becomes negative and this will be the case whenever:

$$\frac{\xi}{\Gamma} > \left(1 - \frac{h^*}{(1+h^*)(1-e^{-rT})}\right)^{-\gamma}$$

or

$$\frac{1}{\xi} < e^{-rT}(1+h^*)\left[\left(1 - \frac{h^*}{(1+h^*)(1-e^{-rT})}\right)^{\gamma} - 1\right] + 1$$

or

$$\xi > \xi^* \stackrel{d}{=} \frac{1}{1 + e^{-rT}(1+h^*)\left[\left(1 - \frac{h^*}{(1+h^*)(1-e^{-rT})}\right)^{\gamma} - 1\right]} \tag{76}$$

Accordingly we only focus on cases where $\xi < \xi^*$.

As might be expected due to the homotheticity of the problem, the optimal reserve ratio does not depend on the level of initial wealth. To complete the solution to the overall problem let us return to the time 0 budget constraint and combine everything together to get:

$$\begin{aligned} X_0 + \frac{y_0^*}{r-g} &= k^{-\frac{1}{\gamma}} E\left[\int_0^{\tau} e^{-ru} du\right] + k^{-\frac{1}{\gamma}} \frac{E\left[e^{-r\tau}\right]}{r} \frac{\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)} + \\ &\quad + h^{opt} \xi k^{-\frac{1}{\gamma}} \frac{E\left[e^{-r\tau}\right]}{r} \frac{\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)} \end{aligned}$$

which shows that k can be determined as:

$$k^{-\frac{1}{\gamma}} = \left[E\left[\int_0^{\tau} e^{-ru} du\right] + \frac{E\left[e^{-r\tau}\right]}{r} \frac{(1+h^{opt}\xi)\Gamma^{-\frac{1}{\gamma}}}{(1+h^*)}\right]^{-1} \left(X_0 + \frac{y_0^*}{r-g}\right)$$

and we have that $\Gamma < 1, \Gamma^{-\frac{1}{\gamma}} > 1$. The final step of the proof is to use the formula for $E\left[e^{-r\tau}\right]$ from section 4.2. ■

Remark 7 *The optimal portfolios can be derived in a manner similar to section 4.2. Their magnitude is much smaller in this case.*

7.5 Propositions and Proofs for section 4.3

We give a sketch of the claims. Formal proofs would proceed along the lines of the respective proofs in section 3.1. We focus only on p_t insurance for simplicity. In this case the Bellman Equation becomes:

$$\begin{aligned} 0 &= \max_{c^*}\left\{\frac{(c^*)^{1-\gamma}}{1-\gamma} - V_X c^*\right\} + \max_{p}\left\{\frac{1}{2}\sigma^2 p^2 V_{XX} + V_{Xs}\sigma^2 p\right\} - \\ &\quad - rV + V_X(rX + y^*) + V_y y^* g + V_s \mu + \frac{1}{2}\sigma^2 V_{ss} + \lambda(s_t)\left[\underline{C}\frac{\left(X_t + \frac{y_t^*}{r-g}\right)^{1-\gamma}}{1-\gamma} - V\right] \end{aligned}$$

Once again the optimal consumption and portfolio policies are given by (assuming a C^2- Value function):

$$c^* = V_X^{-\frac{1}{\gamma}}$$

and

$$p = -\frac{V_{Xs}}{V_{XX}}$$

Notice that the portfolio strategy is a "pure" hedging strategy, in the sense that there are no demands due to risk premia (we assumed them away by positing that the sources of risk are uncorrelated with aggregate consumption growth). In this way we are able to distill out the pure hedging component, excluding mean-variance motives. As before we conjecture a value function of the form:

$$V(X_t, y_t, s_t) = C^H(s_t)\frac{\left(X_t + \frac{y_t}{r-g}\right)^{1-\gamma}}{1-\gamma}$$

where $C^H(s_t)$ satisfies the same boundary conditions as in the no-hedging case (section 3.2.). Under this conjecture the optimal portfolio process becomes:

$$p = \frac{C^H_{s_t}(s_t)\left(X_t + \frac{y_t}{r-g}\right)}{\gamma C^H(s_t)}$$

Notice that the sign and magnitude of hedging is influenced by the derivative of $C(s_t)$ with respect to s_t. One can show that this is a decreasing function and thus not surprisingly the hedging demands are negative, i.e. they involve short sales. Plugging back into the value function and simplifying we get a second order non-linear ODE for $C(s_t)$.

$$\gamma C^{H\frac{\gamma-1}{\gamma}} - r\gamma C^H + C^H_s\mu + C^H_{ss}\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2\frac{(1-\gamma)(C^H_s)^2}{\gamma C^H} + \lambda(s_t)\left[\underline{C} - C^H\right] = 0$$

Now defining

$$C^H = (b^H)^\gamma$$

we get

$$\begin{aligned} 0 &= \gamma\left(b^H\right)^{\gamma-1} - r\gamma(b^H)^\gamma + \gamma\mu(b^H)^{\gamma-1}b^H_s + \frac{\sigma^2}{2}\gamma(b^H)^{\gamma-1}\left[(\gamma-1)\frac{\left(b^H_s\right)^2}{b^H} + b^H_{ss}\right] \\ &+ \frac{1}{2}\sigma^2\frac{(1-\gamma)\gamma^2(b^H_s)^2\left[b^H\right]^{2\gamma-2}}{\gamma(b^H)^\gamma} + \lambda(s_t)\left[A - (b^H)^\gamma\right] \end{aligned}$$

or after simplifying:

$$0 = 1 - rb^H + \mu b^H_s + \frac{\sigma^2}{2}b^H_{ss} + \frac{b^H}{\gamma}\lambda(s_t)\left[\frac{\underline{C}}{(b^H)^\gamma} - 1\right]$$

Notice that once again this ODE is very similar to the ODE with no hedging, namely:

$$0 = 1 - rb + b_s\mu + \left[(\gamma-1)\frac{(b_s)^2}{b} + b_{ss}\right]\frac{1}{2}\sigma^2 + \frac{b}{\gamma}\lambda(s_t)\left[\frac{\underline{C}}{b^\gamma} - 1\right]$$

up to the absence of the term $\frac{1}{2}\sigma^2(\gamma-1)\frac{(b_s)^2}{b}$. In terms of the b–function the optimal consumption and portfolio policies become:

$$c^*_t = \frac{\left(X_t + \frac{y_t}{r-g}\right)}{b^H(s_t)}$$

$$p_t = \frac{b_s^H(s_t)\left(X_t + \frac{y_t^*}{r-g}\right)}{b^H(s_t)}$$

Lemma 6 *Under perfect hedging "excess" consumption, i.e.*

$$c_t - \kappa y_t = c_t^*., \ t < \tau$$

follows the process:

$$\frac{dc_t^*}{c_t^*} = \left(\frac{\lambda(s_t)}{\gamma}\left[\frac{\underline{C}}{C(s_t)} - 1\right]\right) dt$$

Proof. The proof proceeds in a virtually identical way as in section 4.2. ■